Course 421: Algebraic Topology
Section 4: Covering Maps and Discontinuous Group Actions

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Contents

4 Covering Maps and Discontinuous Group Actions

4.1 Covering Maps and Induced Homomorphisms of the Fundamental Group

Proposition 4.1 *Let $p: \tilde{X} \to X$ be a covering map over a topological space X, let $\alpha: [0,1] \to X$ and $\beta: [0,1] \to X$ be paths in X, where $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$, and let $\tilde{\alpha}: [0,1] \to \tilde{X}$ and $\tilde{\beta}: [0,1] \to \tilde{X}$ be paths in $\tilde{X}$ such that $p \circ \tilde{\alpha} = \alpha$ and $p \circ \tilde{\beta} = \beta$. Suppose that $\tilde{\alpha}(0) = \tilde{\beta}(0)$ and that $\alpha \simeq \beta \text{ rel } \{0,1\}$. Then $\tilde{\alpha}(1) = \tilde{\beta}(1)$ and $\tilde{\alpha} \simeq \tilde{\beta} \text{ rel } \{0,1\}$.*

Proof Let x_0 and x_1 be the points of X given by

$$x_0 = \alpha(0) = \beta(0), \qquad x_1 = \alpha(1) = \beta(1).$$

Now $\alpha \simeq \beta \text{ rel } \{0,1\}$, and therefore there exists a homotopy $F: [0,1]\times[0,1] \to X$ such that

$$F(t,0) = \alpha(t) \text{ and } F(t,1) = \beta(t) \text{ for all } t \in [0,1],$$

$$F(0,\tau) = x_0 \text{ and } F(1,\tau) = x_1 \text{ for all } \tau \in [0,1].$$

It then follows from the Monodromy Theorem (Theorem 3.5) that there exists a continuous map $G: [0,1] \times [0,1] \to \tilde{X}$ such that $p \circ G = F$ and $G(0,0) = \tilde{\alpha}(0)$. Then $p(G(0,\tau)) = x_0$ and $p(G(1,\tau)) = x_1$ for all $\tau \in [0,1]$. A straightforward application of Proposition 3.2 shows that any continuous lift of a constant path must itself be a constant path. Therefore $G(0,\tau) = \tilde{x}_0$ and $G(1,\tau) = \tilde{x}_1$ for all $\tau \in [0,1]$, where

$$\tilde{x}_0 = G(0,0) = \tilde{\alpha}(0), \qquad \tilde{x}_1 = G(1,0).$$

However

$$G(0,0) = G(0,1) = \tilde{x}_0 = \tilde{\alpha}(0) = \tilde{\beta}(0),$$

$$p(G(t,0)) = F(t,0) = \alpha(t) = p(\tilde{\alpha}(t))$$

and

$$p(G(t,1)) = F(t,1) = \beta(t) = p(\tilde{\beta}(t))$$

for all $t \in [0,1]$. Now Proposition 3.2 ensures that the lifts $\tilde{\alpha}$ and $\tilde{\beta}$ of the paths α and β are uniquely determined by their starting points. It follows that $G(t,0) = \tilde{\alpha}(t)$ and $G(t,1) = \tilde{\beta}(t)$ for all $t \in [0,1]$. In particular,

$$\tilde{\alpha}(1) = G(0,1) = \tilde{x}_1 = G(1,1) = \tilde{\beta}(1).$$

Moreover the map $G: [0,1] \times [0,1] \to \tilde{X}$ is a homotopy between the paths $\tilde{\alpha}$ and $\tilde{\beta}$ which satisfies $G(0,\tau) = \tilde{x}_0$ and $G(1,\tau) = \tilde{x}_1$ for all $\tau \in [0,1]$. It follows that $\tilde{\alpha} \simeq \tilde{\beta}$ rel $\{0,1\}$, as required. ∎

Corollary 4.2 *Let $p: \tilde{X} \to X$ be a covering map over a topological space X, and let $\tilde{x}_0$ be a point of $\tilde{X}$. Then the homomorphism*

$$p_{\#}: \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, p(\tilde{x}_0))$$

of fundamental groups induced by the covering map p is injective.

Proof Let σ_0 and σ_1 be loops in $\tilde{X}$ based at the point $\tilde{x}_0$, representing elements $[\sigma_0]$ and $[\sigma_1]$ of $\pi_1(\tilde{X}, \tilde{x}_0)$. Suppose that $p_{\#}[\sigma_0] = p_{\#}[\sigma_1]$. Then $p \circ \sigma_0 \simeq p \circ \sigma_1$ rel $\{0,1\}$. Also $\sigma_0(0) = \tilde{x}_0 = \sigma_1(0)$. Therefore $\sigma_0 \simeq \sigma_1$ rel $\{0,1\}$, by Proposition 4.1, and thus $[\sigma_0] = [\sigma_1]$. We conclude that the homomorphism $p_{\#}: \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, p(\tilde{x}_0))$ is injective. ∎

Corollary 4.3 *Let $p: \tilde{X} \to X$ be a covering map over a topological space X, let $\tilde{x}_0$ be a point of $\tilde{X}$, and let γ be a loop in X based at $p(\tilde{x}_0)$. Then $[\gamma] \in p_{\#}(\pi_1(\tilde{X}, \tilde{x}_0))$ if and only if there exists a loop $\tilde{\gamma}$ in $\tilde{X}$, based at the point $\tilde{x}_0$, such that $p \circ \tilde{\gamma} = \gamma$.*

Proof If $\gamma = p \circ \tilde{\gamma}$ for some loop $\tilde{\gamma}$ in $\tilde{X}$ based at $\tilde{x}_0$ then $[\gamma] = p_{\#}[\tilde{\gamma}]$, and therefore $[\gamma] \in p_{\#}(\pi_1(\tilde{X}, \tilde{x}_0))$.

Conversely suppose that $[\gamma] \in p_{\#}(\pi_1(\tilde{X}, \tilde{x}_0))$. We must show that there exists some loop $\tilde{\gamma}$ in $\tilde{X}$ based at $\tilde{x}_0$ such that $\gamma = p \circ \tilde{\gamma}$. Now there exists a loop σ in $\tilde{X}$ based at the point $\tilde{x}_0$ such that $[\gamma] = p_{\#}([\sigma])$ in $\pi_1(X, p(\tilde{x}_0))$. Then $\gamma \simeq p \circ \sigma$ rel $\{0,1\}$. It follows from the Path Lifting Theorem for covering maps (Theorem 3.4) that there exists a unique path $\tilde{\gamma}: [0,1] \to \tilde{X}$ in $\tilde{X}$ for which $\tilde{\gamma}(0) = \tilde{x}_0$ and $p \circ \tilde{\gamma} = \gamma$. It then follows from Proposition 4.1 that $\tilde{\gamma}(1) = \sigma(1)$ and $\tilde{\gamma} \simeq \sigma$ rel $\{0,1\}$. But $\sigma(1) = \tilde{x}_0$. Therefore the path $\tilde{\gamma}$ is the required loop in $\tilde{X}$ based the point $\tilde{x}_0$ which satisfies $p \circ \tilde{\gamma} = \gamma$. ∎

Corollary 4.4 *Let $p: \tilde{X} \to X$ be a covering map over a topological space X, let w_0 and w_1 be points of $\tilde{X}$ satisfying $p(w_0) = p(w_1)$, and let $\alpha: [0,1] \to \tilde{X}$ be a path in $\tilde{X}$ from w_0 to w_1. Suppose that $[p \circ \alpha] \in p_{\#}(\pi_1(\tilde{X}, w_0))$. Then the path α is a loop in $\tilde{X}$, and thus $w_0 = w_1$.*

Proof It follows from Corollary 4.3 that there exists a loop β based at w_0 satisfying $p \circ \beta = p \circ \alpha$. Then $\alpha(0) = \beta(0)$. Now Proposition 3.2 ensures that the lift to $\tilde{X}$ of any path in X is uniquely determined by its starting point. It follows that $\alpha = \beta$. But then the path α must be a loop in $\tilde{X}$, and therefore $w_0 = w_1$, as required. ∎

Corollary 4.5 *Let $p: \tilde{X} \to X$ be a covering map over a topological space X. Let $\alpha: [0,1] \to X$ and $\beta: [0,1] \to X$ be paths in X such that $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$, and let $\alpha.\beta^{-1}$ be the loop in X defined such that*

$$(\alpha.\beta^{-1})(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2}; \\ \beta(2-2t) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Let $\tilde{\alpha}: [0,1] \to \tilde{X}$ and $\tilde{\beta}: [0,1] \to \tilde{X}$ be the unique paths in $\tilde{X}$ such that $p \circ \tilde{\alpha} = \alpha$, and $p \circ \tilde{\beta} = \beta$. Suppose that $\tilde{\alpha}(0) = \tilde{\beta}(0)$. Then $\tilde{\alpha}(1) = \tilde{\beta}(1)$ if and only if $[\alpha.\beta^{-1}] \in p_\#(\pi_1(\tilde{X}, \tilde{x}_0))$, where $\tilde{x}_0 = \tilde{\alpha}(0) = \tilde{\beta}(0)$.

Proof Suppose that $\tilde{\alpha}(1) = \tilde{\beta}(1)$. Then the concatenation $\tilde{\alpha}.\tilde{\beta}^{-1}$ is a loop in $\tilde{X}$ based at $\tilde{x}_0$, and $[\alpha.\beta^{-1}] = p_\#([\tilde{\alpha}.\tilde{\beta}^{-1}])$, and therefore $[\alpha.\beta^{-1}] \in p_\#(\pi_1(\tilde{X}, \tilde{x}_0))$.

