3 Covering Maps and the Monodromy Theorem

3.1 Covering Maps

Definition Let $X$ and $\tilde{X}$ be topological spaces and let $p: \tilde{X} \to X$ be a continuous map. An open subset $U$ of $X$ is said to be evenly covered by the map $p$ if and only if $p^{-1}(U)$ is a disjoint union of open sets of $\tilde{X}$ each of which is mapped homeomorphically onto $U$ by $p$. The map $p: \tilde{X} \to X$ is said to be a covering map if $p: \tilde{X} \to X$ is surjective and in addition every point of $X$ is contained in some open set that is evenly covered by the map $p$.

If $p: \tilde{X} \to X$ is a covering map, then we say that $\tilde{X}$ is a covering space of $X$.

Example Let $S^1$ be the unit circle in $\mathbb{R}^2$. Then the map $p: \mathbb{R} \to S^1$ defined by
\[ p(t) = (\cos 2\pi t, \sin 2\pi t) \]
is a covering map. Indeed let $n$ be a point of $S^1$. Consider the open set $U$ in $S^1$ containing $n$ defined by $U = S^1 \setminus \{-n\}$. Now $n = (\cos 2\pi t_0, \sin 2\pi t_0)$ for some $t_0 \in \mathbb{R}$. Then $p^{-1}(U)$ is the union of the disjoint open sets $J_n$ for all integers $n$, where
\[ J_n = \{ t \in \mathbb{R} : t_0 + n - \frac{1}{2} < t < t_0 + n + \frac{1}{2} \}. \]
Each of the open sets $J_n$ is mapped homeomorphically onto $U$ by the map $p$. This shows that $p: \mathbb{R} \to S^1$ is a covering map.

Example The map $p: \mathbb{C} \to \mathbb{C} \setminus \{0\}$ defined by $p(z) = \exp(z)$ is a covering map. Indeed, given any $\theta \in [-\pi, \pi]$ let us define
\[ U_\theta = \{ z \in \mathbb{C} \setminus \{0\} : \arg(-z) \neq \theta \}. \]
Then $p^{-1}(U_\theta)$ is the disjoint union of the open sets
\[ \{ z \in \mathbb{C} : |\Im z - \theta - 2\pi n| < \pi \}, \]
for all integers $n$, and $p$ maps each of these open sets homeomorphically onto $U_\theta$. Thus $U_\theta$ is evenly covered by the map $p$.

Example Consider the map $\alpha: (-2, 2) \to S^1$, where $\alpha(t) = (\cos 2\pi t, \sin 2\pi t)$ for all $t \in (-2, 2)$. It can easily be shown that there is no open set $U$ containing the point $(1, 0)$ that is evenly covered by the map $\alpha$. Indeed
suppose that there were to exist such an open set \( U \). Then there would exist some \( \delta \) satisfying \( 0 < \delta < \frac{1}{2} \) such that \( U_\delta \subset U \), where

\[
U_\delta = \{(\cos 2\pi t, \sin 2\pi t) : -\delta < t < \delta\}.
\]

The open set \( U_\delta \) would then be evenly covered by the map \( \alpha \). However the connected components of \( \alpha^{-1}(U_\delta) \) are \((-2, -2 + \delta), (-1 - \delta, -1 + \delta), (-\delta, \delta), (1 - \delta, 1 + \delta) \) and \((2 - \delta, 2)\), and neither \((-2, -2 + \delta)\) nor \((2 - \delta, 2)\) is mapped homeomorphically onto \( U_\delta \) by \( \alpha \).

**Lemma 3.1** Let \( p: \tilde{X} \to X \) be a covering map. Then \( p(V) \) is open in \( X \) for every open set \( V \) in \( \tilde{X} \). In particular, a covering map \( p: \tilde{X} \to X \) is a homeomorphism if and only if it is a bijection.

**Proof** Let \( V \) be open in \( \tilde{X} \), and let \( x \in p(V) \). Then \( x = p(v) \) for some \( v \in V \). Now there exists an open set \( U \) containing the point \( x \) which is evenly covered by the covering map \( p \). Then \( p^{-1}(U) \) is a disjoint union of open sets, each of which is mapped homeomorphically onto \( U \) by the covering map \( p \). One of these open sets contains \( v \); let \( \tilde{U} \) be this open set, and let \( N_x = p(V \cap \tilde{U}) \). Now \( N_x \) is open in \( X \), since \( V \cap \tilde{U} \) is open in \( \tilde{U} \) and \( p|\tilde{U} \) is a homeomorphism from \( \tilde{U} \) to \( U \). Also \( x \in N_x \) and \( N_x \subset p(V) \). It follows that \( p(V) \) is the union of the open sets \( N_x \) as \( x \) ranges over all points of \( p(V) \), and thus \( p(V) \) is itself an open set, as required. The result that a bijective covering map is a homeomorphism then follows directly from the fact that a continuous bijection is a homeomorphism if and only if it maps open sets to open sets.

