Course 421: Algebraic Topology
Section 10: Exact Sequences of Homology Groups

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Contents

10 Exact Sequences of Homology Groups

10.1 Homology Groups of Simplicial Pairs

A *simplicial pair* (K, L) consists of a simplicial complex K together with a subcomplex L. The qth chain group $C_q(L)$ of the subcomplex L may be regarded as a subgroup of the qth chain group $C_q(K)$ of the simplicial complex K, and the inclusion map $i: L \hookrightarrow K$ induces inclusion homomorphisms $i_q: C_q(L) \hookrightarrow C_q(K)$. We define the qth chain group $C_q(K, L)$ of the simplicial pair to be the quotient group $C_q(K)/C_q(L)$. The boundary homomorphism $\partial_q: C_q(K) \to C_{q-1}(L)$ maps the subgroup $C_q(L)$ into $C_{q-1}(L)$, and therefore induces a homomorphism $\partial_q: C_q(K, L) \to C_{q-1}(K, L)$. For each integer q, let $u_q: C_q(K) \to C_q(K, L)$ be the quotient homomorphism from $C_q(K)$ to $C_q(K, L)$. Then $\partial_q \circ u_q = u_{q-1} \circ \partial_q$ for all integers q. (This is an immediate consequence of the fact that the homomorphism $\partial_q: C_q(K, L) \to C_{q-1}(K, L)$ is by definition the homomorphism induced by the boundary homomorphism $\partial_q: C_q(K) \to C_{q-1}(K)$ of K.)

Now $\partial_{q-1} \circ \partial_q \circ u_q = \partial_{q-1} \circ u_{q-1} \circ \partial_q = u_{q-2} \circ \partial_{q-1} \circ \partial_q = 0$. Moreover the quotient homomorphism $u_q: C_q(K) \to C_q(K, L)$ is surjective. It follows that the composition of the homomorphisms $\partial_q: C_q(K, L) \to C_{q-1}(K, L)$ and $\partial_{q-1}: C_{q-1}(K, L) \to C_{q-2}(K, L)$ is the zero homomorphism. Therefore the sequence of groups $(C_q(K, L) : q \in \mathbb{Z})$ and homomorphisms $(\partial_q: C_q(K, L) \to C_{q-1}(K, L) : q \in \mathbb{Z})$ constitutes a chain complex $C_*(K, L)$, whose groups are the chain groups of the simplicial pair (K, L). We shall refer to the homomorphisms $\partial_q: C_q(K, L) \to C_{q-1}(K, L)$ as the *boundary homomorphisms* of the simplicial pair (K, L).

The sequence of quotient homomorphisms $(u_q: C_q(K) \to C_q(K, L) : q \in \mathbb{Z})$ define a chain map $u_*: C_*(K) \to C_*(K, L)$ between the chain complexes $C_*(K)$ and $C_*(K, L)$. The image $u_q(c)$ of a q-chain $c \in C_q(K)$ of K under the quotient homomorphism is the coset $c + C_q(L)$ of $C_q(L)$ in $C_q(K)$ that contains c. Moreover $\partial_q(c + C_q(L)) = \partial_q c + C_{q-1}(L)$. We define

$$\begin{aligned}
Z_q(K, L) &= \ker(\partial_q: C_q(K, L) \to C_{q-1}(K, L)) \\
&= \{c + C_q(L) : c \in C_q(K) \text{ and } \partial_q c \in C_{q-1}(L)\}, \\
B_q(K, L) &= \mathrm{image}(\partial_{q+1}: C_{q+1}(K, L) \to C_q(K, L)) \\
&= \{\partial_{q+1}(e) + C_q(L) : e \in C_{q+1}(K)\}.
\end{aligned}$$

Then $B_q(K, L) \subset Z_q(K, L)$. We define $H_q(K, L) = Z_q(K, L)/B_q(K, L)$.

Let z be an element of $Z_q(K, L)$, and let c and c' be elements of $C_q(K)$ for which $z = c + C_q(L) = c' + C_q(L)$. Then $c - c' \in C_q(L)$, $\partial_q c \in C_{q-1}(L)$ and $\partial_q c' \in C_{q-1}(L)$. But $\partial_{q-1}\partial_q c = \partial_{q-1}\partial_q c' = 0$ and $\partial_q c - \partial_q c' = \partial_q(c - c')$.

It follows that $\partial_q c \in Z_{q-1}(L)$, $\partial_q c' \in Z_{q-1}(L)$ and $\partial_q c - \partial_q c' \in B_{q-1}(L)$, and therefore $[\partial_q c] = [\partial_q c']$. It follows that there is a well-defined homomorphism from $Z_q(K, L)$ to $H_{q-1}(L)$ that maps $c + C_q(L)$ to $[\partial_q c]$. The subgroup $B_q(K, L)$ is contained in the kernel of this homomorphism. The homomorphism therefore induces a homomorphism $\partial_*: H_q(K, L) \to H_{q-1}(L)$. This homomorphism sends the homology class of $c + C_q(L)$ in $H_q(K, L)$ to the homology class of $\partial_q c$ in $C_q(L)$ for all $c \in C_q(K)$ satisfying $\partial_q c \in C_{q-1}(L)$.

Proposition 10.1 (The Homology Exact Sequence of a Simplicial Pair) *Let K be a simplicial complex, and let L be a subcomplex of K. Then the sequence*

$$\cdots \xrightarrow{\partial_*} H_q(L) \xrightarrow{i_*} H_q(K) \xrightarrow{u_*} H_q(K, L) \xrightarrow{\partial_*} H_{q-1}(L) \xrightarrow{i_*} H_{q-1}(K) \xrightarrow{u_*} \cdots$$

of homology groups is exact, where $\partial_*: H_q(K, L) \to H_{q-1}(L))$ is the homomorphism that sends the homology class of $c + C_q(L)$ in $H_q(K, L)$ to the homology class of ∂_c in $H_{q-1}(L)$ for all $c \in C_q(K)$ satisfying $\partial_q c \in C_{q-1}(L)$.

Proof The sequence $0 \longrightarrow C_*(L) \xrightarrow{i_*} C_*(K) \xrightarrow{u_*} C_*(K, L) \longrightarrow 0$ is a short exact sequence of chain complexes. It follows from Proposition 9.5 that there is a corresponding (infinite) sequence of homology groups. Moreover the homomorphism from $H_q(K, L)$ to $H_{q-1}(L)$ defined as in the statement of that proposition is the homomorphism $\partial_q: H_q(K, L) \to H_{q-1}(L)$ defined as described above. ∎

Corollary 10.2 *Let K be a simplicial complex, and let L be a subcomplex of K. Suppose that $H_{q+1}(K, L) = H_q(K, L) = 0$ for some integer q. Then $i_*: H_q(L) \to H_q(K)$ is an isomorphism.*

Corollary 10.3 *Let K be a simplicial complex, and let L be a subcomplex of K. Suppose that $H_q(K) = H_{q-1}(K) = 0$ for some integer q. Then $\partial_*: H_q(K, L) \to H_{q-1}(L)$ is an isomorphism.*

Corollary 10.4 *Let K be a simplicial complex, and let L be a subcomplex of K. Suppose that $H_q(L) = H_{q-1}(L) = 0$ for some integer q. Then $u_*: H_q(K) \to H_q(K, L)$ is an isomorphism.*

Example Let K be the simplicial complex consisting of all the faces of an n-dimensional simplex, and let L be the subcomplex consisting of all the proper faces of this simplex. Then $C_q(L) = C_q(K)$ when $q \neq n$, and therefore $C_q(K, L) = 0$ when $q \neq n$. Also $C_n(K, L) \cong \mathbb{Z}$. It follows that $H_n(K, L) \cong \mathbb{Z}$, $H_q(K, L) = 0$ when $q \neq n$. Also it follows from Proposition 6.4 that $H_q(K) = 0$ when $q > 0$.

