Course 421: Algebraic Topology
Section 5: Simplicial Complexes

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5 Simplicial Complexes

5.1 Geometrical Independence

Definition Points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ in some Euclidean space $\mathbb{R}^k$ are said to be *geometrically independent* (or *affine independent*) if the only solution of the linear system

$$\left\{ \begin{array}{rcl} \sum_{j=0}^q \lambda_j \mathbf{v}_j & = & \mathbf{0}, \\ \sum_{j=0}^q \lambda_j & = & 0 \end{array} \right.$$

is the trivial solution $\lambda_0 = \lambda_1 = \cdots = \lambda_q = 0$.

It is straightforward to verify that $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are geometrically independent if and only if the vectors $\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_q - \mathbf{v}_0$ are linearly independent. It follows from this that any set of geometrically independent points in $\mathbb{R}^k$ has at most $k+1$ elements. Note also that if a set consists of geometrically independent points in $\mathbb{R}^k$, then so does every subset of that set.

Definition A *q-simplex* in $\mathbb{R}^k$ is defined to be a set of the form

$$\left\{ \sum_{j=0}^q t_j \mathbf{v}_j : 0 \leq t_j \leq 1 \text{ for } j = 0, 1, \ldots, q \text{ and } \sum_{j=0}^q t_j = 1 \right\},$$

where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are geometrically independent points of $\mathbb{R}^k$. The points $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are referred to as the *vertices* of the simplex. The non-negative integer q is referred to as the *dimension* of the simplex.

Note that a 0-simplex in $\mathbb{R}^k$ is a single point of $\mathbb{R}^k$, a 1-simplex in $\mathbb{R}^k$ is a line segment in $\mathbb{R}^k$, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron.

Let σ be a q-simplex in $\mathbb{R}^k$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. If $\mathbf{x}$ is a point of σ then there exist real numbers $t_0, t_1, \ldots, t_q$ such that

$$\sum_{j=0}^q t_j \mathbf{v}_j = \mathbf{x}, \quad \sum_{j=0}^q t_j = 1 \text{ and } 0 \leq t_j \leq 1 \text{ for } j = 0, 1, \ldots, q.$$

Moreover $t_0, t_1, \ldots, t_q$ are uniquely determined: if $\sum_{j=0}^q s_j \mathbf{v}_j = \sum_{j=0}^q t_j \mathbf{v}_j$ and $\sum_{j=0}^q s_j = 1 = \sum_{j=0}^q t_j$, then $\sum_{j=0}^q (t_j - s_j)\mathbf{v}_j = \mathbf{0}$ and $\sum_{j=0}^q (t_j - s_j) = 0$, hence $t_j - s_j = 0$ for all j, since $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are geometrically independent. We refer to $t_0, t_1, \ldots, t_q$ as the *barycentric coordinates* of the point $\mathbf{x}$ of σ.

Lemma 5.1 *Let q be a non-negative integer, let σ be a q-simplex in $\mathbb{R}^m$, and let τ be a q-simplex in $\mathbb{R}^n$, where $m \geq q$ and $n \geq q$. Then σ and τ are homeomorphic.*

Proof Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the vertices of σ, and let $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_q$ be the vertices of τ. The required homeomorphism $h\colon \sigma \to \tau$ is given by

$$h\left(\sum_{j=0}^{q} t_j \mathbf{v}_j\right) = \sum_{j=0}^{q} t_j \mathbf{w}_j$$

for all $t_0, t_1, \ldots, t_q$ satisfying $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$ and $\sum_{j=0}^{q} t_j = 1$. ∎

A homeomorphism between two q-simplices defined as in the above proof is referred to as a *simplicial homeomorphism*.

5.2 Simplicial Complexes in Euclidean Spaces

Definition Let σ and τ be simplices in $\mathbb{R}^k$. We say that τ is a *face* of σ if the set of vertices of τ is a subset of the set of vertices of σ. A face of σ is said to be a *proper face* if it is not equal to σ itself. An r-dimensional face of σ is referred to as an *r-face* of σ. A 1-dimensional face of σ is referred to as an *edge* of σ.

