Course 421: Algebraic Topology
Section 7: Homology Calculations

David R. Wilkins

Contents

7 Homology Calculations

7.1 The Homology Groups of an Octohedron

Let K be the simplicial complex consisting of the triangular faces, edges and vertices of an octohedron in $\mathbb{R}^3$ with vertices P_1, P_2, P_3, P_4, P_5 and P_6, where

$$P_1 = (0,0,1), \quad P_2 = (1,0,0), \quad P_3 = (0,1,0),$$

$$P_4 = (-1,0,0), \quad P_5 = (0,-1,0), \quad P_6 = (0,0,-1)$$

This octohedron consists of the four triangular faces $P_1P_2P_3$, $P_1P_3P_4$, $P_1P_4P_5$ and $P_1P_5P_2$ of the pyramid whose base is the square $P_2P_3P_4P_5$ and whose apex is P_1, together with the four triangular faces $P_6P_2P_3$, $P_6P_3P_4$, $P_6P_4P_5$ and $P_6P_5P_2$ of the pyramid whose base is $P_2P_3P_4P_5$ and whose apex is P_6.

A typical 2-chain c_2 of K is a linear combination, with integer coefficients, of eight oriented 2-simplices that represent the triangular faces of the octohedron. Thus we can write

$$c_2 = \sum_{i=1}^{8} n_i \sigma_i,$$

where $n_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, 8$ and

$$\sigma_1 = \langle P_1, P_2, P_3 \rangle, \quad \sigma_2 = \langle P_1, P_3, P_4 \rangle, \quad \sigma_3 = \langle P_1, P_4, P_5 \rangle,$$

$$\sigma_4 = \langle P_1, P_5, P_2 \rangle, \quad \sigma_5 = \langle P_6, P_3, P_2 \rangle, \quad \sigma_6 = \langle P_6, P_4, P_3 \rangle,$$

$$\sigma_7 = \langle P_6, P_5, P_4 \rangle, \quad \sigma_8 = \langle P_6, P_2, P_5 \rangle.$$

(The orientation on each of these triangles has been chosen such that the vertices of the triangle are listed in anticlockwise order when viewed from a point close to the centre of triangle that lies outside the octohedron.)

Similarly a typical 1-chain c_1 of K is a linear combination, with integer coefficients, of twelve 1-simplices that represent the edges of the octohedron. Thus we can write

$$c_1 = \sum_{j=1}^{12} m_j \rho_j,$$

where $m_j \in \mathbb{Z}$ for $j = 1, 2, \ldots, 12$ and

$$\rho_1 = \langle P_1, P_2 \rangle, \quad \rho_2 = \langle P_1, P_3 \rangle, \quad \rho_3 = \langle P_1, P_4 \rangle, \quad \rho_4 = \langle P_1, P_5 \rangle,$$

$$\rho_5 = \langle P_2, P_3 \rangle, \quad \rho_6 = \langle P_3, P_4 \rangle, \quad \rho_7 = \langle P_4, P_5 \rangle, \quad \rho_8 = \langle P_5, P_2 \rangle,$$

$$\rho_9 = \langle P_2, P_6 \rangle, \quad \rho_{10} = \langle P_3, P_6 \rangle, \quad \rho_{11} = \langle P_4, P_6 \rangle, \quad \rho_{12} = \langle P_5, P_6 \rangle,$$

A typical 0-chain c_0 takes the form

$$c_0 = \sum_{k=1}^{6} r_k \langle P_k \rangle,$$

where $r_k \in \mathbb{Z}$ for $k = 1, 2, \ldots, 6$.

