6 Simplicial Homology Groups

6.1 The Chain Groups of a Simplicial Complex

Let $K$ be a simplicial complex. For each non-negative integer $q$, let $\Delta_q(K)$ be the additive group consisting of all formal sums of the form

$$n_1(v^1_0, v^1_1, \ldots, v^1_q) + n_2(v^2_0, v^2_1, \ldots, v^2_q) + \cdots + n_s(v^s_0, v^s_1, \ldots, v^s_q),$$

where $n_1, n_2, \ldots, n_s$ are integers and $v^r_0, v^r_1, \ldots, v^r_q$ are (not necessarily distinct) vertices of $K$ that span a simplex of $K$ for $r = 1, 2, \ldots, s$. (In more formal language, the group $\Delta_q(K)$ is the free Abelian group generated by the set of all $(q+1)$-tuples of the form $(v_0, v_1, \ldots, v_q)$, where $v_0, v_1, \ldots, v_q$ span a simplex of $K$.)

We recall some basic facts concerning permutations. A permutation of a set $S$ is a bijection mapping $S$ onto itself. The set of all permutations of some set $S$ is a group; the group multiplication corresponds to composition of permutations. A transposition is a permutation of a set $S$ which interchanges two elements of $S$, leaving the remaining elements of the set fixed. If $S$ is finite and has more than one element then any permutation of $S$ can be expressed as a product of transpositions. In particular any permutation of the set $\{0, 1, \ldots, q\}$ can be expressed as a product of transpositions $(j-1, j)$ that interchange $j-1$ and $j$ for some $j$.

Associated to any permutation $\pi$ of a finite set $S$ is a number $\epsilon_\pi$, known as the parity or signature of the permutation, which can take on the values $\pm 1$. If $\pi$ can be expressed as the product of an even number of transpositions, then $\epsilon_\pi = +1$; if $\pi$ can be expressed as the product of an odd number of transpositions then $\epsilon_\pi = -1$. The function $\pi \mapsto \epsilon_\pi$ is a homomorphism from the group of permutations of a finite set $S$ to the multiplicative group $\{+1, -1\}$ (i.e., $\epsilon_{\pi \rho} = \epsilon_\pi \epsilon_\rho$ for all permutations $\pi$ and $\rho$ of the set $S$). Note in particular that the parity of any transposition is $-1$.

**Definition** The $q$th chain group $C_q(K)$ of the simplicial complex $K$ is defined to be the quotient group $\Delta_q(K)/\Delta_0^q(K)$, where $\Delta_0^q(K)$ is the subgroup of $\Delta_q(K)$ generated by elements of the form $(v_0, v_1, \ldots, v_q)$ where $v_0, v_1, \ldots, v_q$ are not all distinct, and by elements of the form

$$(v_{\pi(0)}, v_{\pi(1)}, \ldots, v_{\pi(q)}) - \epsilon_\pi(v_0, v_1, \ldots, v_q)$$

where $\pi$ is some permutation of $\{0, 1, \ldots, q\}$ with parity $\epsilon_\pi$. For convenience, we define $C_q(K) = \{0\}$ when $q < 0$ or $q > \dim K$, where $\dim K$ is the dimension of the simplicial complex $K$. An element of the chain group $C_q(K)$ is referred to as $q$-chain of the simplicial complex $K$. 

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We denote by \( \langle v_0, v_1, \ldots, v_q \rangle \) the element \( \Delta^0_q(K) + \langle v_0, v_1, \ldots, v_q \rangle \) of \( C_q(K) \) corresponding to \( (v_0, v_1, \ldots, v_q) \). The following results follow immediately from the definition of \( C_q(K) \).

**Lemma 6.1** Let \( v_0, v_1, \ldots, v_q \) be vertices of a simplicial complex \( K \) that span a simplex of \( K \). Then

- \( \langle v_0, v_1, \ldots, v_q \rangle = 0 \) if \( v_0, v_1, \ldots, v_q \) are not all distinct,
- \( \langle v_{\pi(0)}, v_{\pi(1)}, \ldots, v_{\pi(q)} \rangle = \epsilon_\pi \langle v_0, v_1, \ldots, v_q \rangle \) for any permutation \( \pi \) of the set \( \{0, 1, \ldots, q\} \).

