Course 421: Algebraic Topology
Section 3: Covering Maps and the Monodromy Theorem

David R. Wilkins

Contents

3 Covering Maps and the Monodromy Theorem

3.1 Covering Maps

Definition Let X and $\tilde{X}$ be topological spaces and let $p\colon \tilde{X} \to X$ be a continuous map. An open subset U of X is said to be *evenly covered* by the map p if and only if $p^{-1}(U)$ is a disjoint union of open sets of $\tilde{X}$ each of which is mapped homeomorphically onto U by p. The map $p\colon \tilde{X} \to X$ is said to be a *covering map* if $p\colon \tilde{X} \to X$ is surjective and in addition every point of X is contained in some open set that is evenly covered by the map p.

If $p\colon \tilde{X} \to X$ is a covering map, then we say that $\tilde{X}$ is a *covering space* of X.

Example Let S^1 be the unit circle in $\mathbb{R}^2$. Then the map $p\colon \mathbb{R} \to S^1$ defined by

$$p(t) = (\cos 2\pi t, \sin 2\pi t)$$

is a covering map. Indeed let $\mathbf{n}$ be a point of S^1. Consider the open set U in S^1 containing $\mathbf{n}$ defined by $U = S^1 \setminus \{-\mathbf{n}\}$. Now $\mathbf{n} = (\cos 2\pi t_0, \sin 2\pi t_0)$ for some $t_0 \in \mathbb{R}$. Then $p^{-1}(U)$ is the union of the disjoint open sets J_n for all integers n, where

$$J_n = \{t \in \mathbb{R} : t_0 + n - \tfrac{1}{2} < t < t_0 + n + \tfrac{1}{2}\}.$$

Each of the open sets J_n is mapped homeomorphically onto U by the map p. This shows that $p\colon \mathbb{R} \to S^1$ is a covering map.

Example The map $p\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$ defined by $p(z) = \exp(z)$ is a covering map. Indeed, given any $\theta \in [-\pi, \pi]$ let us define

$$U_\theta = \{z \in \mathbb{C} \setminus \{0\} : \arg(-z) \neq \theta\}.$$

Then $p^{-1}(U_\theta)$ is the disjoint union of the open sets

$$\{z \in \mathbb{C} : |\operatorname{Im} z - \theta - 2\pi n| < \pi\},$$

for all integers n, and p maps each of these open sets homeomorphically onto U_θ. Thus U_θ is evenly covered by the map p.

Example Consider the map $\alpha\colon (-2, 2) \to S^1$, where $\alpha(t) = (\cos 2\pi t, \sin 2\pi t)$ for all $t \in (-2, 2)$. It can easily be shown that there is no open set U containing the point $(1, 0)$ that is evenly covered by the map α. Indeed suppose that there were to exist such an open set U. Then there would exist some δ satisfying $0 < \delta < \frac{1}{2}$ such that $U_\delta \subset U$, where

$$U_\delta = \{(\cos 2\pi t, \sin 2\pi t) : -\delta < t < \delta\}.$$

The open set U_δ would then be evenly covered by the map α. However the connected components of $\alpha^{-1}(U_\delta)$ are $(-2, -2+\delta)$, $(-1-\delta, -1+\delta)$, $(-\delta, \delta)$, $(1-\delta, 1+\delta)$ and $(2-\delta, 2)$, and neither $(-2, -2+\delta)$ nor $(2-\delta, 2)$ is mapped homeomorphically onto U_δ by α.

Lemma 3.1 *Let $p: \tilde{X} \to X$ be a covering map. Then $p(V)$ is open in X for every open set V in $\tilde{X}$. In particular, a covering map $p: \tilde{X} \to X$ is a homeomorphism if and only if it is a bijection.*

Proof Let V be open in $\tilde{X}$, and let $x \in p(V)$. Then $x = p(v)$ for some $v \in V$. Now there exists an open set U containing the point x which is evenly covered by the covering map p. Then $p^{-1}(U)$ is a disjoint union of open sets, each of which is mapped homeomorphically onto U by the covering map p. One of these open sets contains v; let $\tilde{U}$ be this open set, and let $N_x = p(V \cap \tilde{U})$. Now N_x is open in X, since $V \cap \tilde{U}$ is open in $\tilde{U}$ and $p|\tilde{U}$ is a homeomorphism from $\tilde{U}$ to U. Also $x \in N_x$ and $N_x \subset p(V)$. It follows that $p(V)$ is the union of the open sets N_x as x ranges over all points of $p(V)$, and thus $p(V)$ is itself an open set, as required. The result that a bijective covering map is a homeomorphism then follows directly from the fact that a continuous bijection is a homeomorphism if and only if it maps open sets to open sets. ∎