Conversely suppose that $\tilde{\alpha}$ and $\tilde{\beta}$ are paths in $\tilde{X}$ satisfying $p \circ \tilde{\alpha} = \alpha$, $p \circ \tilde{\beta} = \beta$ and $\tilde{\alpha}(0) = \tilde{\beta}(0) = \tilde{x}_0$, and that $[\alpha.\beta^{-1}] \in p_\#(\pi_1(\tilde{X}, \tilde{x}_0))$. We must show that $\tilde{\alpha}(1) = \tilde{\beta}(1)$. Let $\gamma: [0,1] \to X$ be the loop based at $p(\tilde{x}_0)$ given by $\gamma = \alpha.\beta^{-1}$. Thus

$$\gamma(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2}; \\ \beta(2-2t) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Then $[\gamma] \in p_\#(\pi_1(\tilde{X}, \tilde{x}_0))$. It follows from Corollary 4.3 that there exists a loop $\tilde{\gamma}$ in $\tilde{X}$ based at $\tilde{x}_0$ such that $p \circ \tilde{\gamma} = \gamma$. Let $\hat{\alpha}: [0,1] \to \tilde{X}$ and Let $\hat{\beta}: [0,1] \to \tilde{X}$ be the paths in $\tilde{X}$ defined such that $\hat{\alpha}(t) = \tilde{\gamma}(\frac{1}{2}t)$ and $\hat{\beta}(t) = \tilde{\gamma}(1 - \frac{1}{2}t)$ for all $t \in [0,1]$. Then

$$\tilde{\alpha}(0) = \hat{\alpha}(0) = \tilde{\beta}(0) = \hat{\beta}(0) = \tilde{x}_0,$$

$p \circ \hat{\alpha} = \alpha = p \circ \tilde{\alpha}$ and $p \circ \hat{\beta} = \beta = p \circ \tilde{\beta}$. But Proposition 3.2 ensures that the lift to $\tilde{X}$ of any path in X is uniquely determined by its starting point. Therefore $\tilde{\alpha} = \hat{\alpha}$ and $\tilde{\beta} = \hat{\beta}$. It follows that

$$\tilde{\alpha}(1) = \hat{\alpha}(1) = \tilde{\gamma}(\tfrac{1}{2}) = \hat{\beta}(1) = \tilde{\beta}(1),$$

as required. ∎

Theorem 4.6 *Let $p: \tilde{X} \to X$ be a covering map over a topological space X. Suppose that $\tilde{X}$ is path-connected and that X is simply-connected. Then the covering map $p: \tilde{X} \to X$ is a homeomorphism.*

Proof We show that the map $p: \tilde{X} \to X$ is a bijection. This map is surjective (since covering maps are by definition surjective). We must show that it is injective. Let w_0 and w_1 be points of $\tilde{X}$ with the property that $p(w_0) = p(w_1)$. Then there exists a path $\alpha: [0,1] \to \tilde{X}$ with $\alpha(0) = w_0$ and $\alpha(1) = w_1$, since $\tilde{X}$ is path-connected. Then $p \circ \alpha$ is a loop in X based at the point x_0, where $x_0 = p(w_0)$. However $\pi_1(X, p(w_0))$ is the trivial group, since X is simply-connected. It follows from Corollary 4.4 that the path α is a loop in $\tilde{X}$ based at w_0, and therefore $w_0 = w_1$. This shows that the the covering map $p: \tilde{X} \to X$ is injective. Thus the map $p: \tilde{X} \to X$ is a bijection, and thus has a well-defined inverse $p^{-1}: X \to \tilde{X}$. It now follows from Lemma 3.1 that $p: \tilde{X} \to X$ is a homeomorphism, as required. ∎

Let $p: \tilde{X} \to X$ be a covering map over some topological space X, and let x_0 be some chosen basepoint of X. We shall investigate the dependence of the subgroup $p_\#(\pi_1(\tilde{X}, \tilde{x}))$ of $\pi_1(X, x_0)$ on the choice of the point $\tilde{x}$ in $\tilde{X}$, where $\tilde{x}$ is chosen such that $p(\tilde{x}) = x_0$. We first introduce some concepts from group theory.

Let G be a group, and let H be a subgroup of G. Given any $g \in G$, let gHg^{-1} denote the subset of G defined by

$$gHg^{-1} = \{g' \in G : g' = ghg^{-1} \text{ for some } h \in H\}.$$

It is easy to verify that gHg^{-1} is a subgroup of G.

Definition Let G be a group, and let H and H' be subgroups of G. We say that H and H' are *conjugate* if and only if there exists some $g \in G$ for which $H' = gHg^{-1}$.

Note that if $H' = gHg^{-1}$ then $H = g^{-1}H'g$. The relation of conjugacy is an equivalence relation on the set of all subgroups of the group G. Moreover conjugate subgroups of G are isomorphic, since the homomorphism sending $h \in H$ to ghg^{-1} is an isomorphism from H to gHg^{-1} whose inverse is the homorphism sending $h' \in gHg^{-1}$ to $g^{-1}h'g$.

A subgroup H of a group G is said to be a *normal subgroup* of G if $ghg^{-1} \in H$ for all $h \in H$ and $g \in G$. If H is a normal subgroup of G then $gHg^{-1} \subset H$ for all $g \in G$. But then $g^{-1}Hg \subset H$ and $H = g(g^{-1}Hg)g^{-1}$ for all $g \in G$, and therefore $H \subset gHg^{-1}$ for all $g \in G$. It follows from this that a subgroup H of G is a normal subgroup if and only if $gHg^{-1} = H$ for all $g \in G$. Thus a subgroup H of G is a normal subgroup if and only if there is no other subgroup of G conjugate to H.

Lemma 4.7 *Let $p\colon \tilde{X} \to X$ be a covering map over a topological space X. Let x_0 be a point of X, and let w_0 and w_1 be points of $\tilde{X}$ for which $p(w_0) = x_0 = p(w_1)$. Let H_0 and H_1 be the subgroups of $\pi_1(X, x_0)$ defined by*

$$H_0 = p_\#(\pi_1(\tilde{X}, w_0)), \quad H_1 = p_\#(\pi_1(\tilde{X}, w_1)).$$

Suppose that the covering space $\tilde{X}$ is path-connected. Then the subgroups H_0 and H_1 of $\pi_1(X, x_0)$ are conjugate. Moreover if H is any subgroup of $\pi_1(X, x_0)$ which is conjugate to H_0 then there exists an element w of $\tilde{X}$ for which $p(w) = x$ and $p_\#(\pi_1(\tilde{X}, w)) = H$.

Proof Let $\alpha\colon [0,1] \to \tilde{X}$ be a path in $\tilde{X}$ for which $\alpha(0) = w_0$ and $\alpha(1) = w_1$. (Such a path exists since $\tilde{X}$ is path-connected.) Then each loop σ in $\tilde{X}$ based at w_1 determines a corresponding loop $\alpha.\sigma.\alpha^{-1}$ in $\tilde{X}$ based at w_0, where

$$(\alpha.\sigma.\alpha^{-1})(t) \equiv \begin{cases} \alpha(3t) & \text{if } 0 \le t \le \frac{1}{3}; \\ \sigma(3t-1) & \text{if } \frac{1}{3} \le t \le \frac{2}{3}; \\ \alpha(3-3t) & \text{if } \frac{2}{3} \le t \le 1. \end{cases}$$

(This loop traverses the path α from w_0 to w_1, then continues round the loop σ, and traverses the path α in the reverse direction in order to return from w_1 to w_0.) Let $\eta\colon [0,1] \to X$ be the loop in X based at the point x_0 given by $\eta = p \circ \alpha$, and let $\varphi\colon \pi_1(X, x_0) \to \pi_1(X, x_0)$ be the automorphism of the group $\pi_1(X, x_0)$ defined such that $\varphi([\gamma]) = [\eta][\gamma][\eta]^{-1}$ for all loops γ in X based at the point x_0. Then $p \circ (\alpha.\sigma.\alpha^{-1}) = \eta.(p \circ \sigma).\eta^{-1}$, and therefore $p_\#([\alpha.\sigma.\alpha^{-1}]) = [\eta]p_\#([\sigma])[\eta]^{-1} = \varphi(p_\#([\sigma]))$ in $\pi_1(X, x_0)$. It follows that $\varphi(H_1) \subset H_0$. Similarly $\varphi^{-1}(H_0) \subset H_1$, where $\varphi^{-1}([\gamma]) = [\eta]^{-1}[\gamma][\eta]$ for all loops γ in X based at the point x_0. It follows that $\varphi(H_1) = H_0$, and thus the subgroups H_0 and H_1 are conjugate

Now let H be a subgroup of $\pi_1(X, x_0)$ which is conjugate to H_0. Then $H_0 = [\eta]H[\eta]^{-1}$ for some loop η in X based at the point x_0. It follows from the Path Lifting Theorem for covering maps (Theorem 3.4) that there exists a path $\alpha\colon [0,1] \to \tilde{X}$ in $\tilde{X}$ for which $\alpha(0) = w_0$ and $p \circ \alpha = \eta$. Let $w = \alpha(1)$. Then

$$p_\#(\pi_1(\tilde{X}, w)) = [\eta]^{-1}H_0[\eta] = H,$$

as required. ∎

4.2 Discontinuous Group Actions

Definition Let G be a group, and let X be a set. The group G is said to *act* on the set X (on the left) if each element g of G determines a corresponding function $\theta_g\colon X \to X$ from the set X to itself, where

(i) $\theta_{gh} = \theta_g \circ \theta_h$ for all $g, h \in G$;

(ii) the function θ_e determined by the identity element e of G is the identity function of X.

