Course 421: Hilary Term 2003
Homology Theory

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Contents

5 Simplicial Homology Groups

5.1 The Chain Groups of a Simplicial Complex

Let K be a simplicial complex. For each non-negative integer q, let $\Delta_q(K)$ be the additive group consisting of all formal sums of the form

$$n_1(\mathbf{v}_0^1, \mathbf{v}_1^1, \ldots, \mathbf{v}_q^1) + n_2(\mathbf{v}_0^2, \mathbf{v}_1^2, \ldots, \mathbf{v}_q^2) + \cdots + n_s(\mathbf{v}_0^s, \mathbf{v}_1^s, \ldots, \mathbf{v}_q^s),$$

where $n_1, n_2, \ldots, n_s$ are integers and $\mathbf{v}_0^r, \mathbf{v}_1^r, \ldots, \mathbf{v}_q^r$ are (not necessarily distinct) vertices of K that span a simplex of K for $r = 1, 2, \ldots, s$. (In more formal language, the group $\Delta_q(K)$ is the *free Abelian group* generated by the set of all $(q+1)$-tuples of the form $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$, where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K.)

We recall some basic facts concerning *permutations*. A *permutation* of a set S is a bijection mapping S onto itself. The set of all permutations of some set S is a group; the group multiplication corresponds to composition of permutations. A *transposition* is a permutation of a set S which interchanges two elements of S, leaving the remaining elements of the set fixed. If S is finite and has more than one element then any permutation of S can be expressed as a product of transpositions. In particular any permutation of the set $\{0, 1, \ldots, q\}$ can be expressed as a product of transpositions $(j-1, j)$ that interchange $j-1$ and j for some j.

Associated to any permutation π of a finite set S is a number ϵ_π, known as the *parity* or *signature* of the permutation, which can take on the values ± 1. If π can be expressed as the product of an even number of transpositions, then $\epsilon_\pi = +1$; if π can be expressed as the product of an odd number of transpositions then $\epsilon_\pi = -1$. The function $\pi \mapsto \epsilon_\pi$ is a homomorphism from the group of permutations of a finite set S to the multiplicative group $\{+1, -1\}$ (i.e., $\epsilon_{\pi\rho} = \epsilon_\pi \epsilon_\rho$ for all permutations π and ρ of the set S). Note in particular that the parity of any transposition is -1.

Definition The qth *chain group* $C_q(K)$ of the simplicial complex K is defined to be the quotient group $\Delta_q(K)/\Delta_q^0(K)$, where $\Delta_q^0(K)$ is the subgroup of $\Delta_q(K)$ generated by elements of the form $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are not all distinct, and by elements of the form

$$(\mathbf{v}_{\pi(0)}, \mathbf{v}_{\pi(1)}, \ldots, \mathbf{v}_{\pi(q)}) - \epsilon_\pi(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$$

where π is some permutation of $\{0, 1, \ldots, q\}$ with parity ϵ_π. For convenience, we define $C_q(K) = \{0\}$ when $q < 0$ or $q > \dim K$, where $\dim K$ is the dimension of the simplicial complex K. An element of the chain group $C_q(K)$ is referred to as q-*chain* of the simplicial complex K.

We denote by $\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$ the element $\Delta_q^0(K) + (\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ of $C_q(K)$ corresponding to $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$. The following results follow immediately from the definition of $C_q(K)$.

Lemma 5.1 *Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be vertices of a simplicial complex K that span a simplex of K. Then*

- $\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle = 0$ *if $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are not all distinct,*
- $\langle \mathbf{v}_{\pi(0)}, \mathbf{v}_{\pi(1)}, \ldots, \mathbf{v}_{\pi(q)} \rangle = \epsilon_\pi \langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$ *for any permutation π of the set $\{0, 1, \ldots, q\}$.*

Example If $\mathbf{v}_0$ and $\mathbf{v}_1$ are the endpoints of some line segment then

$$\langle \mathbf{v}_0, \mathbf{v}_1 \rangle = -\langle \mathbf{v}_1, \mathbf{v}_0 \rangle.$$

If $\mathbf{v_0}$, $\mathbf{v}_1$ and $\mathbf{v}_2$ are the vertices of a triangle in some Euclidean space then

$$\begin{aligned} \langle \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2 \rangle &= \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_0 \rangle = \langle \mathbf{v}_2, \mathbf{v}_0, \mathbf{v}_1 \rangle = -\langle \mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_0 \rangle \\ &= -\langle \mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_1 \rangle = -\langle \mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_2 \rangle. \end{aligned}$$

Definition An *oriented q-simplex* is an element of the chain group $C_q(K)$ of the form $\pm\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$, where $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are distinct and span a simplex of K.