Note that any simplex is a face of itself. Also the vertices and edges of any simplex are by definition faces of the simplex.

Definition A finite collection K of simplices in $\mathbb{R}^k$ is said to be a *simplicial complex* if the following two conditions are satisfied:—

- if σ is a simplex belonging to K then every face of σ also belongs to K,
- if σ_1 and σ_2 are simplices belonging to K then either $\sigma_1 \cap \sigma_2 = \emptyset$ or else $\sigma_1 \cap \sigma_2$ is a common face of both σ_1 and σ_2.

The *dimension* of a simplicial complex K is the greatest non-negative integer n with the property that K contains an n-simplex. The union of all the simplices of K is a compact subset $|K|$ of $\mathbb{R}^k$ referred to as the *polyhedron* of K. (The polyhedron is compact since it is both closed and bounded in $\mathbb{R}^k$.)

Example Let K_σ consist of some n-simplex σ together with all of its faces. Then K_σ is a simplicial complex of dimension n, and $|K_\sigma| = \sigma$.

Lemma 5.2 *Let K be a simplicial complex, and let X be a topological space. A function $f\colon |K| \to X$ is continuous on the polyhedron $|K|$ of K if and only if the restriction of f to each simplex of K is continuous on that simplex.*

Proof If a topological space can be expressed as a finite union of closed subsets, then a function is continuous on the whole space if and only if its restriction to each of the closed subsets is continuous on that closed set. The required result is a direct application of this general principle. ∎

We shall denote by $\operatorname{Vert} K$ the set of vertices of a simplicial complex K (i.e., the set consisting of all vertices of all simplices belonging to K). A collection of vertices of K is said to *span* a simplex of K if these vertices are the vertices of some simplex belonging to K.

Definition Let K be a simplicial complex in $\mathbb{R}^k$. A *subcomplex* of K is a collection L of simplices belonging to K with the following property:—

- if σ is a simplex belonging to L then every face of σ also belongs to L.

Note that every subcomplex of a simplicial complex K is itself a simplicial complex.

Definition Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the vertices of a q-simplex σ in some Euclidean space $\mathbb{R}^k$. We define the *interior* of the simplex σ to be the set of all points of σ that are of the form $\sum_{j=0}^{q} t_j \mathbf{v}_j$, where $t_j > 0$ for $j = 0, 1, \ldots, q$ and $\sum_{j=0}^{q} t_j = 1$. One can readily verify that the interior of the simplex σ consists of all points of σ that do not belong to any proper face of σ. (Note that, if $\sigma \in \mathbb{R}^k$, then the interior of a simplex defined in this fashion will not coincide with the topological interior of σ unless $\dim \sigma = k$.)

Note that any point of a simplex σ belongs to the interior of a unique face of σ. Indeed let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be the vertices of σ, and let $\mathbf{x} \in \sigma$. Then $\mathbf{x} = \sum_{j=0}^{q} t_j \mathbf{v}_j$, where $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$ and $\sum_{j=0}^{q} t_j = 1$. The unique face of σ containing $\mathbf{x}$ in its interior is then the face spanned by those vertices $\mathbf{v}_j$ for which $t_j > 0$.

Lemma 5.3 *Let K be a finite collection of simplices in some Euclidean space $\mathbb{R}^k$, and let $|K|$ be the union of all the simplices in K. Then K is a simplicial complex (with polyhedron $|K|$) if and only if the following two conditions are satisfied:—*

- *K contains the faces of its simplices,*
- *every point of $|K|$ belongs to the interior of a unique simplex of K.*

Proof Suppose that K is a simplicial complex. Then K contains the faces of its simplices. We must show that every point of $|K|$ belongs to the interior of a unique simplex of K. Let $\mathbf{x} \in |K|$. Then $\mathbf{x}$ belongs to the interior of a face σ of some simplex of K (since every point of a simplex belongs to the interior of some face). But then $\sigma \in K$, since K contains the faces of all its simplices. Thus $\mathbf{x}$ belongs to the interior of at least one simplex of K.