Suppose that $n \geq 2$. It follows from Corollary 10.3 that $\partial_*: H_q(K,L) \to H_{q-1}(L)$ is an isomorphism for $q \geq 2$. Therefore $H_{n-1}(L) \cong \mathbb{Z}$, and $H_q(L) = 0$ for $q \neq 0, n-1$.

Now suppose that $n = 1$. We have an exact sequence

$$0 \longrightarrow H_1(K,L) \xrightarrow{\partial_*} H_0(L) \longrightarrow H_0(K) \longrightarrow 0.$$

Now $H_1(K,L) \cong \mathbb{Z}$ when $n = 1$. Also $H_0(K) \cong \mathbb{Z}$. From the exactness of the above sequence we can deduce that $H_0(L) \cong H_1(K,L) \oplus H_0(K) \cong \mathbb{Z} \oplus \mathbb{Z}$. This result is consistent with the fact that, in this case, L is a 0-dimensional simplicial complex consisting of two vertices.

10.2 Homology Groups of some Closed Surfaces

Lemma 10.5 *Let K be a 2-dimensional simplicial complex, and let L and M be subcomplexes of L, where $K = L \cup M$. Suppose that M consists of a triangle of K, together with all its edges and vertices, and that $L \cap M$ matches one of the following descriptions:*

(i) *$L \cap M$ consists of a single vertex of the triangle;*

(ii) *$L \cap M$ consists of a single edge of the triangle together with the endpoints of that edge;*

(iii) *$L \cap M$ consists of two edges of the triangle together with the endpoints of those edges.*

Then $H_q(K,L) = 0$ for all integers q, and therefore the inclusion map $i: L \hookrightarrow K$ induces isomorphisms $i_*: H_q(L) \to H_q(K)$ of homology groups.

Proof Let the triangle have vertices $\mathbf{v}_0$, $\mathbf{v}_1$ and $\mathbf{v}_2$, and let $\tau \in C_2(K)$ and $\rho_0, \rho_1, \rho_2 \in C_1(K)$ be defined by

$$\tau = \langle \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2 \rangle,$$

$$\rho_0 = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle, \quad \rho_1 = \langle \mathbf{v}_2, \mathbf{v}_0 \rangle, \quad \rho_2 = \langle \mathbf{v}_0, \mathbf{v}_1 \rangle.$$

Then $\partial_2 \tau = \rho_0 + \rho_1 + \rho_2$ in $C_1(K)$.

Consider first the case where $L \cap M$ is as described in (i). We label the vertices of the triangle so that $L \cap M$ consists of the single vertex $\mathbf{v}_0$. In this case

$$\begin{aligned} C_2(K,L) &= \{n\tau + C_2(L) : n \in \mathbb{Z}\}, \\ C_1(K,L) &= \{n_0\rho_0 + n_1\rho_1 + n_2\rho_2 + C_1(L) : n_0, n_1 \in \mathbb{Z}\}, \\ C_0(K,L) &= \{r_1\langle \mathbf{v}_1 \rangle + r_2\langle \mathbf{v}_2 \rangle + C_0(L) : r \in \mathbb{Z}\}. \end{aligned}$$

Now $\partial_2\tau \in \rho_0 + \rho_1 + \rho_2 + C_1(L)$, and

$$\partial_1(n_0\rho_0 + n_1\rho_1 + n_2\rho_2) \in (n_2 - n_0)\langle \mathbf{v}_1\rangle + (n_0 - n_1)\langle \mathbf{v}_2\rangle + C_0(L)$$

for all $n_0, n_1, n_2 \in \mathbb{Z}$. It follows that $B_2(K, L) = Z_2(K, L) = 0$,

$$Z_1(K, L) = B_1(K, L) = \{n(\rho_0 + \rho_1 + \rho_2) + C_1(L) : n \in \mathbb{Z}\},$$

and $Z_0(K, L) = B_0(K, L) = C_0(K, L)$. Therefore $H_q(K, L) = 0$ for all integers q in the case when $L \cap M$ consists of a single vertex of the triangle.

Consider next the case where $L \cap M$ is as described in (ii). We label the vertices of the triangle so that $L \cap M$ consists of the single edge ρ_2, together with its endpoints $\mathbf{v}_0$ and $\mathbf{v}_1$. In this case

$$\begin{aligned} C_2(K, L) &= \{n\tau + C_2(L) : n \in \mathbb{Z}\}, \\ C_1(K, L) &= \{n_0\rho_0 + n_1\rho_1 + C_1(L) : n_0, n_1 \in \mathbb{Z}\}, \\ C_0(K, L) &= \{r\langle \mathbf{v}_2\rangle + C_0(L) : r \in \mathbb{Z}\}. \end{aligned}$$

Now $\partial_2\tau \in \rho_0 + \rho_1 + C_1(L)$, and

$$\partial_1(n_0\rho_0 + n_1\rho_1) \in (n_0 - n_1)\langle \mathbf{v}_2\rangle + C_0(L)$$

for all $n_0, n_1 \in \mathbb{Z}$. It follows that $B_2(K, L) = Z_2(K, L) = 0$,

$$Z_1(K, L) = B_1(K, L) = \{n(\rho_0 + \rho_1) + C_1(L) : n \in \mathbb{Z}\},$$

and $Z_0(K, L) = B_0(K, L) = C_0(K, L)$. Therefore $H_q(K, L) = 0$ for all integers q in the case when $L \cap M$ consists of a single edge of the triangle together with its endpoints.

Finally consider the case where $L \cap M$ is as described in (iii). We label the vertices of the triangle so that $L \cap M$ consists of the edges ρ_1 and ρ_2, together with the vertices $\mathbf{v}_0$, $\mathbf{v}_1$ and $\mathbf{v}_2$ of the triangle. In this case

$$\begin{aligned} C_2(K, L) &= \{n\tau + C_2(L) : n \in \mathbb{Z}\}, \\ C_1(K, L) &= \{n_0\rho_0 + C_1(L) : n_0 \in \mathbb{Z}\}, \\ C_0(K, L) &= 0. \end{aligned}$$

In this case $\partial_2 \colon C_2(K, L) \to C_1(K, L)$ is an isomorphism that sends $\tau + C_2(L)$ to $\rho_0 + C_1(L)$, $B_2(K, L) = Z_2(K, L) = 0$, $B_1(K, L) = Z_1(K, L) = C_1(K, L)$ and $B_0(K, L) = Z_0(K, L) = C_0(K, L) = 0$. Therefore $H_q(K, L) = 0$ for all integers q in the case when $L \cap M$ consists of two edges of the triangle, together with the vertices of the triangle.

The exact sequence of homology groups of the simplicial pair (K, L) (Proposition 10.1) then ensures that the inclusion map $i \colon L \hookrightarrow K$ induces isomorphisms $i_* \colon H_q(L) \to H_q(K)$ of homology groups, as required. ∎

Lemma 10.6 *Let K be a 2-dimensional simplicial complex, and let L and M be subcomplexes of L, where $K = L \cup M$. Suppose that M consists of a triangle of K, together with all its edges and vertices, and that $L \cap M$ consists of all the edges and vertices of this triangle. Then $H_2(K, L) \cong \mathbb{Z}$, and $H_q(K, L) = 0$ for all integers q satisfying $q \neq 2$. Moreover $H_0(L) \cong H_0(K)$ and there are short exact sequences*

$$0 \longrightarrow H_2(L) \xrightarrow{i_*} H_2(K) \longrightarrow J \longrightarrow 0,$$

$$0 \longrightarrow I \longrightarrow H_1(L) \xrightarrow{i_*} H_1(K) \longrightarrow 0,$$

where $i_*: H_q(L) \to H_q(K)$ is induced by the inclusion map $i: L \hookrightarrow K$ for all $q \in \mathbb{Z}$, and