An oriented simplex of K can be thought of as consisting of a simplex of K (namely the simplex spanned by the prescribed vertices), together with one of two possible 'orientations' on that simplex. Any ordering of the vertices determines an orientation of the simplex; any even permutation of the ordering of the vertices preserves the orientation on the simplex, whereas any odd permutation of this ordering reverses orientation.

Any q-chain of a simplicial complex K can be expressed as a sum of the form

$$n_1\sigma_1 + n_2\sigma_2 + \cdots + n_s\sigma_s$$

where $n_1, n_2, \ldots, n_s$ are integers and $\sigma_1, \sigma_2, \ldots, \sigma_s$ are *oriented* q-simplices of K. If we reverse the orientation on one of these simplices σ_i then this reverses the sign of the corresponding coefficient n_i. If $\sigma_1, \sigma_2, \ldots, \sigma_s$ represent distinct simplices of K then the coefficients $n_1, n_2, \ldots, n_s$ are uniquely determined.

Example Let $\mathbf{v_0}$, $\mathbf{v}_1$ and $\mathbf{v}_2$ be the vertices of a triangle in some Euclidean space. Let K be the simplicial complex consisting of this triangle, together with its edges and vertices. Every 0-chain of K can be expressed uniquely in the form

$$n_0\langle \mathbf{v}_0\rangle + n_1\langle \mathbf{v}_1\rangle + n_2\langle \mathbf{v}_2\rangle$$

for some $n_0, n_1, n_2 \in \mathbb{Z}$. Similarly any 1-chain of K can be expressed uniquely in the form

$$m_0\langle \mathbf{v}_1, \mathbf{v}_2\rangle + m_1\langle \mathbf{v}_2, \mathbf{v}_0\rangle + m_2\langle \mathbf{v}_0, \mathbf{v}_1\rangle$$

for some $m_0, m_1, m_2 \in \mathbb{Z}$, and any 2-chain of K can be expressed uniquely as $n\langle \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\rangle$ for some integer n.

Lemma 5.2 *Let K be a simplicial complex, and let A be an additive group. Suppose that, to each $(q+1)$-tuple $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ of vertices spanning a simplex of K, there corresponds an element $\alpha(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ of A, where*

- $\alpha(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) = 0$ *unless $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are all distinct,*
- $\alpha(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ *changes sign on interchanging any two adjacent vertices $\mathbf{v}_{j-1}$ and $\mathbf{v}_j$.*

Then there exists a well-defined homomorphism from $C_q(K)$ to A which sends $\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\rangle$ to $\alpha(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. This homomorphism is uniquely determined.

Proof The given function defined on $(q+1)$-tuples of vertices of K extends to a well-defined homomorphism $\alpha\colon \Delta_q(K) \to A$ given by

$$\alpha\left(\sum_{r=1}^{s} n_r(\mathbf{v}_0^r, \mathbf{v}_1^r, \ldots, \mathbf{v}_q^r)\right) = \sum_{r=1}^{s} n_r \alpha(\mathbf{v}_0^r, \mathbf{v}_1^r, \ldots, \mathbf{v}_q^r)$$

for all $\displaystyle\sum_{r=1}^{s} n_r(\mathbf{v}_0^r, \mathbf{v}_1^r, \ldots, \mathbf{v}_q^r) \in \Delta_q(K)$. Moreover $(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) \in \ker\alpha$ unless $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are all distinct. Also

$$(\mathbf{v}_{\pi(0)}, \mathbf{v}_{\pi(1)}, \ldots, \mathbf{v}_{\pi(q)}) - \varepsilon_\pi(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) \in \ker\alpha$$

for all permutations π of $\{0, 1, \ldots, q\}$, since the permutation π can be expressed as a product of transpositions $(j-1, j)$ that interchange $j-1$ with j for some j and leave the rest of the set fixed, and the parity ε_π of π is given by $\varepsilon_\pi = +1$ when the number of such transpositions is even, and by $\varepsilon_\pi = -1$ when the number of such transpositions is odd. Thus the generators of $\Delta_q^0(K)$ are contained in $\ker\alpha$, and hence $\Delta_q^0(K) \subset \ker\alpha$. The required homomorphism $\tilde{\alpha}\colon C_q(K) \to A$ is then defined by the formula

$$\tilde{\alpha}\left(\sum_{r=1}^{s} n_r\langle \mathbf{v}_0^r, \mathbf{v}_1^r, \ldots, \mathbf{v}_q^r\rangle\right) = \sum_{r=1}^{s} n_r \alpha(\mathbf{v}_0^r, \mathbf{v}_1^r, \ldots, \mathbf{v}_q^r). \qquad \blacksquare$$

5.2 Boundary Homomorphisms

Let K be a simplicial complex. We introduce below *boundary homomorphisms* $\partial_q: C_q(K) \to C_{q-1}(K)$ between the chain groups of K. If σ is an oriented q-simplex of K then $\partial_q(\sigma)$ is a $(q-1)$-chain which is a formal sum of the $(q-1)$-faces of σ, each with an orientation determined by the orientation of σ.