Let G be a group acting on a set X. Given any element x of X, the *orbit* $[x]_G$ of x (under the group action) is defined to be the subset $\{\theta_g(x) : g \in G\}$ of X, and the *stabilizer* of x is defined to the the subgroup $\{g \in G : \theta_g(x) = x\}$ of the group G. Thus the orbit of an element x of X is the set consisting of all points of X to which x gets mapped under the action of elements of the group G. The stabilizer of x is the subgroup of G consisting of all elements of this group that fix the point x. The group G is said to act *freely* on X if $\theta_g(x) \neq x$ for all $x \in X$ and $g \in G$ satisfying $g \neq e$. Thus the group G acts freely on X if and only if the stabilizer of every element of X is the trivial subgroup of G.

Let e be the identity element of G. Then $x = \theta_e(x)$ for all $x \in X$, and therefore $x \in [x]_G$ for all $x \in X$, where $[x]_G = \{\theta_g(x) : g \in G\}$.

Let x and y be elements of G for which $[x]_G \cap [y]_G$ is non-empty, and let $z \in [x]_G \cap [y]_G$. Then there exist elements h and k of G such that $z = \theta_h(x) = \theta_k(y)$. Then $\theta_g(z) = \theta_{gh}(x) = \theta_{gk}(y)$, $\theta_g(x) = \theta_{gh^{-1}}(z)$ and $\theta_g(y) = \theta_{gk^{-1}}(z)$ for all $g \in G$, and therefore $[x]_G = [z]_G = [y]_G$. It follows from this that the group action partitions the set X into orbits, so that each element of X determines an orbit which is the unique orbit for the action of G on X to which it belongs. We denote by X/G the set of orbits for the action of G on X.

Now suppose that the group G acts on a topological space X. Then there is a surjective function $q: X \to X/G$, where $q(x) = [x]_G$ for all $x \in X$. This surjective function induces a quotient topology on the set of orbits: a subset U of X/G is open in this quotient topology if and only if $q^{-1}(U)$ is an open set in X (see Lemma 1.9). We define the *orbit space* X/G for the action of G on X to be the topological space whose underlying set is the set of orbits for the action of G on X, the topology on X/G being the quotient topology induced by the function $q: X \to X/G$. This function $q: X \to X/G$ is then an identification map: we shall refer to it as the *quotient map* from X to X/G.

We shall be concerned here with situations in which a group action on a topological space gives rise to a covering map. The relevant group actions are those where the group acts *freely and properly discontinuously* on the topological space.

Definition Let G be a group with identity element e, and let X be a topological space. The group G is said to act *freely and properly discontinuously* on X if each element g of G determines a corresponding continuous map $\theta_g\colon X \to X$, where the following conditions are satisfied:

(i) $\theta_{gh} = \theta_g \circ \theta_h$ for all $g, h \in G$;

(ii) the continuous map θ_e determined by the identity element e of G is the identity map of X;

(iii) given any point x of X, there exists an open set U in X such that $x \in U$ and $\theta_g(U) \cap U = \emptyset$ for all $g \in G$ satisfying $g \neq e$.

Let G be a group which acts freely and properly discontinuously on a topological space X. Given any element g of G, the corresponding continuous function $\theta_g\colon X \to X$ determined by X is a homeomorphism. Indeed it follows from conditions (i) and (ii) in the above definition that $\theta_{g^{-1}} \circ \theta_g$ and $\theta_g \circ \theta_{g^{-1}}$ are both equal to the identity map of X, and therefore $\theta_g\colon X \to X$ is a homeomorphism with inverse $\theta_{g^{-1}}\colon X \to X$.

Remark The terminology 'freely and properly discontinuously' is traditional, but is hardly ideal. The adverb 'freely' refers to the requirement that $\theta_g(x) \neq x$ for all $x \in X$ and for all $g \in G$ satisfying $g \neq e$. The adverb 'discontinuously' refers to the fact that, given any point x of G, the elements of the orbit $\{\theta_g(x) : g \in G\}$ of x are separated; it does not signify that the functions defining the action are in any way discontinuous or badly-behaved. The adverb 'properly' refers to the fact that, given any compact subset K of X, the number of elements of g for which $K \cap \theta_g(K) \neq \emptyset$ is finite. Moreover the definitions of *properly discontinuous actions* in textbooks and in sources of reference are not always in agreement: some say that an action of a group G on a topological space X (where each group element determines a corresponding homeomorphism of the topological space) is *properly discontinuous* if, given any $x \in X$, there exists an open set U in X such that the number of elements g of the group for which $g(U) \cap U \neq \emptyset$ is finite; others say that the action is *properly discontinuous* if it satisfies the conditions given in the definition above for a group acting freely and properly discontinuously on the set. William Fulton, in his textbook *Algebraic topology: a first course* (Springer, 1995), introduced the term 'evenly' in place of 'freely and properly discontinuously', but this change in terminology does not appear to have been generally adopted.

Proposition 4.8 *Let G be a group acting freely and properly discontinuously on a topological space X. Then the quotient map $q\colon X \to X/G$ from X to the corresponding orbit space X/G is a covering map.*

Proof The quotient map $q: X \to X/G$ is surjective. Let V be an open set in X. Then $q^{-1}(q(V))$ is the union $\bigcup_{g \in G} \theta_g(V)$ of the open sets $\theta_g(V)$ as g ranges over the group G, since $q^{-1}(q(V))$ is the subset of X consisting of all elements of X that belong to the orbit of some element of V. But any union of open sets in a topological space is an open set. We conclude therefore that if V is an open set in X then $q(V)$ is an open set in X/G.

Let x be a point of X. Then there exists an open set U in X such that $x \in U$ and $\theta_g(U) \cap U = \emptyset$ for all $g \in G$ satisfying $g \neq e$. Now $q^{-1}(q(U)) = \bigcup_{g \in G} \theta_g(U)$. We claim that the sets $\theta_g(U)$ are disjoint. Let g and h be elements of G. Suppose that $\theta_g(U) \cap \theta_h(U) \neq \emptyset$. Then $\theta_{h^{-1}}(\theta_g(U) \cap \theta_h(U)) \neq \emptyset$. But $\theta_{h^{-1}}: X \to X$ is a bijection, and therefore

$$\theta_{h^{-1}}(\theta_g(U) \cap \theta_h(U)) = \theta_{h^{-1}}(\theta_g(U)) \cap \theta_{h^{-1}}(\theta_h(U)) = \theta_{h^{-1}g}(U) \cap U,$$

and therefore $\theta_{h^{-1}g}(U) \cap U \neq \emptyset$. It follows that $h^{-1}g = e$, where e denotes the identity element of G, and therefore $g = h$. Thus if g and h are elements of g, and if $g \neq h$, then $\theta_g(U) \cap \theta_h(U) = \emptyset$. We conclude therefore that the preimage $q^{-1}(q(U))$ of $q(U)$ is the disjoint union of the sets $\theta_g(U)$ as g ranges over the group G. Moreover each these sets $\theta_g(U)$ is an open set in X.

Now $U \cap [u]_G = \{u\}$ for all $u \in U$, since $[u]_G = \{\theta_g(u) : g \in G\}$ and $U \cap \theta_g(U) = \emptyset$ when $g \neq e$. Thus if u and v are elements of U, and if $q(u) = q(v)$ then $[u]_G = [v]_G$ and therefore $u = v$. It follows that the restriction $q|U: U \to X/G$ of the quotient map q to U is injective, and therefore q maps U bijectively onto $q(U)$. But q maps open sets onto open sets, and any continuous bijection that maps open sets onto open sets is a homeomorphism. We conclude therefore that the restriction of $q: X \to X/G$ to the open set U maps U homeomorphically onto $q(U)$. Moreover, given any element g of G, the quotient map q satisfies $q = q \circ \theta_{g^{-1}}$, and the homeomorphism $\theta_{g^{-1}}$ maps $\theta_g(U)$ homeomorphically onto U. It follows that the quotient map q maps $\theta_g(U)$ homeomorphically onto $q(U)$ for all $g \in U$. We conclude therefore that $q(U)$ is an evenly covered open set in X/G whose preimage $q^{-1}(q(U))$ is the disjoint union of the open sets $\theta_g(U)$ as g ranges over the group G. It follows that the quotient map $q: X \to X/G$ is a covering map, as required. ∎

Theorem 4.9 *Let G be a group acting freely and properly discontinuously on a path-connected topological space X, let $q: X \to X/G$ be the quotient map from X to the orbit space X/G, and let x_0 be a point of X. Then there exists a surjective homomorphism $\lambda: \pi_1(X/G, q(x_0)) \to G$ with the property that $\tilde{\gamma}(1) = \theta_{\lambda([\gamma])}(x_0)$ for any loop γ in X/G based at $q(x_0)$, where $\tilde{\gamma}$ denotes the unique path in X for which $\tilde{\gamma}(0) = x_0$ and $q \circ \tilde{\gamma} = \gamma$. The kernel of this homomorphism is the subgroup $q_\#(\pi_1(X, x_0))$ of $\pi_1(X/G, q(x_0))$.*