**Example** If \( v_0 \) and \( v_1 \) are the endpoints of some line segment then

\[
\langle v_0, v_1 \rangle = -\langle v_1, v_0 \rangle.
\]

If \( v_0, v_1 \) and \( v_2 \) are the vertices of a triangle in some Euclidean space then

\[
\begin{align*}
\langle v_0, v_1, v_2 \rangle &= \langle v_1, v_2, v_0 \rangle = \langle v_2, v_0, v_1 \rangle = -\langle v_2, v_1, v_0 \rangle \\
&= -\langle v_0, v_2, v_1 \rangle = -\langle v_1, v_0, v_2 \rangle.
\end{align*}
\]

**Definition** An *oriented q-simplex* is an element of the chain group \( C_q(K) \) of the form \( \pm \langle v_0, v_1, \ldots, v_q \rangle \), where \( v_0, v_1, \ldots, v_q \) are distinct and span a simplex of \( K \).

An oriented simplex of \( K \) can be thought of as consisting of a simplex of \( K \) (namely the simplex spanned by the prescribed vertices), together with one of two possible ‘orientations’ on that simplex. Any ordering of the vertices determines an orientation of the simplex; any even permutation of the ordering of the vertices preserves the orientation on the simplex, whereas any odd permutation of this ordering reverses orientation.

Any q-chain of a simplicial complex \( K \) can be expressed as a sum of the form

\[
n_1\sigma_1 + n_2\sigma_2 + \cdots + n_s\sigma_s
\]

where \( n_1, n_2, \ldots, n_s \) are integers and \( \sigma_1, \sigma_2, \ldots, \sigma_s \) are oriented q-simplices of \( K \). If we reverse the orientation on one of these simplices \( \sigma_i \) then this reverses the sign of the corresponding coefficient \( n_i \). If \( \sigma_1, \sigma_2, \ldots, \sigma_s \) represent distinct simplices of \( K \) then the coefficients \( n_1, n_2, \ldots, n_s \) are uniquely determined.

**Example** Let \( v_0, v_1 \) and \( v_2 \) be the vertices of a triangle in some Euclidean space. Let \( K \) be the simplicial complex consisting of this triangle, together
with its edges and vertices. Every 0-chain of \( K \) can be expressed uniquely in the form

\[
  n_0(v_0) + n_1(v_1) + n_2(v_2)
\]

for some \( n_0, n_1, n_2 \in \mathbb{Z} \). Similarly any 1-chain of \( K \) can be expressed uniquely in the form

\[
  m_0(v_1, v_2) + m_1(v_2, v_0) + m_2(v_0, v_1)
\]

for some \( m_0, m_1, m_2 \in \mathbb{Z} \), and any 2-chain of \( K \) can be expressed uniquely as \( n(v_0, v_1, v_2) \) for some integer \( n \).

**Lemma 6.2** Let \( K \) be a simplicial complex, and let \( A \) be an additive group. Suppose that, to each \((q + 1)\)-tuple \((v_0, v_1, \ldots, v_q)\) of vertices spanning a simplex of \( K \), there corresponds an element \( \alpha(v_0, v_1, \ldots, v_q) \) of \( A \), where

- \( \alpha(v_0, v_1, \ldots, v_q) = 0 \) unless \( v_0, v_1, \ldots, v_q \) are all distinct,
- \( \alpha(v_0, v_1, \ldots, v_q) \) changes sign on interchanging any two adjacent vertices \( v_j \) with \( v_j \) for all \( j \),
- \( \sum_r n_r(v_0, v_1, \ldots, v_q) \) is well-defined in \( A \).

Then there exists a well-defined homomorphism from \( C_q(K) \) to \( A \) which sends \((v_0, v_1, \ldots, v_q)\) to \( \alpha(v_0, v_1, \ldots, v_q) \) whenever \( v_0, v_1, \ldots, v_q \) span a simplex of \( K \). This homomorphism is uniquely determined.