Let σ be a q-simplex with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. For each integer j between 0 and q we denote by $\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle$ the oriented $(q-1)$-face

$$\langle \mathbf{v}_0, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_q \rangle$$

of the simplex σ obtained on omitting $\mathbf{v}_j$ from the set of vertices of σ. In particular

$$\langle \hat{\mathbf{v}}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle \equiv \langle \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle, \qquad \langle \mathbf{v}_0, \ldots, \mathbf{v}_{q-1}, \hat{\mathbf{v}}_q \rangle \equiv \langle \mathbf{v}_0, \ldots, \mathbf{v}_{q-1} \rangle.$$

Similarly if j and k are integers between 0 and q, where $j < k$, we denote by

$$\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \hat{\mathbf{v}}_k, \ldots \mathbf{v}_q \rangle$$

the oriented $(q-2)$-face $\langle \mathbf{v}_0, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_{k-1}, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_q \rangle$ of the simplex σ obtained on omitting $\mathbf{v}_j$ and $\mathbf{v}_k$ from the set of vertices of σ.

We now define a 'boundary homomorphism' $\partial_q: C_q(K) \to C_{q-1}(K)$ for each integer q. Define $\partial_q = 0$ if $q \leq 0$ or $q > \dim K$. (In this case one or other of the groups $C_q(K)$ and $C_{q-1}(K)$ is trivial.) Suppose then that $0 < q \leq \dim K$. Given vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ spanning a simplex of K, let

$$\alpha(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) = \sum_{j=0}^{q} (-1)^j \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle.$$

Inspection of this formula shows that $\alpha(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ changes sign whenever two adjacent vertices $\mathbf{v}_{i-1}$ and $\mathbf{v}_i$ are interchanged.

Suppose that $\mathbf{v}_j = \mathbf{v}_k$ for some j and k satisfying $j < k$. Then

$$\alpha(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) = (-1)^j \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle + (-1)^k \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q \rangle,$$

since the remaining terms in the expression defining $\alpha(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ contain both $\mathbf{v}_j$ and $\mathbf{v}_k$. However $(\mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q)$ can be transformed to $(\mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q)$ by making $k - j - 1$ transpositions which interchange $\mathbf{v}_j$ successively with the vertices $\mathbf{v}_{j+1}, \mathbf{v}_{j+2}, \ldots, \mathbf{v}_{k-1}$. Therefore

$$\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q \rangle = (-1)^{k-j-1} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle.$$

Thus $\alpha(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q) = 0$ unless $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are all distinct. It now follows immediately from Lemma 5.2 that there is a well-defined homomorphism $\partial_q : C_q(K) \to C_{q-1}(K)$, characterized by the property that

$$\partial_q (\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) = \sum_{j=0}^{q} (-1)^j \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K.

Lemma 5.3 *$\partial_{q-1} \circ \partial_q = 0$ for all integers q.*

Proof The result is trivial if $q < 2$, since in this case $\partial_{q-1} = 0$. Suppose that $q \geq 2$. Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ be vertices spanning a simplex of K. Then

$$\begin{aligned} \partial_{q-1}\partial_q (\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) &= \sum_{j=0}^{q} (-1)^j \partial_{q-1} (\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle) \\ &= \sum_{j=0}^{q} \sum_{k=0}^{j-1} (-1)^{j+k} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_k, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle \\ &\quad + \sum_{j=0}^{q} \sum_{k=j+1}^{q} (-1)^{j+k-1} \langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \hat{\mathbf{v}}_k, \ldots, \mathbf{v}_q \rangle \\ &= 0 \end{aligned}$$

(since each term in this summation over j and k cancels with the corresponding term with j and k interchanged). The result now follows from the fact that the homomorphism $\partial_{q-1} \circ \partial_q$ is determined by its values on all oriented q-simplices of K. ∎