3.2 Path Lifting and the Monodromy Theorem

Let $p: \tilde{X} \to X$ be a covering map over a topological space X. Let Z be a topological space, and let $f: Z \to X$ be a continuous map from Z to X. A continuous map $\tilde{f}: Z \to \tilde{X}$ is said to be a *lift* of the map $f: Z \to X$ if and only if $p \circ \tilde{f} = f$. We shall prove various results concerning the existence and uniqueness of such lifts.

Proposition 3.2 *Let $p: \tilde{X} \to X$ be a covering map, let Z be a connected topological space, and let $g: Z \to \tilde{X}$ and $h: Z \to \tilde{X}$ be continuous maps. Suppose that $p \circ g = p \circ h$ and that $g(z) = h(z)$ for some $z \in Z$. Then $g = h$.*

Proof Let $Z_0 = \{z \in Z : g(z) = h(z)\}$. Note that Z_0 is non-empty, by hypothesis. We show that Z_0 is both open and closed in Z.

Let z be a point of Z. There exists an open set U in X containing the point $p(g(z))$ which is evenly covered by the covering map p. Then $p^{-1}(U)$ is a disjoint union of open sets, each of which is mapped homeomorphically onto U by the covering map p. One of these open sets contains $g(z)$; let this set be denoted by $\tilde{U}$. Also one of these open sets contains $h(z)$; let this open set be denoted by $\tilde{V}$. Let $N_z = g^{-1}(\tilde{U}) \cap h^{-1}(\tilde{V})$. Then N_z is an open set in Z containing z.

Consider the case when $z \in Z_0$. Then $g(z) = h(z)$, and therefore $\tilde{V} = \tilde{U}$. It follows from this that both g and h map the open set N_z into $\tilde{U}$. But $p \circ g = p \circ h$, and $p|\tilde{U}: \tilde{U} \to U$ is a homeomorphism. Therefore $g|N_z = h|N_z$, and thus $N_z \subset Z_0$. We have thus shown that, for each $z \in Z_0$, there exists an open set N_z such that $z \in N_z$ and $N_z \subset Z_0$. We conclude that Z_0 is open.

Next consider the case when $z \in Z \setminus Z_0$. In this case $\tilde{U} \cap \tilde{V} = \emptyset$, since $g(z) \neq h(z)$. But $g(N_z) \subset \tilde{U}$ and $h(N_z) \subset \tilde{V}$. Therefore $g(z') \neq h(z')$ for all $z' \in N_z$, and thus $N_z \subset Z \setminus Z_0$. We have thus shown that, for each $z \in Z \setminus Z_0$, there exists an open set N_z such that $z \in N_z$ and $N_z \subset Z \setminus Z_0$. We conclude that $Z \setminus Z_0$ is open.

The subset Z_0 of Z is therefore both open and closed. Also Z_0 is non-empty by hypothesis. We deduce that $Z_0 = Z$, since Z is connected. Thus $g = h$, as required. ∎

Lemma 3.3 *Let $p: \tilde{X} \to X$ be a covering map, let Z be a topological space, let A be a connected subset of Z, and let $f: Z \to X$ and $g: A \to \tilde{X}$ be continuous maps with the property that $p \circ g = f|A$. Suppose that $f(Z) \subset U$, where U is an open subset of X that is evenly covered by the covering map p. Then there exists a continuous map $\tilde{f}: Z \to \tilde{X}$ such that $\tilde{f}|A = g$ and $p \circ \tilde{f} = f$.*

Proof The open set U is evenly covered by the covering map p, and therefore $p^{-1}(U)$ is a disjoint union of open sets, each of which is mapped homeomorphically onto U by the covering map p. One of these open sets contains $g(a)$ for some $a \in A$; let this set be denoted by $\tilde{U}$. Let $\sigma: U \to \tilde{U}$ be the inverse of the homeomorphism $p|\tilde{U}: \tilde{U} \to U$, and let $\tilde{f} = \sigma \circ f$. Then $p \circ \tilde{f} = f$. Also $p \circ \tilde{f}|A = p \circ g$ and $\tilde{f}(a) = g(a)$. It follows from Proposition 3.2 that $\tilde{f}|A = g$, since A is connected. Thus $\tilde{f}: Z \to \tilde{X}$ is the required map. ∎