Proof Let $\gamma\colon [0,1] \to X/G$ be a loop in the orbit space with $\gamma(0) = \gamma(1) = q(x_0)$. It follows from the Path Lifting Theorem for covering maps (Theorem 3.4) that there exists a unique path $\tilde\gamma\colon [0,1] \to X$ for which $\tilde\gamma(0) = x_0$ and $q \circ \tilde\gamma = \gamma$. Now $\tilde\gamma(0)$ and $\tilde\gamma(1)$ must belong to the same orbit, since $q(\tilde\gamma(0)) = \gamma(0) = \gamma(1) = q(\tilde\gamma(1))$. Therefore there exists some element g of G such that $\tilde\gamma(1) = \theta_g(x_0)$. This element g is uniquely determined, since the group G acts freely on X. Moreover the value of g is determined by the based homotopy class $[\gamma]$ of γ in $\pi_1(X, q(x_0))$. Indeed it follows from Proposition 4.1 that if σ is a loop in X/G based at $q(x_0)$, if $\tilde\sigma$ is the lift of σ starting at x_0 (so that $q \circ \tilde\sigma = \sigma$ and $\tilde\sigma(0) = x_0$), and if $[\gamma] = [\sigma]$ in $\pi_1(X/G, q(x_0))$ (so that $\gamma \simeq \sigma$ rel $\{0,1\}$), then $\tilde\gamma(1) = \tilde\sigma(1)$. We conclude therefore that there exists a well-defined function

$$\lambda\colon \pi_1(X/G, q(x_0)) \to G,$$

which is characterized by the property that $\tilde\gamma(1) = \theta_{\lambda([\gamma])}(x_0)$ for any loop γ in X/G based at $q(x_0)$, where $\tilde\gamma$ denotes the unique path in X for which $\tilde\gamma(0) = x_0$ and $q \circ \tilde\gamma = \gamma$.

Now let $\alpha\colon [0,1] \to X/G$ and $\beta\colon [0,1] \to X/G$ be loops in X/G based at x_0, and let $\tilde\alpha\colon [0,1] \to X$ and $\tilde\beta\colon [0,1] \to X$ be the lifts of α and β respectively starting at x_0, so that $q \circ \tilde\alpha = \alpha$, $q \circ \tilde\beta = \beta$ and $\tilde\alpha(0) = \tilde\beta(0) = x_0$. Then $\tilde\alpha(1) = \theta_{\lambda([\alpha])}(x_0)$ and $\tilde\beta(1) = \theta_{\lambda([\beta])}(x_0)$. Then the path $\theta_{\lambda([\alpha])} \circ \tilde\beta$ is also a lift of the loop β, and is the unique lift of β starting at $\tilde\alpha(1)$. Let $\alpha.\beta$ be the concatenation of the loops α and β, where

$$(\alpha.\beta)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2}; \\ \beta(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Then the unique lift of $\alpha.\beta$ to X starting at x_0 is the path $\sigma\colon [0,1] \to X$, where

$$\sigma(t) = \begin{cases} \tilde\alpha(2t) & \text{if } 0 \le t \le \frac{1}{2}; \\ \theta_{\lambda([\alpha])}(\tilde\beta(2t-1)) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

It follows that

$$\begin{aligned} \theta_{\lambda([\alpha][\beta])}(x_0) &= \theta_{\lambda([\alpha.\beta])}(x_0) = \sigma(1) = \theta_{\lambda([\alpha])}(\tilde\beta(1)) \\ &= \theta_{\lambda([\alpha])}(\theta_{\lambda([\beta])}(x_0)) = \theta_{\lambda([\alpha])\lambda([\beta])}(x_0) \end{aligned}$$

and therefore $\lambda([\alpha][\beta]) = \lambda([\alpha])\lambda([\beta])$. Therefore the function

$$\lambda\colon \pi_1(X/G, q(x_0)) \to G$$

is a homomorphism.

Let $g \in G$. Then there exists a path α in X from x_0 to $\theta_g(x_0)$, since the space X is path-connected. Then $q \circ \alpha$ is a loop in X/G based at $q(x_0)$, and $g = \lambda([q \circ \alpha])$. This shows that the homomorphism λ is surjective.

Let $\gamma\colon [0,1] \to X/G$ be a loop in X/G based at $q(x_0)$. Suppose that $[\gamma] \in \ker \lambda$. Then $\tilde{\gamma}(1) = \theta_e(x_0) = x_0$, and therefore $\tilde{\gamma}$ is a loop in X based at x_0. Moreover $[\gamma] = q_\#[\tilde{\gamma}]$, and therefore $[\gamma] \in q_\#(\pi_1(X, x_0))$. On the other hand, if $[\gamma] \in q_\#(\pi_1(X, x_0))$ then $\gamma = q \circ \tilde{\gamma}$ for some loop $\tilde{\gamma}$ in X based at x_0 (see Corollary 4.3). But then $x_0 = \tilde{\gamma}(1) = \theta_{\lambda([\gamma])}(x_0)$, and therefore $\lambda([\gamma]) = e$, where e is the identity element of G. Thus $\ker \lambda = q_\#(\pi_1(X, x_0))$, as required. ∎

Corollary 4.10 *Let G be a group acting freely and properly discontinuously on a path-connected topological space X, let $q\colon X \to X/G$ be the quotient map from X to the orbit space X/G, and let x_0 be a point of X. Then $q_\#(\pi_1(X, x_0))$ is a normal subgroup of the fundamental group $\pi_1(X/G, q(x_0))$ of the orbit space, and*

$$\frac{\pi_1(X/G, q(x_0))}{q_\#(\pi_1(X, x_0))} \cong G.$$

Proof The subgroup $q_\#(\pi_1(X, x_0))$ is the kernel of the homomorphism

$$\lambda\colon \pi_1(X/G, q(x_0)) \to G$$

described in the statement of Theorem 4.9. It is therefore a normal subgroup of $\pi_1(X/G, q(x_0))$, since the kernel of any homomorphism is a normal subgroup. The homomorphism λ is surjective, and the image of any group homomorphism is isomorphism of the quotient of its domain by its kernel. The result follows. ∎

Corollary 4.11 *Let G be a group acting freely and properly discontinuously on a simply-connected topological space X, let $q\colon X \to X/G$ be the quotient map from X to the orbit space X/G, and let x_0 be a point of X. Then $\pi_1(X/G, q(x_0)) \cong G$.*

Proof This is a special case of Corollary 4.10. ∎

Example The group $\mathbb{Z}$ of integers under addition acts freely and properly discontinuously on the real line $\mathbb{R}$. Indeed each integer n determines a corresponding homeomorphism $\theta_n\colon \mathbb{R} \to \mathbb{R}$, where $\theta_n(x) = x + n$ for all $x \in \mathbb{R}$. Moreover $\theta_m \circ \theta_n = \theta_{m+n}$ for all $m, n \in \mathbb{Z}$, and θ_0 is the identity map of $\mathbb{R}$. If $U = (-\frac{1}{2}, \frac{1}{2})$ then $\theta_n(U) \cap U = \emptyset$ for all non-zero integers n. The real line $\mathbb{R}$ is simply-connected. It follows from Corollary 4.11 that $\pi_1(\mathbb{R}/\mathbb{Z}, b) \cong \mathbb{Z}$ for any point b of $\mathbb{R}/\mathbb{Z}$.

Now the orbit space $\mathbb{R}/\mathbb{Z}$ is homeomorphic to a circle. Indeed let $q: \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the quotient map. Then the surjective function $p: \mathbb{R} \to S^1$ which sends $t \in \mathbb{R}$ to $(\cos 2\pi t, \sin 2\pi t)$ induces a continuous map $h: \mathbb{R}/\mathbb{Z} \to S^1$ defined on the orbit space which satisfies $h \circ q = p$, since the quotient map q is an identification map. Moreover real numbers t_1 and t_2 satisfy $p(t_1) = p(t_2)$ if and only if $q(t_1) = q(t_2)$. It follows that the induced map $h: \mathbb{R}/\mathbb{Z} \to S^1$ is a bijection. This map also maps open sets to open sets, for if W is any open set in the orbit space $\mathbb{R}/\mathbb{Z}$ then $q^{-1}(W)$ is an open set in $\mathbb{R}$, and therefore $p(q^{-1}(W))$ is an open set in S^1, since the covering map $p: \mathbb{R} \to S^1$ maps open sets to open sets (Lemma 3.1). But $p(q^{-1}(W)) = h(W)$ for all open sets W in $\mathbb{R}/\mathbb{Z}$. Thus the continuous bijection $h: \mathbb{R}/\mathbb{Z} \to S^1$ maps open sets to open sets, and is therefore a homeomorphism. Thus Corollary 4.11 generalizes the result of Theorem 3.6.

Example The group $\mathbb{Z}^n$ of ordered n-tuples of integers under addition acts freely and properly discontinuously on $\mathbb{R}^n$, where

$$\theta_{(m_1, m_2, \ldots, m_n)}(x_1, x_2, \ldots, x_n) = (x_1 + m_1, x_2 + m_2, \ldots, x_n + m_n)$$

for all $(m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$ and $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. The orbit space $\mathbb{R}^n/\mathbb{Z}^n$ is an n-dimensional torus, homeomorphic to the product of n circles. It follows from Corollary 4.11 that the fundamental group of this n-dimensional torus is isomorphic to the group $\mathbb{Z}^n$.