5.3 The Homology Groups of a Simplicial Complex

Let K be a simplicial complex. A q-chain z is said to be a *q-cycle* if $\partial_q z = 0$. A q-chain b is said to be a *q-boundary* if $b = \partial_{q+1} c'$ for some $(q+1)$-chain c'. The group of q-cycles of K is denoted by $Z_q(K)$, and the group of q-boundaries of K is denoted by $B_q(K)$. Thus $Z_q(K)$ is the kernel of the boundary homomorphism $\partial_q : C_q(K) \to C_{q-1}(K)$, and $B_q(K)$ is the image of the boundary homomorphism $\partial_{q+1} : C_{q+1}(K) \to C_q(K)$. However $\partial_q \circ \partial_{q+1} = 0$, by Lemma 5.3. Therefore $B_q(K) \subset Z_q(K)$. But these groups are subgroups of the Abelian group $C_q(K)$. We can therefore form the quotient group $H_q(K)$, where $H_q(K) = Z_q(K)/B_q(K)$. The group $H_q(K)$ is referred to as the *qth homology group* of the simplicial complex K. Note that $H_q(K) = 0$ if $q < 0$

or $q > \dim K$ (since $Z_q(K) = 0$ and $B_q(K) = 0$ in these cases). It can be shown that the homology groups of a simplicial complex are topological invariants of the polyhedron of that complex.

The element $[z] \in H_q(K)$ of the homology group $H_q(K)$ determined by $z \in Z_q(K)$ is referred to as the *homology class* of the q-cycle z. Note that $[z_1 + z_2] = [z_1] + [z_2]$ for all $z_1, z_2 \in Z_q(K)$, and $[z_1] = [z_2]$ if and only if $z_1 - z_2 = \partial_{q+1} c$ for some $(q+1)$-chain c.

Proposition 5.4 *Let K be a simplicial complex. Suppose that there exists a vertex $\mathbf{w}$ of K with the following property:*

- *if vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K then so do $\mathbf{w}, \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$.*

Then $H_0(K) \cong \mathbb{Z}$, and $H_q(K)$ is the zero group for all $q > 0$.

Proof Using Lemma 5.2, we see that there is a well-defined homomorphism $D_q\colon C_q(K) \to C_{q+1}(K)$ characterized by the property that

$$D_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) = \langle \mathbf{w}, \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. Now $\partial_1(D_0(\mathbf{v})) = \mathbf{v} - \mathbf{w}$ for all vertices $\mathbf{v}$ of K. It follows that

$$\sum_{r=1}^{s} n_r \langle \mathbf{v}_r \rangle - \left(\sum_{r=1}^{s} n_r \right) \langle \mathbf{w} \rangle = \sum_{r=1}^{s} n_r (\langle \mathbf{v}_r \rangle - \langle \mathbf{w} \rangle) \in B_0(K)$$

for all $\sum\limits_{r=1}^{s} n_r \langle \mathbf{v}_r \rangle \in C_0(K)$. But $Z_0(K) = C_0(K)$ (since $\partial_0 = 0$ by definition), and thus $H_0(K) = C_0(K)/B_0(K)$. It follows that there is a well-defined surjective homomorphism from $H_0(K)$ to $\mathbb{Z}$ induced by the homomorphism from $C_0(K)$ to $\mathbb{Z}$ that sends $\sum\limits_{r=1}^{s} n_r \langle \mathbf{v}_r \rangle \in C_0(K)$ to $\sum\limits_{r=1}^{s} n_r$. Moreover this induced homomorphism is an isomorphism from $H_0(K)$ to $\mathbb{Z}$.

Now let $q > 0$. Then

$$\begin{aligned}
&\partial_{q+1}(D_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle)) \\
&\qquad = \partial_{q+1}(\langle \mathbf{w}, \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) \\
&\qquad = \langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle + \sum_{j=0}^{q} (-1)^{j+1} \langle \mathbf{w}, \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_j, \ldots, \mathbf{v}_q \rangle \\
&\qquad = \langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle - D_{q-1}(\partial_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle))
\end{aligned}$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. Thus

$$\partial_{q+1}(D_q(c)) + D_{q-1}(\partial_q(c)) = c$$

for all $c \in C_q(K)$. In particular $z = \partial_{q+1}(D_q(z))$ for all $z \in Z_q(K)$, and hence $Z_q(K) = B_q(K)$. It follows that $H_q(K)$ is the zero group for all $q > 0$, as required. ∎

Example The hypotheses of the proposition are satisfied for the complex K_σ consisting of a simplex σ together with all of its faces: we can choose $\mathbf{w}$ to be any vertex of the simplex σ. They are also satisfied for the first barycentric subdivision K'_σ of K_σ: in this case we must choose $\mathbf{w}$ to be the barycentre $\hat{\sigma}$ of the simplex σ. Thus the groups $H_0(K_\sigma)$ and $H_0(K'_\sigma)$ are both isomorphic of $\mathbb{Z}$, and the groups $H_q(K_\sigma)$ and $H_q(K'_\sigma)$ are the zero group for all $q > 0$.