Theorem 3.4 (Path Lifting Theorem) *Let $p: \tilde{X} \to X$ be a covering map, let $\gamma: [0, 1] \to X$ be a continuous path in X, and let w be a point of $\tilde{X}$ satisfying $p(w) = \gamma(0)$. Then there exists a unique continuous path $\tilde{\gamma}: [0, 1] \to \tilde{X}$ such that $\tilde{\gamma}(0) = w$ and $p \circ \tilde{\gamma} = \gamma$.*

Proof The map $p: \tilde{X} \to X$ is a covering map; therefore there exists an open cover $\mathcal{U}$ of X such that each open set U belonging to X is evenly covered by the map p. Now the collection consisting of the preimages $\gamma^{-1}(U)$ of the open sets U belonging to $\mathcal{U}$ is an open cover of the interval $[0,1]$. But $[0,1]$ is compact, by the Heine-Borel Theorem. It follows from the Lebesgue Lemma that there exists some $\delta > 0$ such that every subinterval of length less than δ is mapped by γ into one of the open sets belonging to $\mathcal{U}$. Partition the interval $[0,1]$ into subintervals $[t_{i-1}, t_i]$, where $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$, and where the length of each subinterval is less than δ. Then each subinterval $[t_{i-1}, t_i]$ is mapped by γ into some open set in X that is evenly covered by the map p. It follows from Lemma 3.3 that once $\tilde{\gamma}(t_{i-1})$ has been determined, we can extend $\tilde{\gamma}$ continuously over the ith subinterval $[t_{i-1}, t_i]$. Thus by extending $\tilde{\gamma}$ successively over $[t_0, t_1]$, $[t_1, t_2]$,..., $[t_{n-1}, t_n]$, we can lift the path $\gamma: [0,1] \to X$ to a path $\tilde{\gamma}: [0,1] \to \tilde{X}$ starting at w. The uniqueness of $\tilde{\gamma}$ follows from Proposition 3.2. ∎

Theorem 3.5 (The Monodromy Theorem) *Let $p: \tilde{X} \to X$ be a covering map, let $H: [0,1] \times [0,1] \to X$ be a continuous map, and let w be a point of $\tilde{X}$ satisfying $p(w) = H(0,0)$. Then there exists a unique continuous map $\tilde{H}: [0,1] \times [0,1] \to \tilde{X}$ such that $\tilde{H}(0,0) = w$ and $p \circ \tilde{H} = H$.*

Proof The unit square $[0,1] \times [0,1]$ is compact. By applying the Lebesgue Lemma to an open cover of the square by preimages of evenly covered open sets in X (as in the proof of Theorem 3.4), we see that there exists some $\delta > 0$ with the property that any square contained in $[0,1] \times [0,1]$ whose sides have length less than δ is mapped by H into some open set in X which is evenly covered by the covering map p. It follows from Lemma 3.3 that if the lift $\tilde{H}$ of H has already been determined over a corner, or along one side, or along two adjacent sides of a square whose sides have length less than δ, then $\tilde{H}$ can be extended over the whole of that square. Thus if we subdivide $[0,1] \times [0,1]$ into squares $S_{j,k}$, where

$$S_{j,k} = \left\{ (s,t) \in [0,1] \times [0,1] : \frac{j-1}{n} \leq s \leq \frac{j}{n} \text{ and } \frac{k-1}{n} \leq t \leq \frac{k}{n} \right\},$$

and $1/n < \delta$, then we can construct a lift $\tilde{H}$ of H by defining $\tilde{H}(0,0) = w$, and then successively extending $\tilde{H}$ in turn over each of these smaller squares. (Indeed the map $\tilde{H}$ can be extended successively over the squares

$$S_{1,1}, S_{1,2}, \ldots, S_{1,n}, S_{2,1}, S_{2,2}, \ldots, S_{2,n}, S_{3,1}, \ldots, S_{n-1,n}, \ldots, S_{n,1}, S_{n,2}, \ldots, S_{n,n}.)$$