**Proof** The given function defined on \((q + 1)\)-tuples of vertices of \( K \) extends to a well-defined homomorphism \( \alpha: \Delta_q(K) \rightarrow A \) given by

\[
  \alpha \left( \sum_{r=1}^s n_r(v_0^r, v_1^r, \ldots, v_q^r) \right) = \sum_{r=1}^s n_r \alpha(v_0^r, v_1^r, \ldots, v_q^r)
\]

for all \( \sum_{r=1}^s n_r(v_0^r, v_1^r, \ldots, v_q^r) \in \Delta_q(K) \). Moreover \((v_0, v_1, \ldots, v_q) \in \ker \alpha \) unless \( v_0, v_1, \ldots, v_q \) are all distinct. Also

\[
  (v_{\pi(0)}, v_{\pi(1)}, \ldots, v_{\pi(q)}) - \varepsilon_\pi(v_0, v_1, \ldots, v_q) \in \ker \alpha
\]

for all permutations \( \pi \) of \( \{0, 1, \ldots, q\} \), since the permutation \( \pi \) can be expressed as a product of transpositions \( (j - 1, j) \) that interchange \( j - 1 \) with \( j \) for some \( j \) and leave the rest of the set fixed, and the parity \( \varepsilon_\pi \) of \( \pi \) is given by \( \varepsilon_\pi = +1 \) when the number of such transpositions is even, and by \( \varepsilon_\pi = -1 \) when the number of such transpositions is odd. Thus the generators of \( \Delta_0^0(K) \) are contained in \( \ker \alpha \), and hence \( \Delta_0^0(K) \subset \ker \alpha \). The required homomorphism \( \tilde{\alpha}: C_q(K) \rightarrow A \) is then defined by the formula

\[
  \tilde{\alpha} \left( \sum_{r=1}^s n_r(v_0^r, v_1^r, \ldots, v_q^r) \right) = \sum_{r=1}^s n_r \alpha(v_0^r, v_1^r, \ldots, v_q^r).
\]
6.2 Boundary Homomorphisms

Let $K$ be a simplicial complex. We introduce below boundary homomorphisms $\partial_q: C_q(K) \rightarrow C_{q-1}(K)$ between the chain groups of $K$. If $\sigma$ is an oriented $q$-simplex of $K$ then $\partial_q(\sigma)$ is a $(q-1)$-chain which is a formal sum of the $(q-1)$-faces of $\sigma$, each with an orientation determined by the orientation of $\sigma$.

Let $\sigma$ be a $q$-simplex with vertices $v_0, v_1, \ldots, v_q$. For each integer $j$ between 0 and $q$ we denote by $\langle v_0, \ldots, \hat{v}_j, \ldots, v_q \rangle$ the oriented $(q-1)$-face
\[
\langle v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_q \rangle
\]
of the simplex $\sigma$ obtained on omitting $v_j$ from the set of vertices of $\sigma$. In particular
\[
\langle \hat{v}_0, v_1, \ldots, v_q \rangle \equiv \langle v_1, \ldots, v_q \rangle, \quad \langle v_0, \ldots, v_{q-1}, \hat{v}_q \rangle \equiv \langle v_0, \ldots, v_{q-1} \rangle.
\]
Similarly if $j$ and $k$ are integers between 0 and $q$, where $j < k$, we denote by
\[
\langle v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_k, \ldots, v_q \rangle
\]
the oriented $(q-2)$-face $\langle v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{k-1}, v_{k+1}, \ldots, v_q \rangle$ of the simplex $\sigma$ obtained on omitting $v_j$ and $v_k$ from the set of vertices of $\sigma$.

We now define a ‘boundary homomorphism’ $\partial_q: C_q(K) \rightarrow C_{q-1}(K)$ for each integer $q$. Define $\partial_q = 0$ if $q \leq 0$ or $q > \dim K$. (In this case one or other of the groups $C_q(K)$ and $C_{q-1}(K)$ is trivial.) Suppose then that $0 < q \leq \dim K$. Given vertices $v_0, v_1, \ldots, v_q$ spanning a simplex of $K$, let
\[
\alpha(v_0, v_1, \ldots, v_q) = \sum_{j=0}^{q} (-1)^j \langle v_0, \ldots, \hat{v}_j, \ldots, v_q \rangle.
\]
Inspection of this formula shows that $\alpha(v_0, v_1, \ldots, v_q)$ changes sign whenever two adjacent vertices $v_{i-1}$ and $v_i$ are interchanged.