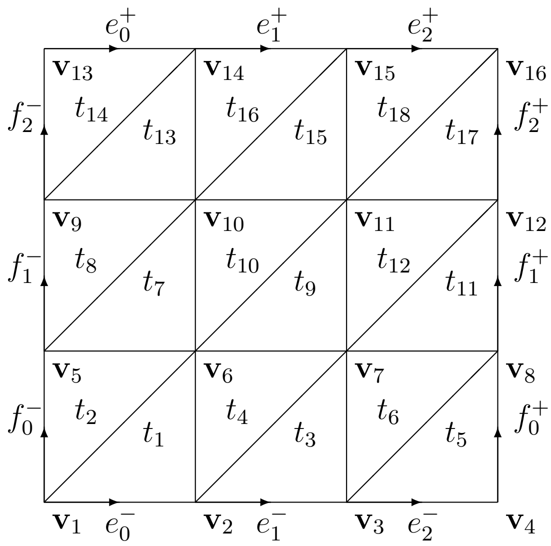
$$J = \ker(\partial_*: H_2(K, L) \to H_1(L)), \quad I = \mathrm{image}(\partial_*: H_2(K, L) \to H_1(L)).$$

Proof Let the triangle have vertices $\mathbf{v}_0$, $\mathbf{v}_1$ and $\mathbf{v}_2$. Then

$$C_2(K, L) = \{n\tau + C_2(L) : n \in \mathbb{Z}\},$$

where $\tau = \langle \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2 \rangle$, and therefore $C_2(K, L) \cong \mathbb{Z}$. Moreover $C_q(L) = C_q(K)$ when $q \neq 2$, and thus $C_q(K, L) = 0$ when $q \neq 2$. It follows that $H_q(K, L) = 0$ when $q \neq 2$, and $H_2(K, L) \cong C_2(K, L) \cong \mathbb{Z}$. The exactness of the short exact sequences then follows from the exact sequence of homology groups of the simplicial pair (K, L) (Proposition 10.1). ∎

Example We calculate the homology groups $H_*(K_S, L_S)$, where the simplicial complex K_S represents a square S, subdivided into eighteen triangles, and L_S is the subcomplex corresponding to the boundary of that square. We let $S = [0, 3] \times [0, 3]$, so that S is the square in the plane with corners at $(0, 0)$, $(3, 0)$, $(3, 3)$ and $(0, 3)$. The subdivision of this square into triangles is as depicted on the following diagram:



The vertices of this simplicial complex K_S are $\mathbf{v}_1, \ldots, \mathbf{v}_{16}$, where

$$\mathbf{v}_1 = (0,0), \quad \mathbf{v}_2 = (1,0), \quad \mathbf{v}_3 = (2,0), \quad \mathbf{v}_4 = (3,0),$$

$$\mathbf{v}_5 = (0,1), \quad \mathbf{v}_6 = (1,1), \quad \mathbf{v}_7 = (2,1), \quad \mathbf{v}_8 = (3,1),$$

$$\mathbf{v}_9 = (0,2), \quad \mathbf{v}_{10} = (1,2), \quad \mathbf{v}_{11} = (2,2), \quad \mathbf{v}_{12} = (3,2),$$

$$\mathbf{v}_{13} = (0,3), \quad \mathbf{v}_{14} = (1,3), \quad \mathbf{v}_{15} = (2,3), \quad \mathbf{v}_{16} = (3,3),$$

We label the exterior edges of the simplicial complex K_S as indicated on the diagram, so that

$$e_0^- = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle, \quad e_1^- = \langle \mathbf{v}_2, \mathbf{v}_3 \rangle, \quad e_2^- = \langle \mathbf{v}_3, \mathbf{v}_4 \rangle,$$

$$e_0^+ = \langle \mathbf{v}_{13}, \mathbf{v}_{14} \rangle, \quad e_1^+ = \langle \mathbf{v}_{14}, \mathbf{v}_{15} \rangle, \quad e_2^+ = \langle \mathbf{v}_{15}, \mathbf{v}_{16} \rangle,$$

$$f_0^- = \langle \mathbf{v}_1, \mathbf{v}_5 \rangle, \quad f_1^- = \langle \mathbf{v}_5, \mathbf{v}_9 \rangle, \quad f_2^- = \langle \mathbf{v}_9, \mathbf{v}_{13} \rangle,$$

$$f_0^+ = \langle \mathbf{v}_4, \mathbf{v}_8 \rangle, \quad f_1^+ = \langle \mathbf{v}_8, \mathbf{v}_{12} \rangle, \quad f_2^+ = \langle \mathbf{v}_{12}, \mathbf{v}_{16} \rangle,$$

We also the vertices, triangles and exterior edges of the simplicial complex K_S as indicated on the diagram. Thus We give each triangle of the simplicial complex K_S the orientation determined by an anticlockwise ordering of its vertices. Then the oriented triangles of K_S are represented by $t_1, \ldots, t_{18}$, where

$$t_1 = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_6 \rangle, \quad t_2 = \langle \mathbf{v}_1, \mathbf{v}_6, \mathbf{v}_5 \rangle, \quad t_3 = \langle \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_7 \rangle,$$

$$t_4 = \langle \mathbf{v}_2, \mathbf{v}_7, \mathbf{v}_6 \rangle, \quad t_5 = \langle \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_8 \rangle, \quad t_6 = \langle \mathbf{v}_3, \mathbf{v}_8, \mathbf{v}_7 \rangle,$$

$$t_7 = \langle \mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_{10} \rangle, \quad t_8 = \langle \mathbf{v}_5, \mathbf{v}_{10}, \mathbf{v}_9 \rangle, \quad t_9 = \langle \mathbf{v}_6, \mathbf{v}_7, \mathbf{v}_{11} \rangle,$$

$$t_{10} = \langle \mathbf{v}_6, \mathbf{v}_{11}, \mathbf{v}_{10} \rangle, \quad t_{11} = \langle \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_{12} \rangle, \quad t_{12} = \langle \mathbf{v}_7, \mathbf{v}_{12}, \mathbf{v}_{11} \rangle,$$

$$t_{13} = \langle \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{14} \rangle, \quad t_{14} = \langle \mathbf{v}_9, \mathbf{v}_{14}, \mathbf{v}_{13} \rangle, \quad t_{15} = \langle \mathbf{v}_{10}, \mathbf{v}_{11}, \mathbf{v}_{15} \rangle,$$

$$t_{16} = \langle \mathbf{v}_{10}, \mathbf{v}_{15}, \mathbf{v}_{14} \rangle, \quad t_{17} = \langle \mathbf{v}_{11}, \mathbf{v}_{12}, \mathbf{v}_{16} \rangle, \quad t_{18} = \langle \mathbf{v}_{11}, \mathbf{v}_{16}, \mathbf{v}_{15} \rangle.$$

Let M_0 be the simplicial complex consisting of the single vertex $\mathbf{v}_1$ and, for each integer k between 1 and 18, let M_k be the subcomplex of K_S consisting of the triangles T_j represented by t_j for $1 \leq j \leq k$, together with all the edges and vertices of those triangles. Then $K_S = M_{18}$. Now examination of the diagrams shows that, for each integer k between 1 and 18, the intersection $T_k \cap \bigcup_{j<k} T_j$ is either a single vertex of T_k, or a single edge of T_k, or the union of two edges of T_k. It follows from Lemma 10.5 that the inclusion map $i_k \colon M_{k-1} \hookrightarrow M_k$ induces isomorphisms $i_{k*} \colon H_*(M_{k-1}) \to H_*(M_k)$ of homology groups for all integers k satisfying $1 \leq k \leq 18$. Therefore $H_q(K_S) \cong H_q(M_0)$ for all integers q, and thus $H_q(K_S) = 0$ when $q > 0$, and $H_0(K_S) \cong \mathbb{Z}$. Moreover $H_0(K_S)$ is generated by the 0-dimensional homology class represented by the single vertex $\mathbf{v}_1$.