We now calculate the boundary of a 2-chain. It follows from the definition of the boundary homomorphism ∂_2 that

$$\partial_2 \sigma_1 = \partial_2 \langle P_1, P_2, P_3 \rangle = \langle P_2 P_3 \rangle - \langle P_1 P_3 \rangle + \langle P_1 P_2 \rangle = \rho_5 - \rho_2 + \rho_1.$$

Similarly

$$\begin{aligned}
\partial_2 \sigma_2 &= \partial_2 \langle P_1, P_3, P_4 \rangle = \rho_6 - \rho_3 + \rho_2, \\
\partial_2 \sigma_3 &= \partial_2 \langle P_1, P_4, P_5 \rangle = \rho_7 - \rho_4 + \rho_3, \\
\partial_2 \sigma_4 &= \partial_2 \langle P_1, P_5, P_2 \rangle = \rho_8 - \rho_1 + \rho_4, \\
\partial_2 \sigma_5 &= \partial_2 \langle P_6, P_3, P_2 \rangle = -\rho_5 + \rho_9 - \rho_{10}, \\
\partial_2 \sigma_6 &= \partial_2 \langle P_6, P_4, P_3 \rangle = -\rho_6 + \rho_{10} - \rho_{11}, \\
\partial_2 \sigma_7 &= \partial_2 \langle P_6, P_5, P_4 \rangle = -\rho_7 + \rho_{11} - \rho_{12}, \\
\partial_2 \sigma_8 &= \partial_2 \langle P_6, P_2, P_5 \rangle = -\rho_8 + \rho_{12} - \rho_9.
\end{aligned}$$

Thus

$$\begin{aligned}
\partial_2 c_2 &= \partial_2 \left(n_1 \sigma_1 + n_2 \sigma_2 + n_3 \sigma_3 + n_4 \sigma_4 + n_5 \sigma_5 + n_6 \sigma_6 + n_7 \sigma_7 + n_8 \sigma_8 \right) \\
&= n_1 \partial_2 \sigma_1 + n_2 \partial_2 \sigma_2 + n_3 \partial_2 \sigma_3 + n_4 \partial_2 \sigma_4 \\
&\qquad + n_5 \partial_2 \sigma_5 + n_6 \partial_2 \sigma_6 + n_7 \partial_2 \sigma_7 + n_8 \partial_2 \sigma_8 \\
&= (n_1 - n_4)\rho_1 + (n_2 - n_1)\rho_2 + (n_3 - n_2)\rho_3 + (n_4 - n_3)\rho_4 \\
&\qquad + (n_1 - n_5)\rho_5 + (n_2 - n_6)\rho_6 + (n_3 - n_7)\rho_7 + (n_4 - n_8)\rho_8 \\
&\qquad + (n_5 - n_8)\rho_9 + (n_6 - n_5)\rho_{10} + (n_7 - n_6)\rho_{11} + (n_8 - n_7)\rho_{12}
\end{aligned}$$

It follows that $\partial_2 c_2 = 0$ if and only if

$$n_1 = n_2 = n_3 = n_4 = n_5 = n_6 = n_7 = n_8.$$

Therefore

$$Z_2(K) = \ker \partial_2 = \{ n\mu : n \in \mathbb{Z} \}, \text{ where } \mu = \sum_{i=1}^{8} \sigma_i.$$

Now $C_3(K) = 0$, and thus $B_2(K) = 0$ (where 0 here denotes the zero group), since the complex K has no 3-simplices. Therefore $H_2(K) \cong Z_2(K) \cong \mathbb{Z}$.

Next we calculate the boundary of a 1-chain. It follows from the definition of the boundary homomorphism ∂_1 that

$$\begin{aligned}
\partial_1 c_1 &= \partial_1 \left(\sum_{j=1}^{12} m_j \rho_j \right) \\
&= m_1(\langle P_2 \rangle - \langle P_1 \rangle) + m_2(\langle P_3 \rangle - \langle P_1 \rangle) \\
&\quad + m_3(\langle P_4 \rangle - \langle P_1 \rangle) + m_4(\langle P_5 \rangle - \langle P_1 \rangle) \\
&\quad + m_5(\langle P_3 \rangle - \langle P_2 \rangle) + m_6(\langle P_4 \rangle - \langle P_3 \rangle) \\
&\quad + m_7(\langle P_5 \rangle - \langle P_4 \rangle) + m_8(\langle P_2 \rangle - \langle P_5 \rangle) \\
&\quad + m_9(\langle P_6 \rangle - \langle P_2 \rangle) + m_{10}(\langle P_6 \rangle - \langle P_3 \rangle) \\
&\quad + m_{11}(\langle P_6 \rangle - \langle P_4 \rangle) + m_{12}(\langle P_6 \rangle - \langle P_5 \rangle) \\
&= -(m_1 + m_2 + m_3 + m_4)\langle P_1 \rangle + (m_1 - m_5 + m_8 - m_9)\langle P_2 \rangle \\
&\quad + (m_2 + m_5 - m_6 - m_{10})\langle P_3 \rangle + (m_3 + m_6 - m_7 - m_{11})\langle P_4 \rangle \\
&\quad + (m_4 + m_7 - m_8 - m_{12})\langle P_5 \rangle + (m_9 + m_{10} + m_{11} + m_{12})\langle P_6 \rangle
\end{aligned}$$