5.4 Simplicial Maps and Induced Homomorphisms

Any simplicial map $\varphi\colon K \to L$ between simplicial complexes K and L induces well-defined homomorphisms $\varphi_q\colon C_q(K) \to C_q(L)$ of chain groups, where

$$\varphi_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) = \langle \varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q) \rangle$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ span a simplex of K. (The existence of these induced homomorphisms follows from a straightforward application of Lemma 5.2.) Note that $\varphi_q\left(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle\right) = 0$ unless $\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$ are all distinct.

Now $\varphi_{q-1} \circ \partial_q = \partial_q \circ \varphi_q$ for each integer q. Therefore $\varphi_q(Z_q(K)) \subset Z_q(L)$ and $\varphi_q(B_q(K)) \subset B_q(L)$ for all integers q. It follows that any simplicial map $\varphi\colon K \to L$ induces well-defined homomorphisms $\varphi_*\colon H_q(K) \to H_q(L)$ of homology groups, where $\varphi_*([z]) = [\varphi_q(z)]$ for all q-cycles $z \in Z_q(K)$. It is a trivial exercise to verify that if K, L and M are simplicial complexes and if $\varphi\colon K \to L$ and $\psi\colon L \to M$ are simplicial maps then the induced homomorphisms of homology groups satisfy $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.

5.5 Connectedness and $H_0(K)$

Lemma 5.5 *Let K be a simplicial complex. Then K can be partitioned into pairwise disjoint subcomplexes $K_1, K_2, \ldots, K_r$ whose polyhedra are the connected components of the polyhedron $|K|$ of K.*

Proof Let $X_1, X_2, \ldots, X_r$ be the connected components of the polyhedron of K, and, for each j, let K_j be the collection of all simplices σ of K for which $\sigma \subset X_j$. If a simplex belongs to K_j for all j then so do all its faces. Therefore $K_1, K_2, \ldots, K_r$ are subcomplexes of K. These subcomplexes are pairwise disjoint since the connected components $X_1, X_2, \ldots, X_r$ of $|K|$ are pairwise disjoint. Moreover, if $\sigma \in K$ then $\sigma \subset X_j$ for some j, since σ is a connected subset of $|K|$, and any connected subset of a topological space is contained in some connected component. But then $\sigma \in K_j$. It follows that $K = K_1 \cup K_2 \cup \cdots \cup K_r$ and $|K| = |K_1| \cup |K_2| \cup \cdots \cup |K_r|$, as required. ∎

The *direct sum* $A_1 \oplus A_2 \oplus \cdots \oplus A_r$ of additive Abelian groups $A_1, A_2, \ldots, A_r$ is defined to be the additive group consisting of all r-tuples $(a_1, a_2, \ldots, a_r)$ with $a_i \in A_i$ for $i = 1, 2, \ldots, r$, where

$$(a_1, a_2, \ldots, a_r) + (b_1, b_2, \ldots, b_r) \equiv (a_1 + b_1, a_2 + b_2, \ldots, a_r + b_r).$$

Lemma 5.6 *Let K be a simplicial complex. Suppose that $K = K_1 \cup K_2 \cup \cdots \cup K_r$, where $K_1, K_2, \ldots K_r$ are pairwise disjoint. Then*

$$H_q(K) \cong H_q(K_1) \oplus H_q(K_2) \oplus \cdots \oplus H_q(K_r)$$

for all integers q.

Proof We may restrict our attention to the case when $0 \leq q \leq \dim K$, since $H_q(K) = \{0\}$ if $q < 0$ or $q > \dim K$. Now any q-chain c of K can be expressed uniquely as a sum of the form $c = c_1 + c_2 + \cdots + c_r$, where c_j is a q-chain of K_j for $j = 1, 2, \ldots, r$. It follows that

$$C_q(K) \cong C_q(K_1) \oplus C_q(K_2) \oplus \cdots \oplus C_q(K_r).$$

Now let z be a q-cycle of K (i.e., $z \in C_q(K)$ satisfies $\partial_q(z) = 0$). We can express z uniquely in the form $z = z_1 + z_2 + \cdots + z_r$, where z_j is a q-chain of K_j for $j = 1, 2, \ldots, r$. Now

$$0 = \partial_q(z) = \partial_q(z_1) + \partial_q(z_2) + \cdots + \partial_q(z_r),$$

and $\partial_q(z_j)$ is a $(q-1)$-chain of K_j for $j = 1, 2, \ldots, r$. It follows that $\partial_q(z_j) = 0$ for $j = 1, 2, \ldots, r$. Hence each z_j is a q-cycle of K_j, and thus

$$Z_q(K) \cong Z_q(K_1) \oplus Z_q(K_2) \oplus \cdots \oplus Z_q(K_r).$$

Now let b be a q-boundary of K. Then $b = \partial_{q+1}(c)$ for some $(q+1)$-chain c of K. Moreover $c = c_1 + c_2 + \cdots c_r$, where $c_j \in C_{q+1}(K_j)$. Thus $b =$