Example Let C_2 be the cyclic group of order 2. Then $C_2 = \{e, a\}$ where e is the identity element, $a \neq e$, $a^2 = e$. Then the group C_2 acts freely and properly discontinuously on the n-dimensional sphere S^n for each non-negative integer n. We represent S^n as the unit sphere centred on the origin in $\mathbb{R}^{n+1}$. The homeomorphism θ_e determined by the identity element e of C_2 is the identity map of S^n; the homeomorphism θ_a determined by the element a of C_2 is the antipodal map that sends each point $\mathbf{x}$ of S^n to $-\mathbf{x}$. The orbit space S^n/C_2 is homeomorphic to real projective n-dimensional space $\mathbb{R}P^n$. The n-dimensional sphere is simply-connected if $n > 1$. It follows from Corollary 4.11 that the fundamental group of $\mathbb{R}P^n$ is isomorphic to the cyclic group C_2 when $n > 1$.

Note that S^0 is a pair of points, and $\mathbb{R}P^0$ is a single point. Also S^1 is a circle (which is not simply-connected) and $\mathbb{R}P^1$ is homeomorphic to a circle. Moreover, for any $b \in S^1$, the homomorphism $q_\#: \pi_1(S^1, b) \to \pi_1(\mathbb{R}P^1, q(b))$ corresponds to the homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$ that sends each integer n to $2n$. This is consistent with the conclusions of Corollary 4.10 in this example.

Example Given a pair (m, n) of integers, let $\theta_{m,n}: \mathbb{R}^2 \to \mathbb{R}^2$ be the homeomorphism of the plane $\mathbb{R}^2$ defined such that $\theta_{m,n}(x, y) = (x + m, (-1)^m y + n)$

for all $(x, y) \in \mathbb{R}^2$. Let (m_1, n_1) and (m_2, n_2) be ordered pairs of integers. Then $\theta_{m_1,n_1} \circ \theta_{m_2,n_2} = \theta_{m_1+m_2,n_1+(-1)^{m_1}n_2}$. Let Γ be the group whose elements are represented as ordered pairs of integers, where the group operation $\#$ on Γ is defined such that

$$(m_1, n_1)\#(m_2, n_2) = (m_1 + m_2, n_1 + (-1)^{m_1} n_2)$$

for all $(m_1, n_1), (m_2, n_2) \in \Gamma$. The group Γ is non-Abelian, and its identity element is $(0, 0)$. This group acts on the plane $\mathbb{R}^2$: given $(m, n) \in \Gamma$ the corresponding symmetry $\theta_{m,n}$ is a translation if m is even, and is a glide reflection if m is odd. Given a pair (m, n) of integers, the corresponding homeomorphism $\theta_{m,n}$ maps an open disk about the point (x, y) onto an open disk of the same radius about the point $\theta_{(m,n)}(x, y)$. It follows that if D is the open disk of radius $\frac{1}{2}$ about the point (x, y), and if $D \cap \theta_{m,n}(D)$ is non-empty, then $(m, n) = (0, 0)$. Thus the group Γ maps freely and properly discontinuously on the plane $\mathbb{R}^2$.

The orbit space $\mathbb{R}^2/\Gamma$ is homeomorphic to a Klein bottle. To see this, note each orbit intersects the closed unit square S, where $S = [0, 1] \times [0, 1]$. If $0 < x < 1$ and $0 < y < 1$ then the orbit of (x, y) intersects the square S in one point, namely the point (x, y). If $0 < x < 1$, then the orbit of $(x, 0)$ intersects the square in two points $(x, 0)$ and $(x, 1)$. If $0 < y < 1$ then the orbit of $(0, y)$ intersects the square S in the two points $(0, y)$ and $(1, 1 - y)$. (Note that $(1, 1-y) = \theta_{1,1}(0, y)$.) And the orbit of any corner of the square S intersects the square in the four corners of the square. The restriction $q|S$ of the quotient map $q: \mathbb{R}^2 \to \mathbb{R}^2/\Gamma$ to the square S is a continuous surjection defined on the square: one can readily verify that it is an identification map. It follows that the orbit space $\mathbb{R}^2/\Gamma$ is homeomorphic to the identification space obtained from the closed square S by identifying together the points $(x, 0)$ and $(x, 1)$ where the real number x satisfies $0 < x < 1$, identifying together the points $(0, y)$ and $(1, 1 - y)$ where the real number y satisfies $0 < y < 1$, and identifying together the four corners of the square: this identification space is the Klein bottle.

The plane $\mathbb{R}^2$ is simply-connected. It follows from Corollary 4.11 that the fundamental group of the Klein bottle is isomorphic to the group Γ defined above.

4.3 Deck Transformations

Definition Let $p: \tilde{X} \to X$ be a covering map over a topological space X. A *deck transformation* of the covering space $\tilde{X}$ is a homeomorphism $g: \tilde{X} \to \tilde{X}$ of $\tilde{X}$ with the property that $p \circ g = p$.

Let $p: \tilde{X} \to X$ be a covering map over some topological space X. The deck transformations of the covering space $\tilde{X}$ constitute a group of homeomorphisms of that covering space (where the group operation is the usual operation of composition of homeomorphisms). We shall denote this group by $\mathrm{Deck}(\tilde{X}|X)$.

Lemma 4.12 *Let $p: \tilde{X} \to X$ be a covering map, where the covering space $\tilde{X}$ is connected. Let $g \in \mathrm{Deck}(\tilde{X}|X)$ be a deck transformation that is not equal to the identity map. Then $g(w) \neq w$ for all $w \in \tilde{X}$.*

Proof The result follows immediately on applying Proposition 3.2. ∎

Proposition 4.13 *Let $p: \tilde{X} \to X$ be a covering map, where the covering space $\tilde{X}$ is connected. Then the group $\mathrm{Deck}(\tilde{X}|X)$ of deck transformations acts freely and properly discontinuously on the covering space $\tilde{X}$.*

Proof Let w be a point of the covering space $\tilde{X}$. Then there exists an evenly-covered open set U in X such that $p(w) \in U$. Then the preimage $p^{-1}(U)$ of U in $\tilde{X}$ is a disjoint union of open subsets, where each of these open subsets is mapped homeomorphically onto U by the covering map. One of these subsets contains the point w: let this open set be $\tilde{U}$. Let $g: \tilde{X} \to \tilde{X}$ be a deck transformation. Suppose that $\tilde{U} \cap g(\tilde{U})$ is non-empty. Then there exist $w_1, w_2 \in \tilde{U}$ such that $g(w_1) = w_2$. But then $p(w_2) = p(g(w_1)) = p(w_1)$, and therefore $w_2 = w_1$, since the covering map p maps $\tilde{U}$ homeomorphically and thus injectively onto U. Thus $g(w_1) = w_1$. It then follows from Lemma 4.12 that the deck transformation g is the identity map. We conclude that $\tilde{U} \cap g(\tilde{U}) = \emptyset$ for all deck transformations g other than the identity map of $\tilde{X}$. This shows that $\mathrm{Deck}(\tilde{X}|X)$ acts freely and properly discontinuously on $\tilde{X}$, as required. ∎

4.4 Local Topological Properties and Local Homeomorphisms

Definition A topological space X is said to be *locally connected* if, given any point x of X, and given any open set U in X with $x \in U$, there exists some connected open set V in X such that $x \in V$ and $V \subset U$.

Definition A topological space X is said to be *locally path-connected* if, given any point x of X, and given any open set U in X with $x \in U$, there exists some path-connected open set V in X such that $x \in V$ and $V \subset U$.

Definition A topological space X is said to be *locally simply-connected* if, given any point x of X, and given any open set U in X with $x \in U$, there exists some simply-connected open set V in X such that $x \in V$ and $V \subset U$.

Definition A topological space X is said to be *contractible* if the identity map of X is homotopic to a constant map that sends the whole of X to a single point of X.

Definition A topological space X is said to be *locally contractible* if, given any point x of X, and given any open set U in X with $x \in U$, there exists some contractible open set V in X such that $x \in V$ and $V \subset U$.

Definition A topological space X is said to be *locally Euclidean* of dimension n if, given any point x of X, there exists some open set V such that $x \in V$ and V is homeomorphic to some open set in n-dimensional Euclidean space $\mathbb{R}^n$.

Note that every locally Euclidean topological space is locally contractible, every locally contractible topological space is locally simply-connected, every locally simply-connected topological space is locally path-connected, and every locally path-connected topological space is locally connected.

Remark A connected topological space need not be locally connected; a locally connected topological space need not be connected. A standard example is the *comb space*. This space is the subset of the plane $\mathbb{R}^2$ consisting of the line segment joining $(0,0)$ to $(1,0)$, the line segment joining $(0,0)$ to $(0,1)$, and the line segments joining $(1/n,0)$ to $(1/n,1)$ for each positive integer n. This space is contractible, and thus simply-connected, path-connected and connected. However there is no connected open subset that contains the point $(0,1)$ and is contained within the open disk of radius 1 about this point, and therefore the space is not locally connected, locally path-connected, locally simply-connected or locally contractible.

Proposition 4.14 *Let X be a connected, locally path-connected topological space. Then X is path-connected.*

Proof Choose a point x_0 of X. Let Z be the subset of X consisting of all points x of X with the property that x can be joined to x_0 by a path. We show that the subset Z is both open and closed in X.

Now, given any point x of X there exists a path connected open set N_x in X such that $x \in N_x$. We claim that if $x \in Z$ then $N_x \subset Z$, and if $x \notin Z$ then $N_x \cap Z = \emptyset$.

Suppose that $x \in Z$. Then, given any point x' of N_x, there exists a path in N_x from x' to x. Moreover it follows from the definition of the set Z that there exists a path in X from x to x_0. These two paths can be concatenated to yield a path in X from x' to x_0, and therefore $x' \in Z$. This shows that $N_x \subset Z$ whenever $x \in Z$.