Suppose that $v_j = v_k$ for some $j$ and $k$ satisfying $j < k$. Then
\[
\alpha(v_0, v_1, \ldots, v_q) = (-1)^j \langle v_0, \ldots, \hat{v}_j, \ldots, v_q \rangle + (-1)^k \langle v_0, \ldots, \hat{v}_k, \ldots, v_q \rangle,
\]
since the remaining terms in the expression defining $\alpha(v_0, v_1, \ldots, v_q)$ contain both $v_j$ and $v_k$. However $(v_0, \ldots, \hat{v}_k, \ldots, v_q)$ can be transformed to $(v_0, \ldots, \hat{v}_j, \ldots, v_q)$ by making $k-j-1$ transpositions which interchange $v_j$ successively with the vertices $v_{j+1}, v_{j+2}, \ldots, v_{k-1}$. Therefore
\[
\langle v_0, \ldots, \hat{v}_k, \ldots, v_q \rangle = (-1)^{k-j-1} \langle v_0, \ldots, \hat{v}_j, \ldots, v_q \rangle.
\]
Thus \( \alpha(v_0, v_1, \ldots, v_q) = 0 \) unless \( v_0, v_1, \ldots, v_q \) are all distinct. It now follows immediately from Lemma 6.2 that there is a well-defined homomorphism \( \partial_q: C_q(K) \rightarrow C_{q-1}(K) \), characterized by the property that

\[
\partial_q (\langle v_0, v_1, \ldots, v_q \rangle) = \sum_{j=0}^{q} (-1)^j \langle v_0, \ldots, \hat{v}_j, \ldots, v_q \rangle
\]

whenever \( v_0, v_1, \ldots, v_q \) span a simplex of \( K \).

**Lemma 6.3** \( \partial_{q-1} \circ \partial_q = 0 \) for all integers \( q \).

**Proof** The result is trivial if \( q < 2 \), since in this case \( \partial_{q-1} = 0 \). Suppose that \( q \geq 2 \). Let \( v_0, v_1, \ldots, v_q \) be vertices spanning a simplex of \( K \). Then

\[
\partial_{q-1} \partial_q (\langle v_0, v_1, \ldots, v_q \rangle) = \sum_{j=0}^{q} (-1)^j \partial_{q-1} (\langle v_0, \ldots, \hat{v}_j, \ldots, v_q \rangle)
\]

\[
= \sum_{j=0}^{q} \sum_{k=0}^{j-1} (-1)^{j+k} \langle v_0, \ldots, \hat{v}_k, \ldots, v_j, \ldots, v_q \rangle
\]

\[
+ \sum_{j=0}^{q} \sum_{k=j+1}^{q} (-1)^{j+k-1} \langle v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_k, \ldots, v_q \rangle
\]

\[
= 0
\]

(since each term in this summation over \( j \) and \( k \) cancels with the corresponding term with \( j \) and \( k \) interchanged). The result now follows from the fact that the homomorphism \( \partial_{q-1} \circ \partial_q \) is determined by its values on all oriented \( q \)-simplices of \( K \). \( \Box \)

### 6.3 The Homology Groups of a Simplicial Complex

Let \( K \) be a simplicial complex. A \( q \)-chain \( z \) is said to be a \( q \)-cycle if \( \partial_q z = 0 \). A \( q \)-chain \( b \) is said to be a \( q \)-boundary if \( b = \partial_{q+1} c' \) for some \( (q+1) \)-chain \( c' \). The group of \( q \)-cycles of \( K \) is denoted by \( Z_q(K) \), and the group of \( q \)-boundaries of \( K \) is denoted by \( B_q(K) \). Thus \( Z_q(K) \) is the kernel of the boundary homomorphism \( \partial_q: C_q(K) \rightarrow C_{q-1}(K) \), and \( B_q(K) \) is the image of the boundary homomorphism \( \partial_{q+1}: C_{q+1}(K) \rightarrow C_q(K) \). However \( \partial_q \circ \partial_{q+1} = 0 \), by Lemma 6.3. Therefore \( B_q(K) \subset Z_q(K) \). But these groups are subgroups of the Abelian group \( C_q(K) \). We can therefore form the quotient group \( H_q(K) \), where \( H_q(K) = Z_q(K) / B_q(K) \). The group \( H_q(K) \) is referred to as the \( q \)th homology group of the simplicial complex \( K \). Note that \( H_q(K) = 0 \) if \( q < 0 \).
or \( q > \dim K \) (since \( Z_q(K) = 0 \) and \( B_q(K) = 0 \) in these cases). It can be shown that the homology groups of a simplicial complex are topological invariants of the polyhedron of that complex.