Let L_S denote the one-dimensional subcomplex of K_S consisting of all edges and vertices of K_S that are contained within the boundary of the square $[0,3] \times [0,3]$. Now $H_0(L_S) \cong \mathbb{Z}$, $H_1(L_S) \cong \mathbb{Z}$, and $H_q(L_S) = 0$ when $q > 1$. The group $H_0(L_S)$ is generated by the homology class representing the vertex $\mathbf{v}_1$, and therefore the homomorphism $i_* \colon H_0(L_S) \to H_0(K_S)$ induced by the inclusion map $i \colon L_S \hookrightarrow K_S$ is an isomorphism. The group $H_1(L_S)$ is generated by the 1-cycle z_S, where

$$z_S = e_0^- + e_1^- + e_2^- + f_0^+ + f_1^+ + f_2^+ - e_2^+ - e_1^+ - e_0^+ - f_2^- - f_1^- - f_0^-.$$

This generating 1-cycle z_S represents the sum of the edges of K_S that lie on the boundary of the square, where the orientation on each edge is consistent with an anticlockwise traversal of the boundary of the square S.

We can use the homology exact sequence of the simplicial pair (K_S, L_S) (Proposition 10.1) in order to evaluate the homology groups $H_*(K_S, L_S)$. The sequence

$$H_1(K_S) \longrightarrow H_1(K_S, L_S) \xrightarrow{\partial_*} H_0(L_S) \xrightarrow{i_{S*}} H_0(K_S)$$

is exact, where the homomorphism $i_{S*} \colon H_0(L_S) \to H_0(K_S)$ is induced by the inclusion map $i_S \colon L_S \hookrightarrow K_S$. We have noted that this homomorphism is an isomorphism. It follows from the exactness of the above sequence that $\partial_* \colon H_1(K_S, L_S) \to H_0(L_S)$ is the zero homomorphism. Therefore its kernel is the whole of $H_1(K_S, L_S)$, and therefore the homomorphism from $H_1(K_S)$ to

$H_1(K_S, L_S)$ is surjective. But we have shown that $H_q(K_S) = 0$ for $q > 0$. It follows that $H_1(K_S, L_S) = 0$.

The homology exact sequence of the pair also ensures that the homomorphism $\partial_*: H_2(K_S, L_S) \to H_1(L_S)$ is an isomophism, since $H_2(K_S) = 0$ and $H_2(L_S) = 0$ (see Corollary 10.3). But $H_1(L_S) \cong \mathbb{Z}$. It follows that $H_2(K_S, L_S) \cong \mathbb{Z}$.

In fact $H_2(K_S, L_S) \cong Z_2(K_S, L_S)$ since $B_2(K_S, L_S) = 0$. Moreover

$$Z_2(K_S, L_S) = \{ny_S + C_2(L_S) : n \in \mathbb{Z}\},$$

where $y_S = \sum_{j=1}^{18} t_j$. Indeed let c be a 2-chain of K_S. Then there are integers $n_1, \ldots, n_{18}$ such that $c = \sum_{j=1}^{18} n_j t_j$. Now any edge belonging to $K_S \setminus L_S$ lies on the boundary of exactly two triangles T_j and $T_{j'}$ of K_S. Moreover orientation on that edge determined by the anticlockwise ordering of the vertices of T_j is opposite to the orientation determined by the anticlockwise ordering of the vertices of $T_{j'}$, and therefore the coefficient of this edge in $\partial_2 c$ is $\pm(n_j - n_{j'})$. It follows that $\partial_2 c \in C_1(L)$ if and only if $n_1 = n_2 = \cdots = n_{18}$, in which case $c_2 = ny_S$ for some integer n. It is then easy to verify that $\partial_2(y_S) = z_S$.

Example We shall make use of the above results to calculate the homology groups of a torus. The two-dimensional torus may be represented as the quotient space obtained from the square $[0, 3] \times [0, 3]$ by identifying the points $(x, 0)$ and $(x, 3)$ for all $x \in [0, 3]$, and also identifying the points $(0, y)$ and $(3, y)$ for all $y \in [0, 3]$. Thus each point on an edge of the square is identified with a corresponding point on the opposite edge of the square. The four corners of the square are identified together, so as to represent a single point of the torus.

Now there exists a simplicial complex K_T, and a simplicial map $p: K_S \to K_T$ where K_S is the simplicial complex triangulating the square $[0, 3] \times [0, 3]$ discussed in the previous example, where the polyhedron $|K_T|$ of K_T is homeomorphic to the torus, and where the induced map $p: |K_S| \to |K_T|$ between polyhedra is an identification map which identifies points on opposite edges of the square S as described above. Moreover this simplicial complex K_T has 18 triangles, 27 edges and 9 vertices. Throughout this example we shall use the notation developed in the previous example to describe the simplical complex K_S and its chain groups and homology groups.

Let the vertices of K_T be labelled as $\mathbf{w}_1, \ldots, \mathbf{w}_9$, where

$$\mathbf{w}_1 \;=\; p(\mathbf{v}_1) = p(\mathbf{v}_4) = p(\mathbf{v}_{13}) = p(\mathbf{v}_{16}),$$

$$
\begin{aligned}
\mathbf{w}_2 &= p(\mathbf{v}_2) = p(\mathbf{v}_{14}),\\
\mathbf{w}_3 &= p(\mathbf{v}_3) = p(\mathbf{v}_{15}),\\
\mathbf{w}_4 &= p(\mathbf{v}_5) = p(\mathbf{v}_8),\\
\mathbf{w}_5 &= p(\mathbf{v}_9) = p(\mathbf{v}_{12}),\\
\mathbf{w}_6 &= p(\mathbf{v}_6),\\
\mathbf{w}_7 &= p(\mathbf{v}_7),\\
\mathbf{w}_8 &= p(\mathbf{v}_{10}),\\
\mathbf{w}_9 &= p(\mathbf{v}_{11}).
\end{aligned}
$$

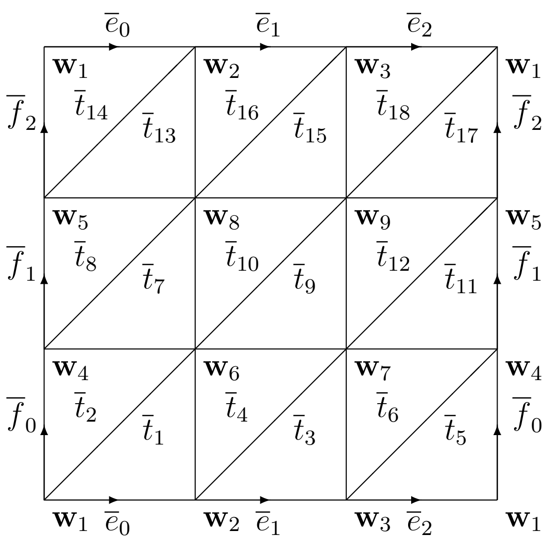
and let

$$
\begin{gathered}
\overline{e}_0 = \langle \mathbf{w}_1, \mathbf{w}_2\rangle, \quad \overline{e}_1 = \langle \mathbf{w}_2, \mathbf{w}_3\rangle, \quad \overline{e}_2 = \langle \mathbf{w}_3, \mathbf{w}_1\rangle,\\
\overline{f}_0 = \langle \mathbf{w}_1, \mathbf{w}_4\rangle, \quad \overline{f}_1 = \langle \mathbf{w}_4, \mathbf{w}_5\rangle, \quad \overline{f}_2 = \langle \mathbf{w}_5, \mathbf{w}_1\rangle,
\end{gathered}
$$

Then

$$
\begin{gathered}
p_\#(e_0^+) = p_\#(e_0^-) = \overline{e}_0, \quad p_\#(e_1^+) = p_\#(e_1^-) = \overline{e}_1,\\
p_\#(e_2^+) = p_\#(e_2^-) = \overline{e}_2, \quad p_\#(f_0^+) = p_\#(f_0^-) = \overline{f}_0,\\
p_\#(f_1^+) = p_\#(f_1^-) = \overline{f}_1, \quad p_\#(f_2^+) = p_\#(f_2^-) = \overline{f}_2,
\end{gathered}
$$

where $p_\#: C_1(K_S) \to C_1(K_T)$ is the homomorphism of chain groups induced by $p: K_S \to K_T$. Also let $\overline{t}_j = p_\#(t_j)$ for $j = 1, 2, \ldots, 18$, where $p_\#: C_2(K_S) \to C_2(K_T)$ is the homomorphism of chain groups induced by the simplicial map $p: K_S \to K_T$. Then the triangulation K_T of the torus may be represented by the following diagram:



Let $L_T = p(L_S)$. Then L_T is the subcomplex of K_T consisting of the five vertices $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, $\mathbf{w}_4$ and $\mathbf{w}_5$, together with the six edges represented by $\overline{e}_0$, $\overline{e}_1$, $\overline{e}_3$, $\overline{f}_0$, $\overline{f}_1$ and $\overline{f}_3$. Now $H_1(L_T) \cong Z_1(L_T)$, since $B_1(L_T) = 0$. Moreover $Z_1(L_T) \cong \mathbb{Z} \oplus \mathbb{Z}$. Indeed

$$Z_1(L_T) = \{n_1 z_1 + n_2 z_2 : n_1, n_2 \in \mathbb{Z}\},$$

where $z_1 = \overline{e}_0 + \overline{e}_1 + \overline{e}_2$ and $z_2 = \overline{f}_0 + \overline{f}_1 + \overline{f}_2$. The simplicial map $p\colon K_S \to K_T$ defines a bijection between the simplices of $K_S \setminus L_S$ and those of $K_T \setminus L_T$. It follows from this that the chain map $p_*\colon C_*(K_S, L_S) \to C_*(K_T, L_T)$ induced by the simplicial map is an isomorphism of chain complexes, and therefore induces isomorphisms

$$p_*\colon H_*(K_S, L_S) \to H_*(K_T, L_T).$$

We conclude that $H_2(K_T, L_T) \cong \mathbb{Z}$, and $H_q(K_T, L_T) = 0$ when $q \neq 2$. Moreover $H_2(K_T, L_T)$ is generated by the homology class of z_T, where

$$y_T = \sum_{j=1}^{18} \overline{t}_j = p_\#(y_S).$$

Also $H_2(L_T) = 0$, because the simplicial complex L_T is one-dimensional.

We now determine the homomorphism $\partial_*\colon H_2(K_T, L_T) \to H_1(L_T)$. Now the following diagram relating homology groups of the square and the torus is commutative:

$$\begin{array}{ccc} H_2(K_S, L_S) & \xrightarrow{\partial_*} & H_1(L_S) \\ \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} \\ H_2(K_T, L_T) & \xrightarrow{\partial_*} & H_1(L_T) \end{array}$$

Moreover

$$H_1(L_S) \cong H_2(K_S, L_S) \cong H_2(K_T, L_T) \cong \mathbb{Z},$$

and the homomorphisms $\partial_*\colon H_2(K_S, L_S) \to H_1(L_S)$ and $p_*\colon H_2(K_S, L_S) \to H_2(K_T, L_T)$ are isomorphisms. Let μ_{K_S, L_S} and μ_{K_T, L_T} be the homology classes in $H_2(K_S, L_S)$ and $H_2(K_T, L_T)$ respectively represented by y_S and y_T. Then $p_*(\mu_{K_S, L_S}) = \mu_{K_T, L_T}$. and $\partial_*(\mu_{K_S, L_S}) = [z_S]$, where

$$z_S = e_0^- + e_1^- + e_2^- + f_0^+ + f_1^+ + f_2^+ - e_2^+ - e_1^+ - e_0^+ - f_2^- - f_1^- - f_0^-.$$

It follows that $\partial_*(\mu_{K_T, L_T}) = p_*(z_S)$.

We now calculate the image of z_S under the homomorphism

$$p_\#\colon C_1(L_S) \to C_1(L_T)$$

induced by the simplicial map $p\colon K_S \to K_T$. We find that

$$\begin{aligned} p_{\#}(z_S) &= p_{\#}(e_0^-) + p_{\#}(e_1^-) + p_{\#}(e_2^-) \\ &\quad + p_{\#}(f_0^+) + p_{\#}(f_1^+) + p_{\#}(f_2^+) \\ &\quad - p_{\#}(e_2^+) - p_{\#}(e_1^+) - p_{\#}(e_0^+) \\ &\quad - p_{\#}(f_2^-) - p_{\#}(f_1^-) - p_{\#}(f_0^-) \\ &= \overline{e}_0 + \overline{e}_1 + \overline{e}_2 \\ &\quad + \overline{f}_0 + \overline{f}_1 + \overline{f}_2 \\ &\quad - \overline{e}_2 - \overline{e}_1 - \overline{e}_0 \\ &\quad - \overline{f}_2 - \overline{f}_1 - \overline{f}_0 \\ &= z_1 + z_2 - z_1 - z_2 = 0. \end{aligned}$$

Therefore $\partial_*(\mu_{K_T,L_T}) = p_*(z_S) = 0$. We conclude from this that

$$\partial_*\colon H_2(K_T, L_T) \to H_1(L_T)$$

is the zero homorphism.

We now have the information required in order to calculate the homology groups of the simplicial complex K_T. The homology exact sequence of the simplicial pair (K_T, L_T) gives rise to the following exact sequence:

$$0 \longrightarrow H_2(K_T) \longrightarrow H_2(K_T, L_T) \xrightarrow{\partial_*} H_1(L_T) \xrightarrow{i_{T*}} H_1(K_T) \longrightarrow 0.$$

Using the exactness of this sequence, together with the result that $\partial_* = 0$, we conclude that $H_2(K_T) \cong H_2(K_T, L_T)$ and $H_1(K_T) \cong H_1(L_T) \cong \mathbb{Z} \oplus \mathbb{Z}$. Indeed

$$H_2(K_T) = \{n[y_T] : n \in \mathbb{Z}\}$$

and

$$H_1(K_T) = \{n_1[z_1] + n_2[z_2] : n_1, n_2 \in \mathbb{Z}\},$$

where y_T, z_1 and z_2 are the 1-cycles of L_T defined above. Thus

$$H_0(K_T) \cong \mathbb{Z}, \quad H_1(K_T) \cong \mathbb{Z} \oplus \mathbb{Z}, \quad H_2(K_T) \cong \mathbb{Z}.$$

Example We shall make use of the above results to calculate the homology groups of a Klein Bottle. The Klein Bottle may be represented as the quotient space obtained from the square $[0,3] \times [0,3]$ by identifying the points $(x, 0)$ and $(x, 3)$ for all $x \in [0, 3]$, and also identifying the points $(0, y)$ and $(3, 3-y)$ for all $y \in [0, 3]$. Thus each point on an edge of the square is identified with some other point on the opposite edge of the square. The four corners of the square are identified together, so as to represent a single point of the Klein Bottle.

Now there exists a simplicial complex K_{KlB}, and a simplicial map $r: K_S \to K_{KlB}$ where K_S is the simplicial complex triangulating the square $[0,3]\times[0,3]$ discussed in the previous example, where the polyhedron $|K_{KlB}|$ of K_{KlB} is homeomorphic to the Klein Bottle, and where the induced map $r: |K_S| \to |K_{KlB}|$ between polyhedra is an identification map which identifies points on opposite edges of the square S as described above. Moreover this simplicial complex K_{KlB} has 18 triangles, 27 edges and 9 vertices. Throughout this example we shall use the notation developed in a previous example to describe the simplical complex K_S and its chain groups and homology groups.