Next suppose that $x \notin Z$. Let $x' \in N_x$. If it were the case that $x' \in Z$, then we would be able to concatenate a path in N_x from x to x' with a path in X from x' to x_0 in order to obtain a path in X from x to x_0. But this is impossible, as $x \notin Z$. Therefore $N_x \cap Z = \emptyset$ whenever $x \notin Z$.

Now the set Z is the union of the open sets N_x as x ranges over all points of Z. It follows that Z is itself an open set. Similarly $X \setminus Z$ is the union of the open sets N_x as x ranges over all points of $X \setminus Z$, and therefore $X \setminus Z$ is itself an open set. It follows that Z is a subset of X that is both open and closed. Moreover $x_0 \in Z$, and therefore Z is non-empty. But the only subsets of X that are both open and closed are $\emptyset$ and X itself, since X is connected. Therefore $Z = X$, and thus every point of X can be joined to the point x_0 by a path in X. We conclude that X is path-connected, as required. ∎

Let P be some property that topological spaces may or may not possess. Suppose that the following conditions are satisfied:—

(i) if a topological space has property P then every topological space homeomorphic to the given space has property P;

(ii) if a topological space has property P then open subset of the given space has property P;

(iii) if a topological space has a covering by open sets, where each of these open sets has property P, then the topological space itself has property P.

Examples of properties satisfying these conditions are the property of being locally connected, the property of being locally path-connected, the property of being locally simply-connected, the property of being locally contractible, and the property of being locally Euclidean.

Properties of topological spaces satisfying conditions (i), (ii) and (iii) above are topological properties that describe the local character of the topological space. Such a property is satisfied by the whole topological space if and only if it is satisfied around every point of that topological space.

Definition Let $f: X \to Y$ be a continuous map between topological spaces X and Y. The map f is said to be a *local homeomorphism* if, given any point x of X, there exists some open set U in X with $x \in U$ such that the function f maps U homeomorphically onto an open set $f(U)$ in Y.

Lemma 4.15 *Every covering map is a local homeomorphism.*

Proof Let $p: \tilde{X} \to X$ be a covering map, and let w be a point of $\tilde{X}$. Then $p(w) \in U$ for some evenly-covered open set U in X. Then $p^{-1}(U)$ is a disjoint union of open sets, where each of these open sets is mapped homeomorphically onto U. One of these open sets contains the point w: let that open set be $\tilde{U}$. Then p maps $\tilde{U}$ homeomorphically onto the open set U. Thus the covering map is a local homeomorphism. ∎

Example Not all local homeomorphisms are covering maps. Let S^1 denote the unit circle in $\mathbb{R}^2$, and let $\alpha: (-2, 2) \to S^1$ denote the continuous map that sends $t \in (-2, 2)$ to $(\cos 2\pi t, \sin 2\pi t)$. Then the map α is a local homeomorphism. But it is not a covering map, since the point $(1, 0)$ does not belong to any evenly covered open set in S^1.

Let $f: X \to Y$ be a local homeomorphism between topological spaces X and Y, and let P be some property of topological spaces that satisfies conditions (i), (ii) and (iii) above. We claim that X has property P if and only if $f(X)$ has property P. Now there exists an open cover $\mathcal{U}$ of X by open sets, where the local homeomorphism f maps each open set U belonging to $\mathcal{U}$ homeomorphically onto an open set $f(U)$ in Y. Then the collection of open sets of the form $f(U)$, as U ranges over $\mathcal{U}$ constitutes an open cover of $f(X)$. Now if X has property P, then each open set U in the open cover $\mathcal{U}$ of X has property P (by condition (ii)). But then set $f(U)$ has property P for each open set U belonging to $\mathcal{U}$ (by condition (i)). But then $f(X)$ itself must have property P (by condition (iii)). Conversely if $f(X)$ has property P, then $f(U)$ has property P for each open set U in the open cover $\mathcal{U}$ of X. But then each open set U belonging to the open cover $\mathcal{U}$ has property P (as U is homeomorphic to $f(U)$). But then X has property P (by condition (iii)).

A covering map $p: \tilde{X} \to X$ between topological spaces is a surjective local homeomorphism. Let P be some property of topological spaces satisfying conditions (i), (ii) and (iii) above. Then the covering space $\tilde{X}$ has property P if and only if the base space X has property P. A number of instances of this principle are collected together in the following proposition.

Proposition 4.16 *Let $p: \tilde{X} \to X$ be a covering map. Then the following are true:*

(i) *$\tilde{X}$ is locally connected if and only if X is locally connected;*

(ii) *$\tilde{X}$ is locally path-connected if and only if X is locally path-connected;*

(iii) $\tilde{X}$ *is locally simply-connected if and only if* X *is locally simply-connected;*

(iv) $\tilde{X}$ *is locally contractible if and only if* X *is locally contractible;*

(v) $\tilde{X}$ *is locally Euclidean if and only if* X *is locally Euclidean.*

Corollary 4.17 *Let* X *be a locally path-connected topological space, and let* $p: \tilde{X} \to X$ *be a covering map over* X*. Suppose that the covering space* $\tilde{X}$ *is connected. Then* $\tilde{X}$ *is path-connected.*

Proof The covering space $\tilde{X}$ is locally path-connected, by Proposition 4.16. It follows from Proposition 4.14 that $\tilde{X}$ is path-connected. ∎

4.5 Lifting of Continuous Maps Into Covering Spaces

Let $p: \tilde{X} \to X$ be a covering map over a topological space X. Let $f: Z \to X$ be a continuous map from some topological space Z into X. If the topological space Z is *locally path-connected* then one can formulate a criterion to determine whether or not there exists a map $\tilde{f}: Z \to \tilde{X}$ for which $p \circ \tilde{f} = f$ (see Theorem 4.19 and Corollary 4.20). This criterion is stated in terms of the homomorphisms of fundamental groups induced by the continuous maps $f: Z \to X$ and $p: \tilde{X} \to X$. We shall use this criterion in order to derive a necessary and sufficient condition for two covering maps over a connected and locally path-connected topological space to be topologically equivalent (see Corollary 4.22). We shall also study the deck transformations of a covering space over some connected and locally path-connected topological space.

Lemma 4.18 *Let* $p: \tilde{X} \to X$ *be a covering map over a topological space* X*, let* Z *be a locally path-connected topological space, and let* $g: Z \to \tilde{X}$ *be a function from* Z *to* $\tilde{X}$*. Suppose that* $p \circ g: Z \to X$ *is continuous, and that* $g \circ \gamma: [0, 1] \to \tilde{X}$ *is continuous for all paths* $\gamma: [0, 1] \to Z$ *in* Z*. Then the function* g *is continuous.*

Proof Let $f: Z \to X$ be the composition function $p \circ g$. Then the function f is a continuous map from Z to X.

Let z be a point of Z. Then there exists an open neighbourhood V of $f(z)$ in X which is evenly covered by the map p. The inverse image $p^{-1}(V)$ of V in the covering space $\tilde{X}$ is a disjoint union of open sets, each of which is mapped homeomorphically onto V by p. One of these open sets contains the point $g(z)$, since $f(z) = p(g(z))$. Let us denote this open set by $\tilde{V}$. Then $g(z) \in \tilde{V}$, and $\tilde{V}$ is mapped homeomorphically onto V by the map p. Let

$s: V \to \tilde{V}$ denote the inverse of the restriction $(p|\tilde{V}): \tilde{V} \to V$ of the covering map p to $\tilde{V}$. Then the map s is continuous, and $p(s(v)) = v$ for all $v \in V$.

Now $f^{-1}(V)$ is an open set in Z containing the point z. But the topological space Z is locally path-connected. Therefore there exists a path-connected open set N_z in Z such that $z \in N_z$ and $N_z \subset f^{-1}(V)$. We claim that $g(N_z) \subset \tilde{V}$. Let z' be a point of N_z. Then there exists a path $\gamma: [0,1] \to N_z$ in N from z to z'. Moreover $f(\gamma([0,1])) \subset V$. Let $\eta: [0,1] \to \tilde{X}$ be the path in $\tilde{X}$ defined such that $\eta(t) = s(f(\gamma(t)))$ for all $t \in [0,1]$. Then $\eta([0,1]) \subset \tilde{V}$, and η is the unique path in $\tilde{X}$ for which $\eta(0) = g(z)$ and $p \circ \eta = f \circ \gamma$. But the composition function $g \circ \gamma$ is a path in $\tilde{X}$, $g(\gamma(0)) = g(z)$ and $p \circ g \circ \gamma = f \circ \gamma$. Therefore $g \circ \gamma = \eta$. It follows that $g(\gamma([0,1])) \subset \tilde{V}$, and therefore $g(z') \in \tilde{V}$. This proves that $g(N_z) \subset \tilde{V}$. Moreover $g(z') = s(f(z'))$ for all $z' \in N_z$, and therefore the restriction $g|N_z: N_z \to \tilde{X}$ of the function g to the open set N_z is continuous.