$b_1 + b_2 + \cdots b_r$, where $b_j \in B_q(K_j)$ is given by $b_j = \partial_{q+1} c_j$ for $j = 1, 2, \ldots, r$. We deduce that

$$B_q(K) \cong B_q(K_1) \oplus B_q(K_2) \oplus \cdots \oplus B_q(K_r).$$

It follows from these observations that there is a well-defined isomorphism

$$\nu\colon H_q(K_1) \oplus H_q(K_2) \oplus \cdots \oplus H_q(K_r) \to H_q(K)$$

which maps $([z_1], [z_2], \ldots, [z_r])$ to $[z_1 + z_2 + \cdots + z_r]$, where $[z_j]$ denotes the homology class of a q-cycle z_j of K_j for $j = 1, 2, \ldots, r$. ∎

Let K be a simplicial complex, and let $\mathbf{y}$ and $\mathbf{z}$ be vertices of K. We say that $\mathbf{y}$ and $\mathbf{z}$ can be joined by an *edge path* if there exists a sequence $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$ of vertices of K with $\mathbf{v}_0 = \mathbf{y}$ and $\mathbf{v}_m = \mathbf{z}$ such that the line segment with endpoints $\mathbf{v}_{j-1}$ and $\mathbf{v}_j$ is an edge belonging to K for $j = 1, 2, \ldots, m$.

Lemma 5.7 *The polyhedron $|K|$ of a simplicial complex K is a connected topological space if and only if any two vertices of K can be joined by an edge path.*

Proof It is easy to verify that if any two vertices of K can be joined by an edge path then $|K|$ is path-connected and is thus connected. (Indeed any two points of $|K|$ can be joined by a path made up of a finite number of straight line segments.)

We must show that if $|K|$ is connected then any two vertices of K can be joined by an edge path. Choose a vertex $\mathbf{v}_0$ of K. It suffices to verify that every vertex of K can be joined to $\mathbf{v}_0$ by an edge path.

Let K_0 be the collection of all of the simplices of K having the property that one (and hence all) of the vertices of that simplex can be joined to $\mathbf{v}_0$ by an edge path. If σ is a simplex belonging to K_0 then every vertex of σ can be joined to $\mathbf{v}_0$ by an edge path, and therefore every face of σ belongs to K_0. Thus K_0 is a subcomplex of K. Clearly the collection K_1 of all simplices of K which do not belong to K_0 is also a subcomplex of K. Thus $K = K_0 \cup K_1$, where $K_0 \cap K_1 = \emptyset$, and hence $|K| = |K_0| \cup |K_1|$, where $|K_0| \cap |K_1| = \emptyset$. But the polyhedra $|K_0|$ and $|K_1|$ of K_0 and K_1 are closed subsets of $|K|$. It follows from the connectedness of $|K|$ that either $|K_0| = \emptyset$ or $|K_1| = \emptyset$. But $\mathbf{v}_0 \in K_0$. Thus $K_1 = \emptyset$ and $K_0 = K$, showing that every vertex of K can be joined to $\mathbf{v}_0$ by an edge path, as required. ∎

Theorem 5.8 *Let K be a simplicial complex. Suppose that the polyhedron $|K|$ of K is connected. Then $H_0(K) \cong \mathbb{Z}$.*

Proof Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r$ be the vertices of the simplicial complex K. Every 0-chain of K can be expressed uniquely as a formal sum of the form

$$n_1\langle \mathbf{u}_1\rangle + n_2\langle \mathbf{u}_2\rangle + \cdots + n_r\langle \mathbf{u}_r\rangle$$

for some integers $n_1, n_2, \ldots, n_r$. It follows that there is a well-defined homomorphism $\varepsilon\colon C_0(K) \to \mathbb{Z}$ defined by

$$\varepsilon\left(n_1\langle \mathbf{u}_1\rangle + n_2\langle \mathbf{u}_2\rangle + \cdots + n_r\langle \mathbf{u}_r\rangle\right) = n_1 + n_2 + \cdots + n_r.$$

Now $\varepsilon(\partial_1(\langle \mathbf{y}, \mathbf{z}\rangle)) = \varepsilon(\langle \mathbf{z}\rangle - \langle \mathbf{y}\rangle) = 0$ whenever $\mathbf{y}$ and $\mathbf{z}$ are endpoints of an edge of K. It follows that $\varepsilon \circ \partial_1 = 0$, and hence $B_0(K) \subset \ker \varepsilon$.

Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$ be vertices of K determining an edge path. Then

$$\langle \mathbf{v}_m\rangle - \langle \mathbf{v}_0\rangle = \partial_1\left(\sum_{j=1}^{m} \langle \mathbf{v}_{j-1}, \mathbf{v}_j\rangle\right) \in B_0(K).$$

Now $|K|$ is connected, and therefore any pair of vertices of K can be joined by an edge path (Lemma 5.7). We deduce that $\langle \mathbf{z}\rangle - \langle \mathbf{y}\rangle \in B_0(K)$ for all vertices $\mathbf{y}$ and $\mathbf{z}$ of K. Thus if $c \in \ker \varepsilon$, where $c = \sum_{j=1}^{r} n_j\langle \mathbf{u}_j\rangle$, then $\sum_{j=1}^{r} n_j = 0$, and hence $c = \sum_{j=2}^{r} n_j(\langle \mathbf{u}_j\rangle - \langle \mathbf{u}_1\rangle)$. But $\langle \mathbf{u}_j\rangle - \langle \mathbf{u}_1\rangle \in B_0(K)$. It follows that $c \in B_0(K)$. We conclude that $\ker \varepsilon \subset B_0(K)$, and hence $\ker \varepsilon = B_0(K)$.

Now the homomorphism $\varepsilon\colon C_0(K) \to \mathbb{Z}$ is surjective and its kernel is $B_0(K)$. Therefore it induces an isomorphism from $C_0(K)/B_0(K)$ to $\mathbb{Z}$. However $Z_0(K) = C_0(K)$ (since $\partial_0 = 0$ by definition). Thus $H_0(K) \equiv C_0(K)/B_0(K) \cong \mathbb{Z}$, as required. ∎

On combining Theorem 5.8 with Lemmas 5.5 and 5.6 we obtain immediately the following result.

Corollary 5.9 *Let K be a simplicial complex. Then*

$$H_0(K) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \qquad (r \text{ times}),$$

where r is the number of connected components of $|K|$.

6 Introduction to Homological Algebra

6.1 Exact Sequences

In homological algebra we consider sequences

$$\cdots \longrightarrow F \xrightarrow{p} G \xrightarrow{q} H \xrightarrow{\cdots}$$

where F, G, H etc. are Abelian groups and p, q etc. are homomorphisms. We denote the trivial group $\{0\}$ by 0, and we denote by $0 \longrightarrow G$ and $G \longrightarrow 0$ the zero homomorphisms from 0 to G and from G to 0 respectively. (These zero homomorphisms are of course the only homomorphisms mapping out of and into the trivial group 0.)

Definition The sequence $F \xrightarrow{p} G \xrightarrow{q} H$ of Abelian groups and homomorphisms is said to be *exact* at G if and only if $\text{image}(p\colon F \to G) = \ker(q\colon G \to H)$. A sequence of Abelian groups and homomorphisms is said to be *exact* if it is exact at each Abelian group occurring in the sequence (so that the image of each homomorphism is the kernel of the succeeding homomorphism).

A *monomorphism* is an injective homomorphism. An *epimorphism* is a surjective homomorphism. An *isomorphism* is a bijective homomorphism.

The following result follows directly from the relevant definitions.

Lemma 6.1 *Let $h\colon G \to H$ be a homomorphism of Abelian groups.*

- *$h\colon G \to H$ is a monomorphism if and only if $0 \longrightarrow G \xrightarrow{h} H$ is an exact sequence.*
- *$h\colon G \to H$ is an epimorphism if and only if $G \xrightarrow{h} H \longrightarrow 0$ is an exact sequence.*
- *$h\colon G \to H$ is an isomorphism if and only if $0 \longrightarrow G \xrightarrow{h} H \longrightarrow 0$ is an exact sequence.*

Let F be a subgroup of an Abelian group G. Then the sequence

$$0 \longrightarrow F \xrightarrow{i} G \xrightarrow{q} G/F \longrightarrow 0,$$

is exact, where G/F is the quotient group, $i\colon F \hookrightarrow G$ is the inclusion homomorphism, and $q\colon G \to G/F$ is the quotient homomorphism. Conversely, given any exact sequence of the form

$$0 \longrightarrow F \xrightarrow{i} G \xrightarrow{q} H \longrightarrow 0,$$

we can regard F as a subgroup of G (on identifying F with $i(F)$), and then H is isomorphic to the quotient group G/F. Exact sequences of this type are referred to as *short exact sequences*.