It follows that the 1-chain c_1 is a 1-cycle if and only if

$$m_1 + m_2 + m_3 + m_4 = 0, \quad m_1 - m_5 + m_8 - m_9 = 0,$$

$$m_2 + m_5 - m_6 - m_{10} = 0, \quad m_3 + m_6 - m_7 - m_{11} = 0,$$

$$m_4 + m_7 - m_8 - m_{12} = 0 \quad \text{and} \quad m_9 + m_{10} + m_{11} + m_{12} = 0.$$

On examining the structure of these equations, we see that, when c_1 is a 1-cycle, it is possible to eliminate five of the integer quantities m_j, expressing them in terms of the remaining quantities. For example, we can eliminate m_4, m_6, m_7, m_8 and m_{12}, expressing these quantities in terms of m_1, m_2, m_3, m_5, m_9 m_{10} and m_{11} by means of the equations

$$\begin{aligned}
m_4 &= -m_1 - m_2 - m_3, \\
m_6 &= m_2 - m_{10} + m_5, \\
m_7 &= m_2 + m_3 - m_{10} - m_{11} + m_5, \\
m_8 &= -m_1 + m_9 + m_5, \\
m_{12} &= -m_9 - m_{10} - m_{11}
\end{aligned}$$

It follows that

$$Z_2(K) = \{m_1 z_1 + m_2 z_2 + m_3 z_3 + m_5 z_5 + m_9 z_9 + m_{10} z_{10} + m_{11} z_{11}\},$$

where

$$\begin{aligned}
z_1 &= \rho_1 - \rho_4 - \rho_8 = -\partial_2\sigma_4, \\
z_2 &= \rho_2 - \rho_4 + \rho_6 + \rho_7 = \partial_2(\sigma_2 + \sigma_3), \\
z_3 &= \rho_3 - \rho_4 + \rho_7 = \partial_2\sigma_3, \\
z_5 &= \rho_5 + \rho_6 + \rho_7 + \rho_8 = \partial_2(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4), \\
z_9 &= \rho_8 + \rho_9 - \rho_{12} = -\partial_2\sigma_8, \\
z_{10} &= -\rho_6 - \rho_7 + \rho_{10} - \rho_{12} = \partial_2(\sigma_6 + \sigma_7), \\
z_{11} &= \rho_{11} - \rho_7 - \rho_{12} = \partial_2\sigma_7.
\end{aligned}$$

From these equations, we see that the generators z_1, z_2, z_3, z_5, z_9, z_{10} and z_{11} of the group $Z_1(K)$ of 1-cycles all belong to the group $B_1(K)$ of 1-boundaries. It follows that $Z_1(K) = B_1(K)$, and therefore $H_1(K) = 0$.

In order to determine $H_0(K)$ it suffices to note that the 0-chains

$$\langle P_2\rangle - \langle P_1\rangle, \quad \langle P_3\rangle - \langle P_1\rangle, \quad \langle P_4\rangle - \langle P_1\rangle, \quad \langle P_5\rangle - \langle P_1\rangle \quad \text{and} \quad \langle P_6\rangle - \langle P_1\rangle$$

are 0-boundaries. Indeed

$$\langle P_2\rangle - \langle P_1\rangle = \partial_1\rho_1, \quad \langle P_3\rangle - \langle P_1\rangle = \partial_1\rho_2, \quad \langle P_4\rangle - \langle P_1\rangle = \partial_1\rho_3,$$

$$\langle P_5\rangle - \langle P_1\rangle = \partial_1\rho_4 \quad \text{and} \quad \langle P_6\rangle - \langle P_1\rangle = \partial_1(\rho_1 + \rho_9).$$