We have now shown that, given any point z of Z, there exists an open set N_z in Z such that $z \in N_z$ and the restriction $g|N_z$ of $g: Z \to \tilde{X}$ to N_z is continuous. It follows from this that the function g is continuous on Z. Indeed let U be an open set in $\tilde{X}$. Then $g^{-1}(U) \cap N_z$ is an open set for all $z \in Z$, since $g|N_z$ is continuous. Moreover $g^{-1}(U)$ is the union of the open sets $g^{-1}(U) \cap N_z$ as z ranges over all points of Z. It follows that $g^{-1}(U)$ is itself an open set in Z. Thus $g: Z \to \tilde{X}$ is continuous, as required. ∎

Theorem 4.19 *Let $p: \tilde{X} \to X$ be a covering map over a topological space X, and let $f: Z \to X$ be a continuous map from some topological space Z into X. Suppose that the topological space Z is both connected and locally path-connected. Suppose also that*

$$f_\#(\pi_1(Z, z_0)) \subset p_\#(\pi_1(\tilde{X}, \tilde{x}_0)),$$

where z_0 and $\tilde{x}_0$ are points of Z and $\tilde{X}$ respectively which satisfy $f(z_0) = p(\tilde{x}_0)$. Then there exists a unique continuous map $\tilde{f}: Z \to \tilde{X}$ for which $\tilde{f}(z_0) = \tilde{x}_0$ and $p \circ \tilde{f} = f$.

Proof Let P denote the set of all ordered pairs (α, ρ), where $\alpha: [0,1] \to Z$ is a path in Z with $\alpha(0) = z_0$, $\rho: [0,1] \to \tilde{X}$ is a path in $\tilde{X}$ with $\rho(0) = \tilde{x}_0$, and $f \circ \alpha = p \circ \rho$. We claim that there is a well-defined function $\tilde{f}: Z \to \tilde{X}$ characterized by the property that $\tilde{f}(\alpha(1)) = \rho(1)$ for all $(\alpha, \rho) \in P$.

The topological space Z is path-connected, by Proposition 4.14. Therefore, given any point z of Z, there exists a path α in Z from z_0 to z. Moreover it follows from the Path Lifting Theorem (Theorem 3.4) that, given any path α in Z from z_0 to z there exists a unique path ρ in $\tilde{X}$ for which $\rho(0) = \tilde{x}_0$

and $p \circ \rho = f \circ \alpha$. It follows that, given any element z of Z, there exists some element (α, ρ) of P for which $\alpha(1) = z$.

Let (α, ρ) and (β, σ) be elements of P. Suppose that $\alpha(1) = \beta(1)$. Then $[(f \circ \alpha).(f \circ \beta)^{-1}] = f_{\#}[\alpha.\beta^{-1}]$. But $f_{\#}(\pi_1(Z, z_0)) \subset p_{\#}(\pi_1(\tilde{X}, \tilde{x}_0)$. Therefore $[(f \circ \alpha).(f \circ \beta)^{-1}] \in p_{\#}(\pi_1(\tilde{X}, \tilde{x}_0))$. It follows from Corollary 4.5 that $\rho(1) = \sigma(1)$. We conclude therefore that if (α, ρ) and (β, σ) are elements of P, and if $\alpha(1) = \beta(1)$, then $\rho(1) = \sigma(1)$. This establishes the existence of a unique function $\tilde{f}: Z \to \tilde{X}$ characterized by the property that $\tilde{f}(\alpha(1)) = \rho(1)$ for all $(\alpha, \rho) \in P$. Now $p(\rho(1)) = f(\alpha(1))$ for all $(\alpha, \rho) \in P$, and therefore $p \circ \tilde{f} = f$. Also $\tilde{f}(z_0) = \tilde{x}_0$, since $(\varepsilon_{z_0}, \varepsilon_{\tilde{x}_0}) \in P$, where ε_{z_0} denotes the constant path in Z based at z_0 and $\varepsilon_{\tilde{x}_0}$ denotes the constant path in $\tilde{X}$ based at $\tilde{x}_0$. Thus it only remains to show that the map $\tilde{f}: Z \to \tilde{X}$ is continuous. In view of Lemma 4.18, it suffices to show that $\tilde{f}$ maps paths in Z to paths in $\tilde{X}$.

Let $\gamma: [0, 1] \to Z$ be a path in Z. We claim that the composition function $\tilde{f} \circ \gamma$ is continuous, and is thus a path in $\tilde{X}$. Let α be a path in Z from z_0 to $\gamma(0)$, let $\rho: [0, 1] \to \tilde{X}$ be the unique path in $\tilde{X}$ satisfying $\rho(0) = \tilde{x}_0$ and $p \circ \rho = f \circ \alpha$, and let $\sigma: [0, 1] \to \tilde{X}$ be the unique path in $\tilde{X}$ satisfying $\sigma(0) = \rho(1)$ and $p \circ \sigma = f \circ \gamma$. Now, for each $\tau \in [0, 1]$, there is a path $\alpha_\tau: [0, 1] \to Z$ from z_0 to $\gamma(\tau)$ defined such that

$$\alpha_\tau(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2}; \\ \gamma((2t-1)\tau) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Then $f \circ \alpha_\tau(t) = p \circ \rho_\tau(t)$ for all $t \in [0, 1]$ where $\rho_\tau: [0, 1] \to \tilde{X}$ is the path in $\tilde{X}$ from $\tilde{x}_0$ to $\sigma(\tau)$ defined such that

$$\rho_\tau(t) = \begin{cases} \rho(2t) & \text{if } 0 \le t \le \frac{1}{2}; \\ \sigma((2t-1)\tau) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

It follows that $(\alpha_\tau, \rho_\tau) \in P$, for all $\tau \in [0, 1]$, and therefore

$$\tilde{f}(\gamma(\tau)) = \tilde{f}(\alpha_\tau(1)) = \rho_\tau(1) = \sigma(\tau)$$

for all $\tau \in [0, 1]$. Thus $\tilde{f} \circ \gamma = \sigma$. We conclude that $\tilde{f} \circ \gamma$ is a path in $\tilde{X}$ for any path γ in Z. It then follows from Lemma 4.18 that the function $\tilde{f}: Z \to \tilde{X}$ is a continuous map from Z to $\tilde{X}$ with the required properties. The uniqueness of this map follows on applying Proposition 3.2. ∎

Corollary 4.20 *Let $p: \tilde{X} \to X$ be a covering map over a topological space X, and let $f: Z \to X$ be a continuous map from some topological space Z into X. Suppose that the covering space $\tilde{X}$ is path-connected and that the topological space Z is both connected and locally path-connected. Let z_0 and w_0 be points* *of Z and $\tilde{X}$ respectively for which $f(z_0) = p(w_0)$. Then there exists a map $\tilde{f}: Z \to \tilde{X}$ satisfying $p \circ \tilde{f} = f$ if and only if there exists a subgroup H of $\pi_1(X, p(w_0))$ such that H is conjugate to $p_\#(\pi_1(\tilde{X}, w_0))$ and $f_\#(\pi_1(Z, z_0)) \subset H$.*

Proof Suppose that there exists a map $\tilde{f}: Z \to \tilde{X}$ for which $p \circ \tilde{f} = f$. Then $f_\#(\pi_1(Z, z_0)) \subset H$, where $H = p_\#(\pi_1(\tilde{X}, \tilde{f}(z_0)))$. Moreover it follows from Lemma 4.7 that the subgroup H of $\pi_1(X, p(w_0))$ is conjugate to $p_\#(\pi_1(\tilde{X}, w_0))$ in $\pi_1(X, p(w_0))$.

Conversely suppose that $f_\#(\pi_1(Z, z_0)) \subset H$, where H is a subgroup of $\pi_1(X, p(w_0))$ that is conjugate to $p_\#(\pi_1(\tilde{X}, w_0))$. It follows from Lemma 4.7 that there exists a point $\tilde{x}$ of $\tilde{X}$ for which $p(\tilde{x}) = p(w_0)$ and $p_\#(\pi_1(\tilde{X}, \tilde{x})) = H$. Then

$$f_\# \left(\pi_1(Z, z_0)\right) \subset p_\#(\pi_1(\tilde{X}, \tilde{x})).$$

It then follows from Theorem 4.19 that there exists a continuous map $\tilde{f}: Z \to \tilde{X}$ for which $p \circ \tilde{f} = f$, as required. ∎

4.6 Isomorphisms of Covering Maps

Definition Let $p_1: \tilde{X}_1 \to X$ and $p_2: \tilde{X}_2 \to X$ be covering maps over some topological space X. We say that the covering maps $p_1: \tilde{X}_1 \to X$ and $p_2: \tilde{X}_2 \to X$ are *topologically isomorphic* if there exists a homeomorphism $h: \tilde{X}_1 \to \tilde{X}_2$ from the covering space $\tilde{X}_1$ to the covering space $\tilde{X}_2$ with the property that $p_1 = p_2 \circ h$.

We can apply Theorem 4.19 in order to derive a criterion for determining whether or not two covering maps over some connected locally path-connected topological space are isomorphic.