Suppose that $\mathbf{x}$ were to belong to the interior of two distinct simplices σ and τ of K. Then $\mathbf{x}$ would belong to some common face $\sigma \cap \tau$ of σ and τ (since K is a simplicial complex). But this common face would be a proper face of one or other of the simplices σ and τ (since $\sigma \neq \tau$), contradicting the fact that $\mathbf{x}$ belongs to the interior of both σ and τ. We conclude that the simplex σ of K containing $\mathbf{x}$ in its interior is uniquely determined, as required.

Conversely, we must show that any collection of simplices satisfying the given conditions is a simplicial complex. Since K contains the faces of all its simplices, it only remains to verify that if σ and τ are any two simplices of K with non-empty intersection then $\sigma \cap \tau$ is a common face of σ and τ.

Let $\mathbf{x} \in \sigma \cap \tau$. Then $\mathbf{x}$ belongs to the interior of a unique simplex ω of K. However any point of σ or τ belongs to the interior of a unique face of that simplex, and all faces of σ and τ belong to K. It follows that ω is a common face of σ and τ, and thus the vertices of ω are vertices of both σ and τ. We deduce that the simplices σ and τ have vertices in common, and that every point of $\sigma \cap \tau$ belongs to the common face ρ of σ and τ spanned by these common vertices. But this implies that $\sigma \cap \tau = \rho$, and thus $\sigma \cap \tau$ is a common face of both σ and τ, as required. ∎

Definition A *triangulation* (K, h) of a topological space X consists of a simplicial complex K in some Euclidean space, together with a homeomorphism $h: |K| \to X$ mapping the polyhedron $|K|$ of K onto X.

The polyhedron of a simplicial complex is a compact Hausdorff space. Thus if a topological space admits a triangulation then it must itself be a compact Hausdorff space.

Lemma 5.4 *Let X be a Hausdorff topological space, let K be a simplicial complex, and let $h: |K| \to X$ be a bijection mapping $|K|$ onto X. Suppose that the restriction of h to each simplex of K is continuous on that simplex. Then the map $h: |K| \to X$ is a homeomorphism, and thus (K, h) is a triangulation of X.*

Proof Each simplex of K is a closed subset of $|K|$, and the number of simplices of K is finite. It follows from Lemma 5.2 that $h: |K| \to X$ is continuous. Also the polyhedron $|K|$ of K is a compact topological space. But every continuous bijection from a compact topological space to a Hausdorff space is a homeomorphism. Thus (K, h) is a triangulation of X. ∎

5.3 Simplicial Maps

Definition A *simplicial map* $\varphi: K \to L$ between simplicial complexes K and L is a function $\varphi: \operatorname{Vert} K \to \operatorname{Vert} L$ from the vertex set of K to that of L such that $\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$ span a simplex belonging to L whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K.

Note that a simplicial map $\varphi: K \to L$ between simplicial complexes K and L can be regarded as a function from K to L: this function sends a simplex σ of K with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ to the simplex $\varphi(\sigma)$ of L spanned by the vertices $\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$.

A simplicial map $\varphi: K \to L$ also induces in a natural fashion a continuous map $\varphi: |K| \to |L|$ between the polyhedra of K and L, where

$$\varphi\left(\sum_{j=0}^{q} t_j \mathbf{v}_j\right) = \sum_{j=0}^{q} t_j \varphi(\mathbf{v}_j)$$

whenever $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$, $\sum\limits_{j=0}^{q} t_j = 1$, and $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. The continuity of this map follows immediately from a straightforward application of Lemma 5.2. Note that the interior of a simplex σ of K is mapped into the interior of the simplex $\varphi(\sigma)$ of L.

There are thus three equivalent ways of describing a simplicial map: as a function between the vertex sets of two simplicial complexes, as a function from one simplicial complex to another, and as a continuous map between the polyhedra of two simplicial complexes. In what follows, we shall describe a simplicial map using the representation that is most appropriate in the given context.