Therefore

$$\sum_{k=1}^{6} r_k\langle P_k\rangle - \left(\sum_{k=1}^{6} r_k\right)\langle P_1\rangle \in B_0(K)$$

for all integers r_1, r_2, r_3, r_4, r_5 and r_6. It follows that $B_0(K) = \ker\varepsilon$, where $\varepsilon\colon C_0(K) \to \mathbb{Z}$ is the homomorphism defined such that

$$\varepsilon\left(\sum_{k=1}^{6} r_k\langle P_k\rangle\right) = \sum_{k=1}^{6} r_k$$

for all integers r_k ($k = 1, 2, \ldots, 6$). Now $Z_0(K) = C_0(K)$ since the homomorphism $\partial_0\colon C_0(K) \to C_{-1}(K)$ is the zero homomorphism mapping $C_0(K)$ to the zero group. It follows that

$$H_0(K) = C_0(K)/B_0(K) = C_0(K)/\ker\varepsilon \cong \mathbb{Z}.$$

(Here we are using the result that the image of a homomorphism is isomorphic to the quotient of the domain of the homomorphism by the kernel of the homomorphism.)

We have thus shown that

$$H_2(K) \cong \mathbb{Z}, \quad H_1(K) = 0, \quad H_0(K) \cong \mathbb{Z}.$$

One can show that $Z_1(K) = B_1(K)$ by employing an alternative approach to that used above. An element z of $Z_1(K)$ is of the form $z = \sum_{j=1}^{12} m_j \rho_j$, where

$$m_1 + m_2 + m_3 + m_4 = 0, \quad m_1 - m_5 + m_8 - m_9 = 0,$$

$$m_2 + m_5 - m_6 - m_{10} = 0, \quad m_3 + m_6 - m_7 - m_{11} = 0,$$

$$m_4 + m_7 - m_8 - m_{12} = 0 \quad \text{and} \quad m_9 + m_{10} + m_{11} + m_{12} = 0.$$

The 1-cycle z belongs to the group $B_1(K)$ if and only if there exists some 2-chain c_2 such that $z = \partial_2 c_2$. It follows that $z \in B_1(K)$ if and only if there exist integers $n_1, n_2, \ldots, n_8$ such that

$$m_1 = n_1 - n_4, \quad m_2 = n_2 - n_1, \quad m_3 = n_3 - n_2, \quad m_4 = n_4 - n_3,$$

$$m_5 = n_1 - n_5, \quad m_6 = n_2 - n_6, \quad m_7 = n_3 - n_7, \quad m_8 = n_4 - n_8,$$

$$m_9 = n_5 - n_8, \quad m_{10} = n_6 - n_5, \quad m_{11} = n_7 - n_6, \quad m_{12} = n_8 - n_7.$$

The integers $n_1, n_2, \ldots, n_8$ solving the above equations are not uniquely determined, since, given one collection of integers $n_1, n_2, \ldots, n_8$ satisfying these equations, another solution can be obtained by adding some fixed integer to each of $n_1, n_2, \ldots, n_8$. It follows from this that if there exists some collection $n_1, n_2, \ldots, n_8$ of integers that solves the above equations, then there exists a solution which satisfies the extra condition $n_1 = 0$. We then find that

$$n_1 = 0, \quad n_2 = m_2, \quad n_3 = m_2 + m_3, \quad n_4 = -m_1,$$

$$n_5 = -m_5, \quad n_6 = m_2 - m_6, \quad n_7 = m_2 + m_3 - m_7, \quad n_8 = -m_1 - m_8.$$

On substituting $n_1, n_2, \ldots, n_8$ into the relevant equations, and making use of the constraints on the values of $m_1, m_2, \ldots, m_{12}$, we find that we do indeed have a solution to the equations that express the integers m_j in terms of the integers n_i. It follows that every 1-cycle of K is a 1-boundary. Thus $Z_1(K) = B_1(K)$, and therefore $H_1(K) = 0$.