Theorem 4.21 *Let X be a topological space which is both connected and locally path-connected, let $\tilde{X}_1$ and $\tilde{X}_2$ be connected topological spaces, and let $p_1: \tilde{X}_1 \to X$ and $p_2: \tilde{X}_2 \to X$ be covering maps over X. Let w_1 and w_2 be points of $\tilde{X}_1$ and $\tilde{X}_2$ respectively for which $p_1(w_1) = p_2(w_2)$. Then there exists a homeomorphism $h: \tilde{X}_1 \to \tilde{X}_2$ from the covering space $\tilde{X}_1$ to the covering space $\tilde{X}_2$ satisfying $p_2 \circ h = p_1$ and $h(w_1) = w_2$ if and only if the subgroups $p_{1\#}(\pi_1(\tilde{X}_1, w_1))$ and $p_{2\#}(\pi_1(\tilde{X}_2, w_2))$ of $\pi_1(X, p_1(w_1))$ coincide.*

Proof Suppose that there exists a homeomorphism $h: \tilde{X}_1 \to \tilde{X}_2$ from the covering space $\tilde{X}_1$ to the covering space $\tilde{X}_2$ for which $p_2 \circ h = p_1$ and $h(w_1) = w_2$. Then $h_\#(\pi_1(\tilde{X}_1, w_1)) = \pi_1(\tilde{X}_2, w_2)$, and therefore

$$p_{1\#}(\pi_1(\tilde{X}_1, w_1)) = p_{2\#}\left(h_\#(\pi_1(\tilde{X}_1, w_1))\right) = p_{2\#}(\pi_1(\tilde{X}_2, w_2)).$$

Conversely suppose that $p_{1\#}(\pi_1(\tilde{X}_1, w_1)) = p_{2\#}(\pi_1(\tilde{X}_2, w_2))$. It follows from Proposition 4.16 that the covering spaces $\tilde{X}_1$ and $\tilde{X}_2$ are both locally path-connected, since X is a locally path-connected topological space. But $\tilde{X}_1$ and $\tilde{X}_2$ are also connected. It follows from Theorem 4.19 that there exist unique continuous maps $h: \tilde{X}_1 \to \tilde{X}_2$ and $k: \tilde{X}_2 \to \tilde{X}_1$ for which $p_2 \circ h = p_1$, $p_1 \circ k = p_2$, $h(w_1) = w_2$ and $k(w_2) = w_1$. But then $p_1 \circ k \circ h = p_1$ and $(k \circ h)(w_1) = w_1$. It follows from this that the composition map $k \circ h$ is the identity map of $\tilde{X}_1$ (since a straightforward application of Theorem 4.19 shows that any continuous map $j: \tilde{X}_1 \to \tilde{X}_1$ which satisfies $p_1 \circ j = p_1$ and $j(w_1) = w_1$ must be the identity map of $\tilde{X}_1$). Similarly the composition map $h \circ k$ is the identity map of $\tilde{X}_2$. Thus $h: \tilde{X}_1 \to \tilde{X}_2$ is a homeomorphism whose inverse is k. Moreover $p_2 \circ h = p_2$. Thus $h: \tilde{X}_1 \to \tilde{X}_2$ is a homeomorphism with the required properties. ∎

Corollary 4.22 *Let X be a topological space which is both connected and locally path-connected, let $\tilde{X}_1$ and $\tilde{X}_2$ be connected topological spaces, and let $p_1: \tilde{X}_1 \to X$ and $p_2: \tilde{X}_2 \to X$ be covering maps over X. Let w_1 and w_2 be points of $\tilde{X}_1$ and $\tilde{X}_2$ respectively for which $p_1(w_1) = p_2(w_2)$. Then the covering maps $p_1: \tilde{X}_1 \to X$ and $p_2: \tilde{X}_2 \to X$ are topologically isomorphic if and only if the subgroups $p_{1\#}(\pi_1(\tilde{X}_1, w_1))$ and $p_{2\#}(\pi_1(\tilde{X}_2, w_2))$ of $\pi_1(X, p_1(w_1))$ are conjugate.*

Proof Suppose that the covering maps $p_1: \tilde{X}_1 \to X$ and $p_2: \tilde{X}_2 \to X$ are topologically isomorphic. Let $h: \tilde{X}_1 \to \tilde{X}_2$ be a homeomorphism for which $p_2 \circ h = p_1$. Then

$$p_{1\#}(\pi_1(\tilde{X}_1, w_1)) = p_{2\#}(\pi_1(\tilde{X}_2, h(w_1))).$$

It follows immediately from Lemma 4.7 that the subgroups $p_{1\#}(\pi_1(\tilde{X}_1, w_1))$ and $p_{2\#}(\pi_1(\tilde{X}_2, w_2))$ of $\pi_1(X, p_1(w_1))$ are conjugate.

Conversely, suppose that the subgroups

$$p_{1\#}(\pi_1(\tilde{X}_1, w_1)) \text{ and } p_{2\#}(\pi_1(\tilde{X}_2, w_2))$$

of $\pi_1(X, p_1(w_1))$ are conjugate. The covering space $\tilde{X}_2$ is both locally path-connected (Proposition 4.16) and connected, and is therefore path-connected (Corollary 4.17). It follows from Lemma 4.7 that there exists a point w of $\tilde{X}_2$ for which $p_2(w) = p_2(w_2) = p_1(w_1)$ and

$$p_{2\#}(\pi_1(\tilde{X}_2, w)) = p_{1\#}(\pi_1(\tilde{X}_1, w_1)).$$

Theorem 4.21 now ensures that there exists a homeomorphism $h: \tilde{X}_1 \to \tilde{X}_2$ from $\tilde{X}_1$ to $\tilde{X}_2$ such that $p_2 \circ h = p_1$ and $h(w_1) = w$. It follows that the covering maps are topologically isomorphic, as required. ∎

4.7 Deck Transformations of Locally Path-Connected Coverings

Proposition 4.23 *Let X be a topological space which is connected and locally path-connected, let $\tilde{X}$ be a connected topological space, let $p: \tilde{X} \to X$ be a covering map over X, and let w_1 and w_2 be points of the covering space $\tilde{X}$. Then there exists a deck transformation $h: \tilde{X} \to \tilde{X}$ sending w_1 to w_2 if and only if $p_\#(\pi_1(\tilde{X}, w_1)) = p_\#(\pi_1(\tilde{X}, w_2))$, in which case the deck transformation sending w_1 to w_2 is uniquely determined.*

Proof The proposition follows immediately on applying Theorem 4.21. ∎

Corollary 4.24 *Let X be a topological space which is connected and locally path-connected, let $p: \tilde{X} \to X$ be a covering map over X, where the covering space $\tilde{X}$ is connected. Suppose that $p_\#(\pi_1(\tilde{X}, w_1))$ is a normal subgroup of $\pi_1(X, p(w_1))$. Then, given any points w_1 and w_2 of the covering space $\tilde{X}$ satisfying $p(w_1) = p(w_2)$, there exists a unique deck transformation $h: \tilde{X} \to \tilde{X}$ satisfying $h(w_1) = w_2$.*

Proof The covering space $\tilde{X}$ is both locally path-connected (by Proposition 4.16) and connected, and is therefore path-connected (by Corollary 4.17). It follows that $p_\#(\pi_1(\tilde{X}, w_1))$ and $p_\#(\pi_1(\tilde{X}, w_2))$ are conjugate subgroups of $\pi_1(X, p(w_1))$ (Lemma 4.7). But then $p_\#(\pi_1(\tilde{X}, w_1)) = p_\#(\pi_1(\tilde{X}, w_2))$, since $p_\#(\pi_1(\tilde{X}, w_1))$ is a normal subgroup of $\pi_1(X, p(w_1))$. The result now follows from Proposition 4.23. ∎

Theorem 4.25 *Let $p: \tilde{X} \to X$ be a covering map over some topological space X which is both connected and locally path-connected, and let x_0 and $\tilde{x}_0$ be points of X and $\tilde{X}$ respectively satisfying $p(\tilde{x}_0) = x_0$. Suppose that $\tilde{X}$ is connected and that $p_\#(\pi_1(\tilde{X}, \tilde{x}_0))$ is a normal subgroup of $\pi_1(X, x_0)$. Then the group $\mathrm{Deck}(\tilde{X}|X)$ of deck transformations is isomorphic to the corresponding quotient group $\pi_1(X, x_0)/p_\#(\pi_1(\tilde{X}, \tilde{x}_0))$.*

Proof Let G be the group $\mathrm{Deck}(\tilde{X}|X)$ of deck transformations of $\tilde{X}$. Then the group G acts freely and properly discontinuously on $\tilde{X}$ (Proposition 4.13). Let $q: X \to \tilde{X}/G$ be the quotient map onto the orbit space X/G. Elements w_1 and w_2 of $\tilde{X}$ satisfy $w_2 = g(w_1)$ for some $g \in G$ if and only if $p(w_1) = p(w_2)$. It follows that there is a continuous map $h: \tilde{X}/G \to X$ for which $h \circ q = p$. This map h is a bijection. Moreover it maps open sets to open sets, for if W is some open set in $\tilde{X}/G$ then $q^{-1}(W)$ is an open set in $\tilde{X}$, and therefore $p(q^{-1}(W))$ is an open set in X, since any covering map maps open sets to open sets (Lemma 3.1). But $p(q^{-1}(W)) = h(W)$. Thus $h: \tilde{X}/G \to X$ is a continuous bijection that maps open sets to open sets, and is therefore a homeomorphism. The fundamental group of the topological space X is thus isomorphic to that of the orbit space $\tilde{X}/G$. It follows from Proposition 4.9 that there exists a surjective homomorphism from $\pi_1(X, x_0)$ to the group G of deck transformations of the covering space. The kernel of this homomorphism is $p_\#(\pi_1(\tilde{X}, \tilde{x}_0))$. The result then follows directly from the fact that the image of a group homomorphism is isomorphic to the quotient of the domain by the kernel of the homomorphism. ∎

Corollary 4.26 *Let $p: \tilde{X} \to X$ be a covering map over some topological space X which is both connected and locally path-connected, and let x_0 be a point of X. Suppose that $\tilde{X}$ is simply-connected. Then* $\mathrm{Deck}(\tilde{X}|X) \cong \pi_1(X, x_0)$.