**3.2 Path Lifting and the Monodromy Theorem**

Let \( p: \tilde{X} \to X \) be a covering map over a topological space \( X \). Let \( Z \) be a topological space, and let \( f: Z \to X \) be a continuous map from \( Z \) to \( X \). A continuous map \( \tilde{f}: Z \to \tilde{X} \) is said to be a lift of the map \( f: Z \to X \) if and only if \( p \circ \tilde{f} = f \). We shall prove various results concerning the existence and uniqueness of such lifts.

**Proposition 3.2** Let \( p: \tilde{X} \to X \) be a covering map, let \( Z \) be a connected topological space, and let \( g: Z \to \tilde{X} \) and \( h: Z \to \tilde{X} \) be continuous maps. Suppose that \( p \circ g = p \circ h \) and that \( g(z) = h(z) \) for some \( z \in Z \). Then \( g = h \).

**Proof** Let \( Z_0 = \{z \in Z : g(z) = h(z)\} \). Note that \( Z_0 \) is non-empty, by hypothesis. We show that \( Z_0 \) is both open and closed in \( Z \).
Let \( z \) be a point of \( Z \). There exists an open set \( U \) in \( X \) containing the point \( p(g(z)) \) which is evenly covered by the covering map \( p \). Then \( p^{-1}(U) \) is a disjoint union of open sets, each of which is mapped homeomorphically onto \( U \) by the covering map \( p \). One of these open sets contains \( g(z) \); let this set be denoted by \( \tilde{U} \). Also one of these open sets contains \( h(z) \); let this open set be denoted by \( \tilde{V} \). Let \( N_z = g^{-1}(\tilde{U}) \cap h^{-1}(\tilde{V}) \). Then \( N_z \) is an open set in \( Z \) containing \( z \).

Consider the case when \( z \in Z_0 \). Then \( g(z) = h(z) \), and therefore \( \tilde{V} = \tilde{U} \). It follows from this that both \( g \) and \( h \) map the open set \( N_z \) into \( \tilde{U} \). But \( p \circ g = p \circ h \), and \( p(\tilde{U}) : \tilde{U} \to U \) is a homeomorphism. Therefore \( g|_{N_z} = h|_{N_z} \), and thus \( N_z \subset Z_0 \). We have thus shown that, for each \( z \in Z_0 \), there exists an open set \( N_z \) such that \( z \in N_z \) and \( N_z \subset Z_0 \). We conclude that \( Z_0 \) is open.

Next consider the case when \( z \in Z \setminus Z_0 \). In this case \( \tilde{U} \cap \tilde{V} = \emptyset \), since \( g(z) \neq h(z) \). But \( g(N_z) \subset \tilde{U} \) and \( h(N_z) \subset \tilde{V} \). Therefore \( g(z') \neq h(z') \) for all \( z' \in N_z \), and thus \( N_z \subset Z \setminus Z_0 \). We have thus shown that, for each \( z \in Z \setminus Z_0 \), there exists an open set \( N_z \) such that \( z \in N_z \) and \( N_z \subset Z \setminus Z_0 \). We conclude that \( Z \setminus Z_0 \) is open.

The subset \( Z_0 \) of \( Z \) is therefore both open and closed. Also \( Z_0 \) is non-empty by hypothesis. We deduce that \( Z_0 = Z \), since \( Z \) is connected. Thus \( g = h \), as required.

**Lemma 3.3** Let \( p : \tilde{X} \to X \) be a covering map, let \( Z \) be a topological space, let \( A \) be a connected subset of \( Z \), and let \( f : Z \to X \) and \( g : A \to \tilde{X} \) be continuous maps with the property that \( p \circ g = f|A \). Suppose that \( f(Z) \subset U \), where \( U \) is an open subset of \( X \) that is evenly covered by the covering map \( p \). Then there exists a continuous map \( \tilde{f} : Z \to \tilde{X} \) such that \( \tilde{f}|A = g \) and \( p \circ \tilde{f} = f \).