7.2 Another Homology Example

Let P_1, P_2, P_3, P_4, P_5 and P_6 be the vertices of a regular hexagon in the plane, listed in cyclic order, and let K be simplicial complex consisting of the triangles $P_1P_2P_3$, $P_3P_4P_5$ and $P5P_6P_1$, together with all the edges and vertices of these triangles. Then

$$C_2(K) = \{n_1\tau_1 + n_2\tau_2 + n_3\tau_3 : n_1, n_2, n_3 \in \mathbb{Z}\},$$

where

$$\tau_1 = \langle P_1P_2P_3\rangle, \quad \tau_2 = \langle P_3P_4P_5\rangle \quad \text{and} \quad \tau_3 = \langle P_5P_6P_1\rangle.$$

Also

$$C_1(K) = \left\{ \sum_{j=1}^{9} m_j\rho_j : m_j \in \mathbb{Z} \text{ for } j = 1, 2, \ldots, 9 \right\},$$

where

$$\rho_1 = \langle P_6P_1\rangle, \quad \rho_2 = \langle P_1P_2\rangle, \quad \rho_3 = \langle P_2P_3\rangle, \quad \rho_4 = \langle P_3P_4\rangle, \quad \rho_5 = \langle P_4P_5\rangle,$$

$$\rho_6 = \langle P_5P_6\rangle, \quad \rho_7 = \langle P_5P_1\rangle, \quad \rho_8 = \langle P_1P_3\rangle \quad \text{and} \quad \rho_9 = \langle P_3P_5\rangle,$$

and

$$C_0(K) = \left\{ \sum_{k=1}^{6} r_k\langle P_k\rangle : m_k \in \mathbb{Z} \text{ for } k = 1, 2, \ldots, 6 \right\}.$$

Also

$$\partial_2\tau_1 = \rho_3 - \rho_8 + \rho_2, \quad \partial_2\tau_2 = \rho_5 - \rho_9 + \rho_4, \quad \partial_2\tau_3 = \rho_1 - \rho_7 + \rho_6,$$

Now

$$\begin{aligned} &\partial_2(n_1\tau_1 + n_2\tau_2 + n_3\tau_3) \\ &\quad = \quad n_3\rho_1 + n_1\rho_2 + n_1\rho_3 + n_2\rho_4 + n_2\rho_5 + n_3\rho_6 - n_3\rho_7 - n_1\rho_8 - n_2\rho_9. \end{aligned}$$

The simplicial complex K has no non-zero 2-cycles, and therefore $Z_2(K) = 0$. It follows that $H_2(K) = 0$.

Let

$$c_1 = \sum_{j=1}^{9} m_j\rho_j.$$

Then

$$\begin{aligned} \partial_1 c_1 \quad = \quad & (m_1 - m_2 + m_7 - m_8)\langle P_1\rangle + (m_2 - m_3)\langle P_2\rangle \\ & + (m_3 - m_4 + m_8 - m_9)\langle P_3\rangle + (m_4 - m_5)\langle P_4\rangle \\ & + (m_5 - m_6 + m_9 - m_7)\langle P_5\rangle + (m_6 - m_1)\langle P_6\rangle \end{aligned}$$

It follows that c_1 is a 1-cycle of K if and only if

$$m_2 = m_3, \quad m_4 = m_5, \quad m_6 = m_1$$

and

$$m_1 + m_7 = m_3 + m_8 = m_5 + m_9.$$

Moreover c_1 is a 1-boundary of K if and only if

$$m_2 = m_3 = -m_8, \quad m_4 = m_5 = -m_9, \quad m_6 = m_1 = -m_7.$$

We see from this that not every 1-cycle of K is a 1-boundary of K. Indeed

$$Z_1(K) = \{n_1\partial_2\tau_1 + n_2\partial_2\tau_2 + n_3\partial_2\tau_3 + nz : n_1, n_2, n_3, n \in \mathbb{Z}\},$$

where $z = \rho_7 + \rho_8 + \rho_9$. Let $\theta: Z_1(K) \to \mathbb{Z}$ be the homomorphism defined such that

$$\theta\left(n_1\partial_2\tau_1 + n_2\partial_2\tau_2 + n_3\partial_2\tau_3 + nz\right) = n$$

for all $n_1, n_2, n_3, n \in \mathbb{Z}$. Now

$$n_1\partial_2\tau_1 + n_2\partial_2\tau_2 + n_3\partial_2\tau_3 + nz \in B_1(K) \quad \text{if and only if} \quad n = 0.$$

It follows that $B_1(K) = \ker\theta$. Therefore the homomorphism θ induces an isomorphism from $H_1(K)$ to $\mathbb{Z}$, where $H_1(K) = Z_1(K)/B_1(K)$. Indeed $H_1(K) = \{n[z] : n \in \mathbb{Z}\}$, where $z = \rho_7 + \rho_8 + \rho_9$ and $[z]$ denotes the homology class of the 1-cycle z.