The element \([z] \in H_q(K)\) of the homology group \( H_q(K)\) determined by \( z \in Z_q(K) \) is referred to as the homology class of the \( q\)-cycle \( z \). Note that \([z_1 + z_2] = [z_1] + [z_2]\) for all \( z_1, z_2 \in Z_q(K)\), and \([z_1] = [z_2]\) if and only if \( z_1 - z_2 = \partial_{q+1} c\) for some \((q + 1)\)-chain \( c\).

**Proposition 6.4** Let \( K \) be a simplicial complex. Suppose that there exists a vertex \( w \) of \( K \) with the following property:

- if vertices \( v_0, v_1, \ldots, v_q \) span a simplex of \( K \) then so do \( w, v_0, v_1, \ldots, v_q \).

Then \( H_0(K) \cong \mathbb{Z} \), and \( H_q(K) \) is the zero group for all \( q > 0 \).

**Proof** Using Lemma 6.2, we see that there is a well-defined homomorphism \( D_q: C_q(K) \to C_{q+1}(K)\) characterized by the property that

\[
D_q(\langle v_0, v_1, \ldots, v_q \rangle) = \langle w, v_0, v_1, \ldots, v_q \rangle
\]

whenever \( v_0, v_1, \ldots, v_q \) span a simplex of \( K \). Now \( \partial_1(D_0(v)) = v - w \) for all vertices \( v \) of \( K \). It follows that

\[
\sum_{r=1}^{s} n_r \langle v_r \rangle - \left( \sum_{r=1}^{s} n_r \right) \langle w \rangle = \sum_{r=1}^{s} n_r (\langle v_r \rangle - \langle w \rangle) \in B_0(K)
\]

for all \( \sum_{r=1}^{s} n_r \langle v_r \rangle \in C_0(K) \). But \( Z_0(K) = C_0(K) \) (since \( \partial_0 = 0 \) by definition), and thus \( H_0(K) = C_0(K) / B_0(K) \). It follows that there is a well-defined surjective homomorphism from \( H_0(K) \) to \( \mathbb{Z} \) induced by the homomorphism from \( C_0(K) \) to \( \mathbb{Z} \) that sends \( \sum_{r=1}^{s} n_r \langle v_r \rangle \in C_0(K) \) to \( \sum_{r=1}^{s} n_r \). Moreover this induced homomorphism is an isomorphism from \( H_0(K) \) to \( \mathbb{Z} \).

Now let \( q > 0 \). Then

\[
\partial_{q+1}(D_q(\langle v_0, v_1, \ldots, v_q \rangle)) = \partial_{q+1}(\langle w, v_0, v_1, \ldots, v_q \rangle)
\]

\[
= \langle v_0, v_1, \ldots, v_q \rangle + \sum_{j=0}^{q} (-1)^{j+1} \langle w, v_0, \ldots, \hat{v}_j, \ldots, v_q \rangle
\]

\[
= \langle v_0, v_1, \ldots, v_q \rangle - D_{q-1}(\partial_q(\langle v_0, v_1, \ldots, v_q \rangle))
\]

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whenever \( v_0, v_1, \ldots, v_q \) span a simplex of \( K \). Thus
\[
\partial_{q+1}(D_q(c)) + D_{q-1}(\partial_q(c)) = c
\]
for all \( c \in C_q(K) \). In particular \( z = \partial_{q+1}(D_q(z)) \) for all \( z \in Z_q(K) \), and hence \( Z_q(K) = B_q(K) \). It follows that \( H_q(K) \) is the zero group for all \( q > 0 \), as required.