**Proof** The open set \( U \) is evenly covered by the covering map \( p \), and therefore \( p^{-1}(U) \) is a disjoint union of open sets, each of which is mapped homeomorphically onto \( U \) by the covering map \( p \). One of these open sets contains \( g(a) \) for some \( a \in A \); let this set be denoted by \( \tilde{U} \). Let \( \sigma : U \to \tilde{U} \) be the inverse of the homeomorphism \( p(\tilde{U}) : \tilde{U} \to U \), and let \( \tilde{f} = \sigma \circ f \). Then \( p \circ \tilde{f} = f \). Also \( p \circ \tilde{f}|A = p \circ g \) and \( \tilde{f}(a) = g(a) \). It follows from Proposition 3.2 that \( \tilde{f}|A = g \), since \( A \) is connected. Thus \( \tilde{f} : Z \to \tilde{X} \) is the required map.

**Theorem 3.4** (Path Lifting Theorem) Let \( p : \tilde{X} \to X \) be a covering map, let \( \gamma : [0, 1] \to X \) be a continuous path in \( X \), and let \( w \) be a point of \( \tilde{X} \) satisfying \( p(w) = \gamma(0) \). Then there exists a unique continuous path \( \tilde{\gamma} : [0, 1] \to \tilde{X} \) such that \( \tilde{\gamma}(0) = w \) and \( p \circ \tilde{\gamma} = \gamma \).
**Proof** The map $p: \tilde{X} \to X$ is a covering map; therefore there exists an open cover $U$ of $X$ such that each open set $U$ belonging to $X$ is evenly covered by the map $p$. Now the collection consisting of the preimages $\gamma^{-1}(U)$ of the open sets $U$ belonging to $U$ is an open cover of the interval $[0, 1]$. But $[0, 1]$ is compact, by the Heine-Borel Theorem. It follows from the Lebesgue Lemma that there exists some $\delta > 0$ such that every subinterval of length less than $\delta$ is evenly covered by the covering map $p$. Partition the interval $[0, 1]$ into subintervals $[t_{i-1}, t_i]$, where $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$, and where the length of each subinterval is less than $\delta$. Then each subinterval $[t_{i-1}, t_i]$ is mapped by $\gamma$ into some open set in $X$ that is evenly covered by the map $p$. It follows from Lemma 3.3 that once $\tilde{\gamma}(t_{i-1})$ has been determined, we can extend $\tilde{\gamma}$ continuously over the $i$th subinterval $[t_{i-1}, t_i]$. Thus by extending $\tilde{\gamma}$ successively over $[t_0, t_1]$, $[t_1, t_2]$, ..., $[t_{n-1}, t_n]$, we can lift the path $\gamma:[0, 1] \to X$ to a path $\tilde{\gamma}:[0, 1] \to \tilde{X}$ starting at $w$. The uniqueness of $\tilde{\gamma}$ follows from Proposition 3.2.

**Theorem 3.5** (The Monodromy Theorem) Let $p: \tilde{X} \to X$ be a covering map, let $H: [0, 1] \times [0, 1] \to X$ be a continuous map, and let $w$ be a point of $X$ satisfying $p(w) = H(0, 0)$. Then there exists a unique continuous map $\tilde{H}: [0, 1] \times [0, 1] \to \tilde{X}$ such that $\tilde{H}(0, 0) = w$ and $p \circ \tilde{H} = H$.

**Proof** The unit square $[0, 1] \times [0, 1]$ is compact. By applying the Lebesgue Lemma to an open cover of the square by preimages of evenly covered open sets in $X$ (as in the proof of Theorem 3.4), we see that there exists some $\delta > 0$ with the property that any square contained in $[0, 1] \times [0, 1]$ whose sides have length less than $\delta$ is mapped by $H$ into some open set in $X$ which is evenly covered by the covering map $p$. It follows from Lemma 3.3 that if the lift $\tilde{H}$ of $H$ has already been determined over a corner, or along one side, or along two adjacent sides of a square whose sides have length less than $\delta$, then $\tilde{H}$ can be extended over the whole of that square. Thus if we subdivide $[0, 1] \times [0, 1]$ into squares $S_{j,k}$, where

$$S_{j,k} = \left\{ (s, t) \in [0, 1] \times [0, 1] : \frac{j-1}{n} \leq s \leq \frac{j}{n} \text{ and } \frac{k-1}{n} \leq t \leq \frac{k}{n} \right\},$$

and $1/n < \delta$, then we can construct a lift $\tilde{H}$ of $H$ by defining $\tilde{H}(0, 0) = w$, and then successively extending $\tilde{H}$ in turn over each of these smaller squares. (Indeed the map $\tilde{H}$ can be extended successively over the squares $S_{1,1}, S_{1,2}, \ldots, S_{1,n}, S_{2,1}, S_{2,2}, \ldots, S_{2,n}, S_{3,1}, \ldots, S_{n-1,n}, \ldots, S_{n,1}, S_{n,2}, \ldots, S_{n,n}$.) The uniqueness of $\tilde{H}$ follows from Proposition 3.2.
3.3 The Fundamental Group of the Circle

Theorem 3.6 $\pi_1(S^1, b) \cong \mathbb{Z}$ for any $b \in S^1$.