It is a straightforward exercise to verify that

$$B_0(K) = \left\{\sum_{k=1}^{6} r_k\langle P_k\rangle : r_k \in \mathbb{Z} \text{ for } k = 1, 2, \ldots, 6 \text{ and } \sum_{k=1}^{6} r_k = 0\right\}.$$

It follows from this that $H_0(K) \cong \mathbb{Z}$. Indeed this result is a consquence of the fact that the polyhedron $|K|$ of the simplicial complex K is connected.

7.3 The Homology Groups of the Boundary of a Simplex

Proposition 7.1 *Let K be the simplicial complex consisting of all the proper faces of an $(n+1)$-dimensional simplex σ, where $n > 0$. Then*

$$H_0(K) \cong \mathbb{Z}, \quad H_n(K) \cong Z, \quad H_q(K) = 0 \text{ when } q \neq 0, n.$$

Proof Let M be the simplicial complex consisting of the $(n+1)$-dimensional simplex σ, together with all its faces. Then K is a subcomplex of M, and $C_q(K) = C_q(M)$ when $q \leq n$.

It follows from Proposition 6.4 that $H_0(M) \cong \mathbb{Z}$ and $H_q(M) = 0$ when $q > 0$. (Here 0 denotes the zero group.) Now $Z_q(K) = Z_q(M)$ when $q \leq n$, and $B_q(K) = B_q(M)$ when $q < n$. It follows that $H_q(K) = H_q(M)$ when $q < n$. Thus $H_0(K) \cong \mathbb{Z}$ and $H_q(K) = 0$ when $0 < q < n$. Also $H_q(K) = 0$ when $q > n$, since the simplicial complex K is of dimension n. Thus, to determine the homology of the complex K, it only remains to find $H_n(K)$.

Let the $(n+1)$-dimensional simplex σ have vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{n+1}$. Then

$$C_{n+1}(M) = \{n\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{n+1}\rangle : n \in \mathbb{Z}\}.$$

and therefore $B_n(M) = \{nz : n \in \mathbb{Z}\}$, where

$$z = \partial_{n+1}\left(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{n+1}\rangle\right).$$

Now $H_n(M) = 0$ (Proposition 6.4). It follows that $Z_n(M) = B_n(M)$. But $Z_n(K) = Z_m(M)$, since $C_n(K) = C_n(M)$ and the definition of the boundary homomorphism on $C_n(K)$ is consistent with the definition of the boundary homomorphism on $C_n(M)$. Also $B_n(K) = 0$, because the simplicial complex K is of dimension n, and therefore has no non-zero n-boundaries. It follows that

$$H_n(K) \cong Z_n(K) = Z_n(M) = B_n(M) \cong \mathbb{Z}.$$

Indeed $H_n(K) = \{n[z] : n \in \mathbb{Z}\}$, where $[z]$ denotes the homology class of the n-cycle z of K defined above. ∎

Remark Note that the n-cycle z is an n-cycle of the simplicial complex K, since it is a linear combination, with integer coefficients, of oriented n-simplices of K. The n-cycle z is an n-boundary of the large simplicial complex M. However it is not an n-boundary of K. Indeed the n-dimensional simplicial complex K has no non-zero $(n+1)$-chains, therefore has no non-zero n-boundaries. Therefore z represents a non-zero homology class $[z]$ of $H_n(K)$. This homology class generates the homology group $H_n(K)$.

Remark The boundary of a 1-simplex consists of two points. Thus if K is the simplicial complex representing the boundary of a 1-simplex then $H_0(K) \cong \mathbb{Z} \oplus IZ$ (Corollary 6.9), and $H_q(K) = 0$ when $q > 0$.