The uniqueness of $\tilde{H}$ follows from Proposition 3.2. ∎

3.3 The Fundamental Group of the Circle

Theorem 3.6 *$\pi_1(S^1, b) \cong \mathbb{Z}$ for any $b \in S^1$.*

Proof We regard S^1 as the unit circle in $\mathbb{R}^2$. Without loss of generality, we can take $b = (1, 0)$. Now the map $p: \mathbb{R} \to S^1$ which sends $t \in \mathbb{R}$ to $(\cos 2\pi t, \sin 2\pi t)$ is a covering map, and $b = p(0)$. Moreover $p(t_1) = p(t_2)$ if and only if $t_1 - t_2$ is an integer; in particular $p(t) = b$ if and only if t is an integer.

Let α and β be loops in S^1 based at b, and let $\tilde{\alpha}$ and $\tilde{\beta}$ be paths in $\mathbb{R}$ that satisfy $p \circ \tilde{\alpha} = \alpha$ and $p \circ \tilde{\beta} = \beta$. Suppose that α and β represent the same element of $\pi_1(S^1, b)$. Then there exists a homotopy $F: [0, 1] \times [0, 1] \to S^1$ such that $F(t, 0) = \alpha(t)$ and $F(t, 1) = \beta(t)$ for all $t \in [0, 1]$, and $F(0, \tau) = F(1, \tau) = b$ for all $\tau \in [0, 1]$. It follows from the Monodromy Theorem (Theorem 3.5) that this homotopy lifts to a continuous map $G: [0, 1] \times [0, 1] \to \mathbb{R}$ satisfying $p \circ G = F$. Moreover $G(0, \tau)$ and $G(1, \tau)$ are integers for all $\tau \in [0, 1]$, since $p(G(0, \tau)) = b = p(G(1, \tau))$. Also $G(t, 0) - \tilde{\alpha}(t)$ and $G(t, 1) - \tilde{\beta}(t)$ are integers for all $t \in [0, 1]$, since $p(G(t, 0)) = \alpha(t) = p(\tilde{\alpha}(t))$ and $p(G(t, 1)) = \beta(t) = p(\tilde{\beta}(t))$. Now any continuous integer-valued function on $[0, 1]$ is constant, by the Intermediate Value Theorem. In particular the functions sending $\tau \in [0, 1]$ to $G(0, \tau)$ and $G(1, \tau)$ are constant, as are the functions sending $t \in [0, 1]$ to $G(t, 0) - \tilde{\alpha}(t)$ and $G(t, 1) - \tilde{\beta}(t)$. Thus

$$G(0,0) = G(0,1), \qquad G(1,0) = G(1,1),$$

$$G(1,0) - \tilde{\alpha}(1) = G(0,0) - \tilde{\alpha}(0), \qquad G(1,1) - \tilde{\beta}(1) = G(0,1) - \tilde{\beta}(0).$$

On combining these results, we see that

$$\tilde{\alpha}(1) - \tilde{\alpha}(0) = G(1,0) - G(0,0) = G(1,1) - G(0,1) = \tilde{\beta}(1) - \tilde{\beta}(0).$$

We conclude from this that there exists a well-defined function $\lambda: \pi_1(S^1, b) \to \mathbb{Z}$ characterized by the property that $\lambda([\alpha]) = \tilde{\alpha}(1) - \tilde{\alpha}(0)$ for all loops α based at b, where $\tilde{\alpha}: [0, 1] \to \mathbb{R}$ is any path in $\mathbb{R}$ satisfying $p \circ \tilde{\alpha} = \alpha$.

Next we show that λ is a homomorphism. Let α and β be any loops based at b, and let $\tilde{\alpha}$ and $\tilde{\beta}$ be lifts of α and β. The element $[\alpha][\beta]$ of $\pi_1(S^1, b)$ is represented by the product path $\alpha.\beta$, where

$$(\alpha.\beta)(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2}; \\ \beta(2t - 1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Define a continuous path $\sigma: [0, 1] \to \mathbb{R}$ by

$$\sigma(t) = \begin{cases} \tilde{\alpha}(2t) & \text{if } 0 \le t \le \frac{1}{2}; \\ \tilde{\beta}(2t - 1) + \tilde{\alpha}(1) - \tilde{\beta}(0) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

(Note that $\sigma(t)$ is well-defined when $t = \frac{1}{2}$.) Then $p \circ \sigma = \alpha.\beta$ and thus

$$\begin{aligned}\lambda([\alpha][\beta]) &= \lambda([\alpha.\beta]) = \sigma(1) - \sigma(0) = \tilde{\alpha}(1) - \tilde{\alpha}(0) + \tilde{\beta}(1) - \tilde{\beta}(0) \\ &= \lambda([\alpha]) + \lambda([\beta]).\end{aligned}$$

Thus $\lambda: \pi_1(S^1, b) \to \mathbb{Z}$ is a homomorphism.