**Example** The hypotheses of the proposition are satisfied for the complex \( K_\sigma \) consisting of a simplex \( \sigma \) together with all of its faces: we can choose \( w \) to be any vertex of the simplex \( \sigma \).

### 6.4 Simplicial Maps and Induced Homomorphisms

Any simplicial map \( \varphi: K \to L \) between simplicial complexes \( K \) and \( L \) induces well-defined homomorphisms \( \varphi_q: C_q(K) \to C_q(L) \) of chain groups, where
\[
\varphi_q([v_0, v_1, \ldots, v_q]) = [\varphi(v_0), \varphi(v_1), \ldots, \varphi(v_q)]
\]
whenever \( v_0, v_1, \ldots, v_q \) span a simplex of \( K \). (The existence of these induced homomorphisms follows from a straightforward application of Lemma 6.2.) Note that \( \varphi_q([v_0, v_1, \ldots, v_q]) = 0 \) unless \( \varphi(v_0), \varphi(v_1), \ldots, \varphi(v_q) \) are all distinct.

Now \( \varphi_{q-1} \circ \partial_q = \partial_q \circ \varphi_q \) for each integer \( q \). Therefore \( \varphi_q(Z_q(K)) \subset Z_q(L) \) and \( \varphi_q(B_q(K)) \subset B_q(L) \) for all integers \( q \). It follows that any simplicial map \( \varphi: K \to L \) induces well-defined homomorphisms \( \varphi_*: H_q(K) \to H_q(L) \) of homology groups, where \( \varphi_*([z]) = [\varphi_q(z)] \) for all \( q \)-cycles \( z \in Z_q(K) \). It is a trivial exercise to verify that if \( K, L \) and \( M \) are simplicial complexes and if \( \varphi: K \to L \) and \( \psi: L \to M \) are simplicial maps then the induced homomorphisms of homology groups satisfy \( (\psi \circ \varphi)_* = \psi_* \circ \varphi_* \).

### 6.5 Connectedness and \( H_0(K) \)

**Lemma 6.5** Let \( K \) be a simplicial complex. Then \( K \) can be partitioned into pairwise disjoint subcomplexes \( K_1, K_2, \ldots, K_r \) whose polyhedra are the connected components of the polyhedron \( |K| \) of \( K \).

**Proof** Let \( X_1, X_2, \ldots, X_r \) be the connected components of the polyhedron of \( K \), and, for each \( j \), let \( K_j \) be the collection of all simplices \( \sigma \) of \( K \) for which \( \sigma \subset X_j \). If a simplex belongs to \( K_j \) for all \( j \) then so do all its faces. Therefore \( K_1, K_2, \ldots, K_r \) are subcomplexes of \( K \). These subcomplexes are pairwise disjoint since the connected components \( X_1, X_2, \ldots, X_r \) of \( |K| \) are
expressed uniquely as a sum of the form $c = \sum H^r$, since $r = K$, pairwise disjoint. Moreover, if $\sigma \in K$ then $\sigma \subset X_j$ for some $j$, since $\sigma$ is a connected subset of $|K|$, and any connected subset of a topological space is contained in some connected component. But then $\sigma \in K_j$. It follows that $K = K_1 \cup K_2 \cup \cdots \cup K_r$ and $|K| = |K_1| \cup |K_2| \cup \cdots \cup |K_r|$, as required. 

The direct sum $A_1 \oplus A_2 \oplus \cdots \oplus A_r$ of additive Abelian groups $A_1, A_2, \ldots, A_r$ is defined to be the additive group consisting of all $r$-tuples $(a_1, a_2, \ldots, a_r)$ with $a_i \in A_i$ for $i = 1, 2, \ldots, r$, where

$$(a_1, a_2, \ldots, a_r) + (b_1, b_2, \ldots, b_r) \equiv (a_1 + b_1, a_2 + b_2, \ldots, a_r + b_r).$$

**Lemma 6.6** Let $K$ be a simplicial complex. Suppose that $K = K_1 \cup K_2 \cup \cdots \cup K_r$, where $K_1, K_2, \ldots, K_r$ are pairwise disjoint. Then

$$H_q(K) \cong H_q(K_1) \oplus H_q(K_2) \oplus \cdots \oplus H_q(K_r)$$

for all integers $q$.