Let the vertices of K_{KlB} be labelled as $\mathbf{u}_1, \ldots, \mathbf{u}_9$, where

$$
\begin{aligned}
\mathbf{u}_1 &= r(\mathbf{v}_1) = r(\mathbf{v}_4) = r(\mathbf{v}_{13}) = r(\mathbf{v}_{16}),\\
\mathbf{u}_2 &= r(\mathbf{v}_2) = r(\mathbf{v}_{14}),\\
\mathbf{u}_3 &= r(\mathbf{v}_3) = r(\mathbf{v}_{15}),\\
\mathbf{u}_4 &= r(\mathbf{v}_9) = r(\mathbf{v}_8),\\
\mathbf{u}_5 &= r(\mathbf{v}_5) = r(\mathbf{v}_{12}),\\
\mathbf{u}_6 &= r(\mathbf{v}_6),\\
\mathbf{u}_7 &= r(\mathbf{v}_7),\\
\mathbf{u}_8 &= r(\mathbf{v}_{10}),\\
\mathbf{u}_9 &= r(\mathbf{v}_{11}).
\end{aligned}
$$

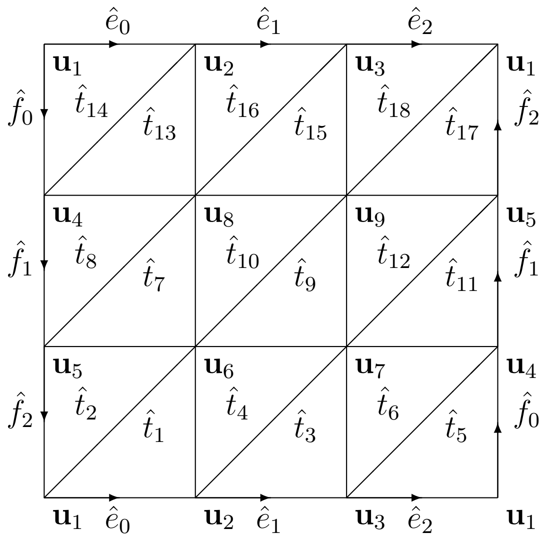
and let

$$
\hat{e}_0 = \langle \mathbf{u}_1, \mathbf{u}_2 \rangle, \quad \hat{e}_1 = \langle \mathbf{u}_2, \mathbf{u}_3 \rangle, \quad \hat{e}_2 = \langle \mathbf{u}_3, \mathbf{u}_1 \rangle,
$$
$$
\hat{f}_0 = \langle \mathbf{u}_1, \mathbf{u}_4 \rangle, \quad \hat{f}_1 = \langle \mathbf{u}_4, \mathbf{u}_5 \rangle, \quad \hat{f}_2 = \langle \mathbf{u}_5, \mathbf{u}_1 \rangle,
$$

Then

$$
r_\#(e_0^+) = r_\#(e_0^-) = \hat{e}_0, \quad r_\#(e_1^+) = r_\#(e_1^-) = \hat{e}_1,
$$
$$
r_\#(e_2^+) = r_\#(e_2^-) = \hat{e}_2, \quad r_\#(f_0^+) = -r_\#(f_2^-) = \hat{f}_0,
$$
$$
r_\#(f_1^+) = -r_\#(f_1^-) = \hat{f}_1, \quad r_\#(f_2^+) = -r_\#(f_0^-) = \hat{f}_2,
$$

where $r_\#: C_1(K_S) \to C_1(K_{KlB})$ is the homomorphism of chain groups induced by $r: K_S \to K_{KlB}$. Also let $\hat{t}_j = r_\#(t_j)$ for $j = 1, 2, \ldots, 18$, where $r_\#: C_2(K_S) \to C_2(K_{KlB})$ is the homomorphism of chain groups induced by the simplicial map $r: K_S \to K_{KlB}$. Then the triangulation K_{KlB} of the Klein Bottle may be represented by the following diagram:



Let $L_{KlB} = r(L_S)$. Then L_{KlB} is the subcomplex of K_{KlB} consisting of the five vertices $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\mathbf{u}_4$ and $\mathbf{u}_5$, together with the six edges represented by $\hat{e}_0$, $\hat{e}_1$, $\hat{e}_3$, $\hat{f}_0$, $\hat{f}_1$ and $\hat{f}_3$. Now $H_1(L_{KlB}) \cong Z_1(L_{KlB})$, since $B_1(L_{KlB}) = 0$. Moreover $Z_1(L_{KlB}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Indeed

$$Z_1(L_{KlB}) = \{n_1 z_1 + n_2 z_2 : n_1, n_2 \in \mathbb{Z}\},$$

where $z_1 = \hat{e}_0 + \hat{e}_1 + \hat{e}_2$ and $z_2 = \hat{f}_0 + \hat{f}_1 + \hat{f}_2$. The simplicial map $r\colon K_S \to K_{KlB}$ defines a bijection between the simplices of $K_S \setminus L_S$ and those of $K_{KlB} \setminus L_{KlB}$. It follows from this that the chain map $r_*\colon C_*(K_S, L_S) \to C_*(K_{KlB}, L_{KlB})$ induced by the simplicial map is an isomorphism of chain complexes, and therefore induces isomorphisms

$$r_*\colon H_*(K_S, L_S) \to H_*(K_{KlB}, L_{KlB}).$$

We conclude that $H_2(K_{KlB}, L_{KlB}) \cong \mathbb{Z}$, and $H_q(K_{KlB}, L_{KlB}) = 0$ when $q \neq 2$. Moreover $H_2(K_{KlB}, L_{KlB})$ is generated by the homology class of z_{KlB}, where

$$y_{KlB} = \sum_{j=1}^{18} \hat{t}_j = r_\#(y_S).$$

Also $H_2(L_{KlB}) = 0$, because the simplicial complex L_{KlB} is one-dimensional.

We now determine the homomorphism $\partial_*\colon H_2(K_{KlB}, L_{KlB}) \to H_1(L_{KlB})$. Now the following diagram relating homology groups of the square and the

Klein Bottle is commutative:

$$\begin{array}{ccc} H_2(K_S, L_S) & \xrightarrow{\partial_*} & H_1(L_S) \\ \Big\downarrow{\scriptstyle r_*} & & \Big\downarrow{\scriptstyle r_*} \\ H_2(K_{KlB}, L_{KlB}) & \xrightarrow{\partial_*} & H_1(L_{KlB}) \end{array}$$

Moreover

$$H_1(L_S) \cong H_2(K_S, L_S) \cong H_2(K_{KlB}, L_{KlB}) \cong \mathbb{Z},$$

and the homomorphisms $\partial_*\colon H_2(K_S, L_S) \to H_1(L_S)$ and $r_*\colon H_2(K_S, L_S) \to H_2(K_{KlB}, L_{KlB})$ are isomorphisms. Let μ_{K_S,L_S} and $\mu_{K_{KlB},L_{KlB}}$ be the homology classes in $H_2(K_S, L_S)$ and $H_2(K_{KlB}, L_{KlB})$ respectively represented by y_S and y_{KlB}. Then $r_*(\mu_{K_S,L_S}) = \mu_{K_{KlB},L_{KlB}}$. and $\partial_*(\mu_{K_S,L_S}) = [z_S]$, where

$$z_S = e_0^- + e_1^- + e_2^- + f_0^+ + f_1^+ + f_2^+ - e_2^+ - e_1^+ - e_0^+ - f_2^- - f_1^- - f_0^-.$$

It follows that $\partial_*(\mu_{K_{KlB},L_{KlB}}) = r_*(z_S)$.

We now calculate the image of z_S under the homomorphism

$$r_\#\colon C_1(L_S) \to C_1(L_{KlB})$$

induced by the simplicial map $r\colon K_S \to K_{KlB}$. We find that

$$\begin{aligned} r_\#(z_S) &= r_\#(e_0^-) + r_\#(e_1^-) + r_\#(e_2^-) \\ &\quad + r_\#(f_0^+) + r_\#(f_1^+) + r_\#(f_2^+) \\ &\quad - r_\#(e_2^+) - r_\#(e_1^+) - r_\#(e_0^+) \\ &\quad - r_\#(f_2^-) - r_\#(f_1^-) - r_\#(f_0^-) \\ &= \hat{e}_0 + \hat{e}_1 + \hat{e}_2 \\ &\quad + \hat{f}_0 + \hat{f}_1 + \hat{f}_2 \\ &\quad - \hat{e}_2 - \hat{e}_1 - \hat{e}_0 \\ &\quad + \hat{f}_0 + \hat{f}_1 + \hat{f}_2 \\ &= z_1 + z_2 - z_1 + z_2 = 2z_2. \end{aligned}$$

Therefore $\partial_*(\mu_{K_{KlB},L_{KlB}}) = r_*(z_S) = 2z_2$. We conclude from this that

$$\partial_*\colon H_2(K_{KlB}, L_{KlB}) \to H_1(L_{KlB})$$

is an injective homomorphism whose image is the subgroup of $H_1(KlB)$ generated by $2[z_2]$.