Proof We regard $S^1$ as the unit circle in $\mathbb{R}^2$. Without loss of generality, we can take $b = (1, 0)$. Now the map $p: \mathbb{R} \to S^1$ which sends $t \in \mathbb{R}$ to $(\cos 2\pi t, \sin 2\pi t)$ is a covering map, and $b = p(0)$. Moreover $p(t_1) = p(t_2)$ if and only if $t_1 - t_2$ is an integer; in particular $p(t) = b$ if and only if $t$ is an integer.

Let $\alpha$ and $\beta$ be loops in $S^1$ based at $b$, and let $\tilde{\alpha}$ and $\tilde{\beta}$ be paths in $\mathbb{R}$ that satisfy $p \circ \tilde{\alpha} = \alpha$ and $p \circ \tilde{\beta} = \beta$. Suppose that $\alpha$ and $\beta$ represent the same element of $\pi_1(S^1, b)$. Then there exists a homotopy $F: [0, 1] \times [0, 1] \to S^1$ such that $F(t, 0) = \alpha(t)$ and $F(t, 1) = \beta(t)$ for all $t \in [0, 1]$, and $F(0, \tau) = F(1, \tau) = b$ for all $\tau \in [0, 1]$. It follows from the Monodromy Theorem (Theorem 3.5) that this homotopy lifts to a continuous map $\tilde{G}: [0, 1] \times [0, 1] \to \mathbb{R}$ satisfying $p \circ \tilde{G} = F$. Moreover $G(0, \tau)$ and $G(1, \tau)$ are integers for all $\tau \in [0, 1]$, since $p(G(0, \tau)) = b = p(G(1, \tau))$. Also $G(t, 0) - \tilde{\alpha}(t)$ and $G(t, 1) - \tilde{\beta}(t)$ are integers for all $t \in [0, 1]$, since $p(G(t, 0)) = \alpha(t) = p(\tilde{\alpha}(t))$ and $p(G(t, 1)) = \beta(t) = p(\tilde{\beta}(t))$. Now any continuous integer-valued function on $[0, 1]$ is constant, by the Intermediate Value Theorem. In particular the functions sending $\tau \in [0, 1]$ to $G(0, \tau)$ and $G(1, \tau)$ are constant, as are the functions sending $t \in [0, 1]$ to $G(t, 0) - \tilde{\alpha}(t)$ and $G(t, 1) - \tilde{\beta}(t)$. Thus

\[
G(0, 0) = G(0, 1), \quad G(1, 0) = G(1, 1),
\]

\[
G(1, 0) - \tilde{\alpha}(1) = G(0, 0) - \tilde{\alpha}(0), \quad G(1, 1) - \tilde{\beta}(1) = G(0, 1) - \tilde{\beta}(0).
\]

On combining these results, we see that

\[
\tilde{\alpha}(1) - \tilde{\alpha}(0) = G(1, 0) - G(0, 0) = G(1, 1) - G(0, 1) = \tilde{\beta}(1) - \tilde{\beta}(0).
\]

We conclude from this that there exists a well-defined function $\lambda: \pi_1(S^1, b) \to \mathbb{Z}$ characterized by the property that $\lambda([\alpha]) = \tilde{\alpha}(1) - \tilde{\alpha}(0)$ for all loops $\alpha$ based at $b$, where $\tilde{\alpha}: [0, 1] \to \mathbb{R}$ is any path in $\mathbb{R}$ satisfying $p \circ \tilde{\alpha} = \alpha$.