**Proof** We may restrict our attention to the case when $0 \leq q \leq \dim K$, since $H_q(K) = \{0\}$ if $q < 0$ or $q > \dim K$. Now any $q$-chain $c$ of $K$ can be expressed uniquely as a sum of the form $c = c_1 + c_2 + \cdots + c_r$, where $c_j$ is a $q$-chain of $K_j$ for $j = 1, 2, \ldots, r$. It follows that

$$C_q(K) \cong C_q(K_1) \oplus C_q(K_2) \oplus \cdots \oplus C_q(K_r).$$

Now let $z$ be a $q$-cycle of $K$ (i.e., $z \in C_q(K)$ satisfies $\partial_q(z) = 0$). We can express $z$ uniquely in the form $z = z_1 + z_2 + \cdots + z_r$, where $z_j$ is a $q$-chain of $K_j$ for $j = 1, 2, \ldots, r$. Now

$$0 = \partial_q(z) = \partial_q(z_1) + \partial_q(z_2) + \cdots + \partial_q(z_r),$$

and $\partial_q(z_j)$ is a $(q-1)$-chain of $K_j$ for $j = 1, 2, \ldots, r$. It follows that $\partial_q(z_j) = 0$ for $j = 1, 2, \ldots, r$. Hence each $z_j$ is a $q$-cycle of $K_j$, and thus

$$Z_q(K) \cong Z_q(K_1) \oplus Z_q(K_2) \oplus \cdots \oplus Z_q(K_r).$$

Now let $b$ be a $q$-boundary of $K$. Then $b = \partial_q(c)$ for some $(q + 1)$-chain $c$ of $K$. Moreover $c = c_1 + c_2 + \cdots + c_r$, where $c_j \in C_{q+1}(K_j)$. Thus $b = b_1 + b_2 + \cdots + b_r$, where $b_j \in B_q(K_j)$ is given by $b_j = \partial_{q+1} c_j$ for $j = 1, 2, \ldots, r$. We deduce that

$$B_q(K) \cong B_q(K_1) \oplus B_q(K_2) \oplus \cdots \oplus B_q(K_r).$$

It follows from these observations that there is a well-defined isomorphism

$$\nu: H_q(K_1) \oplus H_q(K_2) \oplus \cdots \oplus H_q(K_r) \rightarrow H_q(K)$$

which maps $([z_1], [z_2], \ldots, [z_r])$ to $[z_1 + z_2 + \cdots + z_r]$, where $[z_j]$ denotes the homology class of a $q$-cycle $z_j$ of $K_j$ for $j = 1, 2, \ldots, r$. 

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Let $K$ be a simplicial complex, and let $y$ and $z$ be vertices of $K$. We say that $y$ and $z$ can be joined by an edge path if there exists a sequence \( v_0, v_1, \ldots, v_m \) of vertices of $K$ with $v_0 = y$ and $v_m = z$ such that the line segment with endpoints $v_{j-1}$ and $v_j$ is an edge belonging to $K$ for $j = 1, 2, \ldots, m$.

**Lemma 6.7** The polyhedron $|K|$ of a simplicial complex $K$ is a connected topological space if and only if any two vertices of $K$ can be joined by an edge path.

**Proof** It is easy to verify that if any two vertices of $K$ can be joined by an edge path then $|K|$ is path-connected and is thus connected. (Indeed any two points of $|K|$ can be joined by a path made up of a finite number of straight line segments.)

We must show that if $|K|$ is connected then any two vertices of $K$ can be joined by an edge path. Choose a vertex $v_0$ of $K$. It suffices to verify that every vertex of $K$ can be joined to $v_0$ by an edge path.