We now have the information required in order to calculate the homology groups of the simplicial complex K_{KlB}. The homology exact sequence of the simplicial pair (K_{KlB}, L_{KlB}) gives rise to the following exact sequence:

$$0 \longrightarrow H_2(K_{KlB}) \longrightarrow H_2(K_{KlB}, L_{KlB}) \xrightarrow{\partial_*} H_1(L_{KlB}) \xrightarrow{i_{T*}} H_1(K_{KlB}) \longrightarrow 0.$$

Using the exactness of this sequence, together with the result that ∂_* is injective, we conclude that $H_2(K_{KlB}) = 0$. Also

$$H_1(K_{KlB}) \cong H_1(L_{KlB})/\partial_*(H_2(K_{KlB}, L_{KlB})) \cong \mathbb{Z} \oplus \mathbb{Z}_2.$$

Indeed there is an isomorphism $\varphi\colon H_1(L_{KlB}) \to \mathbb{Z} \oplus \mathbb{Z}$ which maps the homology classes of the cycles z_1 and z_2 to $(1,0)$ and $(0,1)$ respectively. Then $\varphi(\partial_*(H_2(K_{KlB}, L_{KlB})))$ is the subgroup of $\mathbb{Z} \oplus \mathbb{Z}$ generated by $(0,2)$, and the corresponding quotient group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$. Thus

$$H_0(K_{KlB}) \cong \mathbb{Z}, \quad H_1(K_{KlB}) \cong \mathbb{Z} \oplus \mathbb{Z}_2, \quad H_2(K_{KlB}) \cong 0.$$

Example We shall make use of the above results to calculate the homology groups of a real projective plane. The real projective plane may be represented as the quotient space obtained from the square $[0,3] \times [0,3]$ by identifying the points $(x,0)$ and $(3-x,3)$ for all $x \in [0,3]$, and also identifying the points $(0,y)$ and $(3,3-y)$ for all $y \in [0,3]$. Thus each point on an edge of the square is identified with some other point on the opposite edge of the square. Also each corner of the square is identified with the corner diagonally opposite, so as to represent a single point of the real projective plane.

Now there exists a simplicial complex K_{PP}, and a simplicial map $s\colon K_S \to K_{PP}$ where K_S is the simplicial complex triangulating the square $[0,3] \times [0,3]$ discussed in the previous example, where the polyhedron $|K_{PP}|$ of K_{PP} is homeomorphic to the real projective plane, and where the induced map $s\colon |K_S| \to |K_{PP}|$ between polyhedra is an identification map which identifies points on opposite edges of the square S as described above. Moreover this simplicial complex K_{PP} has 18 triangles, 27 edges and 10 vertices. Throughout this example we shall use the notation developed in a previous example to describe the simplical complex K_S and its chain groups and homology groups.

Let the vertices of K_{PP} be labelled as $\mathbf{q}_1, \ldots, \mathbf{q}_{10}$, where

$$\begin{aligned}
\mathbf{q}_1 &= s(\mathbf{v}_1) = s(\mathbf{v}_{16}), \\
\mathbf{q}_2 &= s(\mathbf{v}_2) = s(\mathbf{v}_{15}), \\
\mathbf{q}_3 &= s(\mathbf{v}_3) = s(\mathbf{v}_{14}), \\
\mathbf{q}_4 &= s(\mathbf{v}_4) = s(\mathbf{v}_{13}), \\
\mathbf{q}_5 &= s(\mathbf{v}_9) = s(\mathbf{v}_8), \\
\mathbf{q}_6 &= s(\mathbf{v}_5) = s(\mathbf{v}_{12}), \\
\mathbf{q}_7 &= s(\mathbf{v}_6), \\
\mathbf{q}_8 &= s(\mathbf{v}_7),
\end{aligned}$$

$$
\begin{aligned}
\mathbf{q}_9 &= s(\mathbf{v}_{10}), \\
\mathbf{q}_{10} &= s(\mathbf{v}_{11}).
\end{aligned}
$$

and let

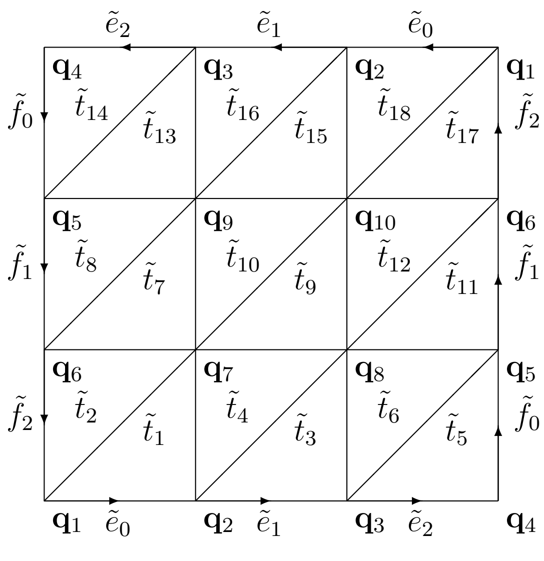
$$
\tilde{e}_0 = \langle \mathbf{q}_1, \mathbf{q}_2 \rangle, \quad \tilde{e}_1 = \langle \mathbf{q}_2, \mathbf{q}_3 \rangle, \quad \tilde{e}_2 = \langle \mathbf{q}_3, \mathbf{q}_4 \rangle,
$$

$$
\tilde{f}_0 = \langle \mathbf{q}_4, \mathbf{q}_5 \rangle, \quad \tilde{f}_1 = \langle \mathbf{q}_5, \mathbf{q}_6 \rangle, \quad \tilde{f}_2 = \langle \mathbf{q}_6, \mathbf{q}_1 \rangle,
$$

Then

$$
\begin{aligned}
-s_\#(e_2^+) = s_\#(e_0^-) = \tilde{e}_0, &\quad -s_\#(e_1^+) = s_\#(e_1^-) = \tilde{e}_1, \\
-s_\#(e_0^+) = s_\#(e_2^-) = \tilde{e}_2, &\quad s_\#(f_0^+) = -s_\#(f_2^-) = \tilde{f}_0, \\
s_\#(f_1^+) = -s_\#(f_1^-) = \tilde{f}_1, &\quad s_\#(f_2^+) = -s_\#(f_0^-) = \tilde{f}_2,
\end{aligned}
$$

where $s_\#: C_1(K_S) \to C_1(K_{PP})$ is the homomorphism of chain groups induced by $s: K_S \to K_{PP}$. Also let $\tilde{t}_j = s_\#(t_j)$ for $j = 1, 2, \ldots, 18$, where $s_\#: C_2(K_S) \to C_2(K_{PP})$ is the homomorphism of chain groups induced by the simplicial map $s: K_S \to K_{PP}$. Then the triangulation K_{PP} of the real projective plane may be represented by the following diagram:



Let $L_{PP} = s(L_S)$. Then L_{PP} is the subcomplex of K_{PP} consisting of the six vertices $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, $\mathbf{q}_4$, $\mathbf{q}_5$ and $\mathbf{q}_6$, together with the six edges represented by $\tilde{e}_0$, $\tilde{e}_1$, $\tilde{e}_3$, $\tilde{f}_0$, $\tilde{f}_1$ and $\tilde{f}_3$. Now $H_1(L_{PP}) \cong Z_1(L_{PP})$, since $B_1(L_{PP}) = 0$. Moreover $Z_1(L_{PP}) \cong \mathbb{Z}$. Indeed

$$
Z_1(L_{PP}) = \{nz_0 : n \in \mathbb{Z}\},
$$

where

$$z_0 = \tilde{e}_0 + \tilde{e}_1 + \tilde{e}_2 + \tilde{f}_0 + \tilde{f}_1 + \tilde{f}_2.$$