Now suppose that $\lambda([\alpha]) = \lambda([\beta])$. Let $F: [0,1] \times [0,1] \to S^1$ be the homotopy between α and β defined by

$$F(t, \tau) = p\left((1-\tau)\tilde{\alpha}(t) + \tau\tilde{\beta}(t)\right),$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are the lifts of α and β respectively starting at 0. Now $\tilde{\beta}(1) = \lambda([\beta]) = \lambda([\alpha]) = \tilde{\alpha}(1)$, and $\tilde{\beta}(0) = \tilde{\alpha}(0) = 0$. Therefore $F(0, \tau) = b = p(\tilde{\alpha}(1)) = F(1, \tau)$ for all $\tau \in [0,1]$. Thus $\alpha \simeq \beta$ rel $\{0, 1\}$, and therefore $[\alpha] = [\beta]$. This shows that $\lambda: \pi_1(S^1, b) \to \mathbb{Z}$ is injective.

The homomorphism λ is surjective, since $n = \lambda([\gamma_n])$ for all $n \in \mathbb{Z}$, where the loop $\gamma_n: [0,1] \to S^1$ is given by $\gamma_n(t) = p(nt) = (\cos 2\pi nt, \sin 2\pi nt)$ for all $t \in [0,1]$. We conclude that $\lambda: \pi_1(S^1, b) \to \mathbb{Z}$ is an isomorphism. ∎

We now show that every continuous map from the closed disk D to itself has at least one fixed point. This is the two-dimensional version of the *Brouwer Fixed Point Theorem.*

Theorem 3.7 *Let $f: D \to D$ be a continuous map which maps the closed disk D into itself. Then $f(\mathbf{x}_0) = \mathbf{x}_0$ for some $\mathbf{x}_0 \in D$.*

Proof Let ∂D denote the boundary circle of D. The inclusion map $i: \partial D \hookrightarrow D$ induces a corresponding homomorphism $i_\#: \pi_1(\partial D, \mathbf{b}) \to \pi_1(D, \mathbf{b})$ of fundamental groups for any $\mathbf{b} \in \partial D$.

Suppose that it were the case that the map f has no fixed point in D. Then one could define a continuous map $r: D \to \partial D$ as follows: for each $\mathbf{x} \in D$, let $r(\mathbf{x})$ be the point on the boundary ∂D of D obtained by continuing the line segment joining $f(\mathbf{x})$ to $\mathbf{x}$ beyond $\mathbf{x}$ until it intersects ∂D at the point $r(\mathbf{x})$. Note that $r|\partial D$ is the identity map of ∂D.

Let $r_\#: \pi_1(D, \mathbf{b}) \to \pi_1(\partial D, \mathbf{b})$ be the homomorphism of fundamental groups induced by $r: D \to \partial D$. Now $(r \circ i)_\#: \pi_1(\partial D, \mathbf{b}) \to \pi_1(\partial D, \mathbf{b})$ is the identity isomorphism of $\pi_1(\partial D, \mathbf{b})$, since $r \circ i: \partial D \to \partial D$ is the identity map. But it follows directly from the definition of induced homomorphisms that $(r \circ i)_\# = r_\# \circ i_\#$. Therefore $i_\#: \pi_1(\partial D, \mathbf{b}) \to \pi_1(D, \mathbf{b})$ is injective, and $r_\#: \pi_1(D, \mathbf{b}) \to \pi_1(\partial D, \mathbf{b})$ is surjective. But this is impossible, since $\pi_1(\partial D, \mathbf{b}) \cong \mathbb{Z}$ (Theorem 3.6) and $\pi_1(D, \mathbf{b})$ is the trivial group. This contradiction shows that the continuous map $f: D \to D$ must have at least one fixed point. ∎