Next we show that $\lambda$ is a homomorphism. Let $\alpha$ and $\beta$ be any loops based at $b$, and let $\tilde{\alpha}$ and $\tilde{\beta}$ be lifts of $\alpha$ and $\beta$. The element $[\alpha][\beta]$ of $\pi_1(S^1, b)$ is represented by the product path $\alpha \beta$, where

\[
(\alpha \beta)(t) = \begin{cases} 
\alpha(2t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\
\beta(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1.
\end{cases}
\]

Define a continuous path $\sigma: [0, 1] \to \mathbb{R}$ by

\[
\sigma(t) = \begin{cases} 
\tilde{\alpha}(2t) & \text{if } 0 \leq t \leq \frac{1}{2}; \\
\tilde{\beta}(2t - 1) + \tilde{\alpha}(1) - \tilde{\beta}(0) & \text{if } \frac{1}{2} \leq t \leq 1.
\end{cases}
\]
(Note that $\sigma(t)$ is well-defined when $t = \frac{1}{2}$.) Then $p \circ \sigma = \alpha \cdot \beta$ and thus
\[
\lambda([\alpha][\beta]) = \lambda([\alpha \cdot \beta]) = \sigma(1) - \sigma(0) = \tilde{\alpha}(1) - \tilde{\alpha}(0) + \tilde{\beta}(1) - \tilde{\beta}(0) = \lambda([\alpha]) + \lambda([\beta]).
\]
Thus $\lambda : \pi_1(S^1, b) \to \mathbb{Z}$ is a homomorphism.

Now suppose that $\lambda([\alpha]) = \lambda([\beta])$. Let $F : [0, 1] \times [0, 1] \to S^1$ be the homotopy between $\alpha$ and $\beta$ defined by
\[
F(t, \tau) = p\left(1 - \tau\right)\tilde{\alpha}(t) + \tau \tilde{\beta}(t),
\]
where $\tilde{\alpha}$ and $\tilde{\beta}$ are the lifts of $\alpha$ and $\beta$ respectively starting at 0. Now $\tilde{\beta}(1) = \lambda([\beta]) = \lambda([\alpha]) = \tilde{\alpha}(1)$, and $\tilde{\beta}(0) = \tilde{\alpha}(0) = 0$. Therefore $F(0, \tau) = b = p(\tilde{\alpha}(1)) = F(1, \tau)$ for all $\tau \in [0, 1]$. Thus $\alpha \simeq \beta$ rel $\{0, 1\}$, and therefore $[\alpha] = [\beta]$. This shows that $\lambda : \pi_1(S^1, b) \to \mathbb{Z}$ is injective.

The homomorphism $\lambda$ is surjective, since $n = \lambda([\gamma_n])$ for all $n \in \mathbb{Z}$, where the loop $\gamma_n : [0, 1] \to S^1$ is given by $\gamma_n(t) = p(nt) = (\cos 2\pi nt, \sin 2\pi nt)$ for all $t \in [0, 1]$. We conclude that $\lambda : \pi_1(S^1, b) \to \mathbb{Z}$ is an isomorphism. □

We now show that every continuous map from the closed disk $D$ to itself has at least one fixed point. This is the two-dimensional version of the Brouwer Fixed Point Theorem.

**Theorem 3.7** Let $f : D \to D$ be a continuous map which maps the closed disk $D$ into itself. Then $f(x_0) = x_0$ for some $x_0 \in D$.

**Proof** Let $\partial D$ denote the boundary circle of $D$. The inclusion map $i : \partial D \hookrightarrow D$ induces a corresponding homomorphism $i_# : \pi_1(\partial D, b) \to \pi_1(D, b)$ of fundamental groups for any $b \in \partial D$.

Suppose that it were the case that the map $f$ has no fixed point in $D$. Then one could define a continuous map $r : D \to \partial D$ as follows: for each $x \in D$, let $r(x)$ be the point on the boundary $\partial D$ of $D$ obtained by continuing the line segment joining $f(x)$ to $x$ beyond $x$ until it intersects $\partial D$ at the point $r(x)$. Note that $r|\partial D$ is the identity map of $\partial D$.

Let $r_# : \pi_1(D, b) \to \pi_1(\partial D, b)$ be the homomorphism of fundamental groups induced by $r : D \to \partial D$. Now $(r \circ i)_# : \pi_1(\partial D, b) \to \pi_1(\partial D, b)$ is the identity isomorphism of $\pi_1(\partial D, b)$, since $r \circ i : \partial D \to \partial D$ is the identity map. But it follows directly from the definition of induced homomorphisms that $(r \circ i)_# = r_# \circ i_#$. Therefore $i_# : \pi_1(\partial D, b) \to \pi_1(D, b)$ is injective, and $r_# : \pi_1(D, b) \to \pi_1(\partial D, b)$ is surjective. But this is impossible, since $\pi_1(\partial D, b) \cong \mathbb{Z}$ (Theorem 3.6) and $\pi_1(D, b)$ is the trivial group. This contradiction shows that the continuous map $f : D \to D$ must have at least one fixed point. □