The simplicial map $s\colon K_S \to K_{PP}$ defines a bijection between the simplices of $K_S \setminus L_S$ and those of $K_{PP} \setminus L_{PP}$. It follows from this that the chain map $s_*\colon C_*(K_S, L_S) \to C_*(K_{PP}, L_{PP})$ induced by the simplicial map is an isomorphism of chain complexes, and therefore induces isomorphisms

$$s_*\colon H_*(K_S, L_S) \to H_*(K_{PP}, L_{PP}).$$

We conclude that $H_2(K_{PP}, L_{PP}) \cong \mathbb{Z}$, and $H_q(K_{PP}, L_{PP}) = 0$ when $q \neq 2$. Moreover $H_2(K_{PP}, L_{PP})$ is generated by the homology class of z_{PP}, where

$$y_{PP} = \sum_{j=1}^{18} \tilde{t}_j = s_\#(y_S).$$

Also $H_2(L_{PP}) = 0$, because the simplicial complex L_{PP} is one-dimensional.

We now determine the homomorphism $\partial_*\colon H_2(K_{PP}, L_{PP}) \to H_1(L_{PP})$. Now the following diagram relating homology groups of the square and the real projective plane is commutative:

$$\begin{array}{ccc} H_2(K_S, L_S) & \xrightarrow{\partial_*} & H_1(L_S) \\ \downarrow{\scriptstyle s_*} & & \downarrow{\scriptstyle s_*} \\ H_2(K_{PP}, L_{PP}) & \xrightarrow{\partial_*} & H_1(L_{PP}) \end{array}$$

Moreover

$$H_1(L_S) \cong H_2(K_S, L_S) \cong H_2(K_{PP}, L_{PP}) \cong \mathbb{Z},$$

and the homomorphisms $\partial_*\colon H_2(K_S, L_S) \to H_1(L_S)$ and $s_*\colon H_2(K_S, L_S) \to H_2(K_{PP}, L_{PP})$ are isomorphisms. Let μ_{K_S,L_S} and $\mu_{K_{PP},L_{PP}}$ be the homology classes in $H_2(K_S, L_S)$ and $H_2(K_{PP}, L_{PP})$ respectively represented by y_S and y_{PP}. Then $s_*(\mu_{K_S,L_S}) = \mu_{K_{PP},L_{PP}}$. and $\partial_*(\mu_{K_S,L_S}) = [z_S]$, where

$$z_S = e_0^- + e_1^- + e_2^- + f_0^+ + f_1^+ + f_2^+ - e_2^+ - e_1^+ - e_0^+ - f_2^- - f_1^- - f_0^-.$$

It follows that $\partial_*(\mu_{K_{PP},L_{PP}}) = s_*(z_S)$.

We now calculate the image of z_S under the homomorphism

$$s_\#\colon C_1(L_S) \to C_1(L_{PP})$$

induced by the simplicial map $s\colon K_S \to K_{PP}$. We find that

$$s_\#(z_S) \quad = \quad s_\#(e_0^-) + s_\#(e_1^-) + s_\#(e_2^-)$$

$$\begin{aligned}
& + s_\#(f_0^+) + s_\#(f_1^+) + s_\#(f_2^+) \\
& - s_\#(e_2^+) - s_\#(e_1^+) - s_\#(e_0^+) \\
& - s_\#(f_2^-) - s_\#(f_1^-) - s_\#(f_0^-) \\
= \; & \tilde{e}_0 + \tilde{e}_1 + \tilde{e}_2 \\
& + \tilde{f}_0 + \tilde{f}_1 + \tilde{f}_2 \\
& + \tilde{e}_0 + \tilde{e}_1 + \tilde{e}_2 \\
& + \tilde{f}_0 + \tilde{f}_1 + \tilde{f}_2 \\
= \; & 2z_0.
\end{aligned}$$

Therefore $\partial_*(\mu_{K_{PP},L_{PP}}) = s_*(z_S) = 2z_0$. We conclude from this that

$$\partial_*: H_2(K_{PP}, L_{PP}) \to H_1(L_{PP})$$

is an injective homomorphism whose image is the subgroup of $H_1(PP)$ generated by $2[z_0]$.

We now have the information required in order to calculate the homology groups of the simplicial complex K_{PP}. The homology exact sequence of the simplicial pair (K_{PP}, L_{PP}) gives rise to the following exact sequence:

$$0 \longrightarrow H_2(K_{PP}) \longrightarrow H_2(K_{PP}, L_{PP}) \xrightarrow{\partial_*} H_1(L_{PP}) \xrightarrow{i_{T*}} H_1(K_{PP}) \longrightarrow 0.$$

Using the exactness of this sequence, together with the result that ∂_* is injective, we conclude that $H_2(K_{PP}) = 0$. Also

$$H_1(K_{PP}) \cong H_1(L_{PP})/\partial_*(H_2(K_{PP}, L_{PP})) \cong \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2.$$

Thus

$$H_0(K_{PP}) \cong \mathbb{Z}, \quad H_1(K_{PP}) \cong \mathbb{Z}_2, \quad H_2(K_{PP}) \cong 0.$$

10.3 The Mayer-Vietoris Sequence

Let K be a simplicial complex and let L and M be subcomplexes of K such that $K = L \cup M$. Let

$$i_q: C_q(L \cap M) \to C_q(L), \qquad j_q: C_q(L \cap M) \to C_q(M),$$

$$u_q: C_q(L) \to C_q(K), \qquad v_q: C_q(M) \to C_q(K)$$

be the inclusion homomorphisms induced by the inclusion maps $i: L \cap M \hookrightarrow L$, $j: L \cap M \hookrightarrow M$, $u: L \hookrightarrow K$ and $v: M \hookrightarrow K$. Then

$$0 \longrightarrow C_*(L \cap M) \xrightarrow{k_*} C_*(L) \oplus C_*(M) \xrightarrow{w_*} C_*(K) \longrightarrow 0$$

is a short exact sequence of chain complexes, where

$$\begin{aligned} k_q(c) &= (i_q(c), -j_q(c)), \\ w_q(c', c'') &= u_q(c') + v_q(c''), \\ \partial_q(c', c'') &= (\partial_q(c'), \partial_q(c'')) \end{aligned}$$

for all $c \in C_q(L \cap M)$, $c' \in C_q(L)$ and $c'' \in C_q(M)$. It follows from Lemma 9.4 that there is a well-defined homomorphism $\alpha_q\colon H_q(K) \to H_{q-1}(L \cap M)$ such that $\alpha_q([z]) = [\partial_q(c')] = -[\partial_q(c'')]$ for any $z \in Z_q(K)$, where c' and c'' are any q-chains of L and M respectively satisfying $z = c' + c''$. (Note that $\partial_q(c') \in Z_{q-1}(L \cap M)$ since $\partial_q(c') \in Z_{q-1}(L)$, $\partial_q(c'') \in Z_{q-1}(M)$ and $\partial_q(c') = -\partial_q(c'')$.) It now follows immediately from Proposition 9.5 that the infinite sequence

$$\cdots \xrightarrow{\alpha_{q+1}} H_q(L \cap M) \xrightarrow{k_*} H_q(L) \oplus H_q(M) \xrightarrow{w_*} H_q(K) \xrightarrow{\alpha_q} H_{q-1}(L \cap M) \xrightarrow{k_*} \cdots,$$

of homology groups is exact. This long exact sequence of homology groups is referred to as the *Mayer-Vietoris sequence* associated with the decomposition of K as the union of the subcomplexes L and M.