Let $K_0$ be the collection of all of the simplices of $K$ having the property that one (and hence all) of the vertices of that simplex can be joined to $v_0$ by an edge path. If $\sigma$ is a simplex belonging to $K_0$ then every vertex of $\sigma$ can be joined to $v_0$ by an edge path, and therefore every face of $\sigma$ belongs to $K_0$. Thus $K_0$ is a subcomplex of $K$. Clearly the collection $K_1$ of all simplices of $K$ which do not belong to $K_0$ is also a subcomplex of $K$. Thus $K = K_0 \cup K_1$, where $K_0 \cap K_1 = \emptyset$, and hence $|K| = |K_0| \cup |K_1|$, where $|K_0| \cap |K_1| = \emptyset$. But the polyhedra $|K_0|$ and $|K_1|$ of $K_0$ and $K_1$ are closed subsets of $|K|$. It follows from the connectedness of $|K|$ that either $|K_0| = \emptyset$ or $|K_1| = \emptyset$. But $v_0 \in K_0$. Thus $K_1 = \emptyset$ and $K_0 = K$, showing that every vertex of $K$ can be joined to $v_0$ by an edge path, as required.

**Theorem 6.8** Let $K$ be a simplicial complex. Suppose that the polyhedron $|K|$ of $K$ is connected. Then $H_0(K) \cong \mathbb{Z}$.

**Proof** Let $u_1, u_2, \ldots, u_r$ be the vertices of the simplicial complex $K$. Every 0-chain of $K$ can be expressed uniquely as a formal sum of the form

$$n_1\langle u_1 \rangle + n_2\langle u_2 \rangle + \cdots + n_r\langle u_r \rangle$$

for some integers $n_1, n_2, \ldots, n_r$. It follows that there is a well-defined homomorphism $\varepsilon : C_0(K) \to \mathbb{Z}$ defined by

$$\varepsilon \left( n_1\langle u_1 \rangle + n_2\langle u_2 \rangle + \cdots + n_r\langle u_r \rangle \right) = n_1 + n_2 + \cdots + n_r.$$
Now \( \varepsilon(\partial_1(\langle y,z \rangle)) = \varepsilon(\langle z \rangle - \langle y \rangle) = 0 \) whenever \( y \) and \( z \) are endpoints of an edge of \( K \). It follows that \( \varepsilon \circ \partial_1 = 0 \), and hence \( B_0(K) \subset \ker \varepsilon \).

Let \( v_0, v_1, \ldots, v_m \) be vertices of \( K \) determining an edge path. Then
\[
\langle v_m \rangle - \langle v_0 \rangle = \partial_1 \left( \sum_{j=1}^{m} \langle v_{j-1}, v_j \rangle \right) \in B_0(K).
\]

Now \( |K| \) is connected, and therefore any pair of vertices of \( K \) can be joined by an edge path (Lemma 6.7). We deduce that \( \langle z \rangle - \langle y \rangle \in B_0(K) \) for all vertices \( y \) and \( z \) of \( K \). Thus if \( c \in \ker \varepsilon \), where \( c = \sum_{j=1}^{r} n_j \langle u_j \rangle \), then \( \sum_{j=1}^{r} n_j = 0 \), and hence \( c = \sum_{j=2}^{r} n_j (\langle u_j \rangle - \langle u_1 \rangle) \). But \( \langle u_j \rangle - \langle u_1 \rangle \in B_0(K) \). It follows that \( c \in B_0(K) \). We conclude that \( \ker \varepsilon \subset B_0(K) \), and hence \( \ker \varepsilon = B_0(K) \).

Now the homomorphism \( \varepsilon : C_0(K) \to \mathbb{Z} \) is surjective and its kernel is \( B_0(K) \). Therefore it induces an isomorphism from \( C_0(K)/B_0(K) \) to \( \mathbb{Z} \). However \( \mathbb{Z}_0(K) = C_0(K) \) (since \( \partial_0 = 0 \) by definition). Thus \( H_0(K) \cong C_0(K)/B_0(K) \cong \mathbb{Z} \), as required.

On combining Theorem 6.8 with Lemmas 6.5 and 6.6 we obtain immediately the following result.

**Corollary 6.9** Let \( K \) be a simplicial complex. Then
\[
H_0(K) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \quad (r \text{ times}),
\]
where \( r \) is the number of connected components of \( |K| \).