We now introduce the concept of a *commutative diagram*. This is a diagram depicting a collection of homomorphisms between various Abelian groups occurring on the diagram. The diagram is said to *commute* if, whenever there are two routes through the diagram from an Abelian group G to an Abelian group H, the homomorphism from G to H obtained by forming the composition of the homomorphisms along one route in the diagram agrees with that obtained by composing the homomorphisms along the other route. Thus, for example, the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p"] & B \arrow[r, "g"] \arrow[d, "q"] & C \arrow[d, "r"] \\ D \arrow[r, "h"] & E \arrow[r, "k"] & F \end{tikzcd}$$

commutes if and only if $q \circ f = h \circ p$ and $r \circ g = k \circ q$.

Proposition 6.2 *Suppose that the following diagram of Abelian groups and homomorphisms*

$$\begin{tikzcd} G_1 \arrow[r, "\theta_1"] \arrow[d, "\psi_1"] & G_2 \arrow[r, "\theta_2"] \arrow[d, "\psi_2"] & G_3 \arrow[r, "\theta_3"] \arrow[d, "\psi_3"] & G_4 \arrow[r, "\theta_4"] \arrow[d, "\psi_4"] & G_5 \arrow[d, "\psi_5"] \\ H_1 \arrow[r, "\phi_1"] & H_2 \arrow[r, "\phi_2"] & H_3 \arrow[r, "\phi_3"] & H_4 \arrow[r, "\phi_4"] & H_5 \end{tikzcd}$$

commutes and that both rows are exact sequences. Then the following results follow:

(i) *if ψ_2 and ψ_4 are monomorphisms and if ψ_1 is a epimorphism then ψ_3 is an monomorphism,*

(ii) *if ψ_2 and ψ_4 are epimorphisms and if ψ_5 is a monomorphism then ψ_3 is an epimorphism.*

Proof First we prove (i). Suppose that ψ_2 and ψ_4 are monomorphisms and that ψ_1 is an epimorphism. We wish to show that ψ_3 is a monomorphism. Let $x \in G_3$ be such that $\psi_3(x) = 0$. Then $\psi_4(\theta_3(x)) = \phi_3(\psi_3(x)) = 0$, and hence $\theta_3(x) = 0$. But then $x = \theta_2(y)$ for some $y \in G_2$, by exactness. Moreover

$$\phi_2(\psi_2(y)) = \psi_3(\theta_2(y)) = \psi_3(x) = 0,$$

hence $\psi_2(y) = \phi_1(z)$ for some $z \in H_1$, by exactness. But $z = \psi_1(w)$ for some $w \in G_1$, since ψ_1 is an epimorphism. Then

$$\psi_2(\theta_1(w)) = \phi_1(\psi_1(w)) = \psi_2(y),$$

and hence $\theta_1(w) = y$, since ψ_2 is a monomorphism. But then

$$x = \theta_2(y) = \theta_2(\theta_1(w)) = 0$$

by exactness. Thus ψ_3 is a monomorphism.

Next we prove (ii). Thus suppose that ψ_2 and ψ_4 are epimorphisms and that ψ_5 is a monomorphism. We wish to show that ψ_3 is an epimorphism. Let a be an element of H_3. Then $\phi_3(a) = \psi_4(b)$ for some $b \in G_4$, since ψ_4 is an epimorphism. Now

$$\psi_5(\theta_4(b)) = \phi_4(\psi_4(b)) = \phi_4(\phi_3(a)) = 0,$$

hence $\theta_4(b) = 0$, since ψ_5 is a monomorphism. Hence there exists $c \in G_3$ such that $\theta_3(c) = b$, by exactness. Then

$$\phi_3(\psi_3(c)) = \psi_4(\theta_3(c)) = \psi_4(b),$$

hence $\phi_3(a - \psi_3(c)) = 0$, and thus $a - \psi_3(c) = \phi_2(d)$ for some $d \in H_2$, by exactness. But ψ_2 is an epimorphism, hence there exists $e \in G_2$ such that $\psi_2(e) = d$. But then

$$\psi_3(\theta_2(e)) = \phi_2(\psi_2(e)) = a - \psi_3(c).$$

Hence $a = \psi_3(c + \theta_2(e))$, and thus a is in the image of ψ_3. This shows that ψ_3 is an epimorphism, as required. ∎

The following result is an immediate corollary of Proposition 6.2.

Lemma 6.3 (Five-Lemma) *Suppose that the rows of the commutative diagram of Proposition 6.2 are exact sequences and that ψ_1, ψ_2, ψ_4 and ψ_5 are isomorphisms. Then ψ_3 is also an isomorphism.*

6.2 Chain Complexes

Definition A *chain complex* C_* is a (doubly infinite) sequence $(C_i : i \in \mathbb{Z})$ of Abelian groups, together with homomorphisms $\partial_i \colon C_i \to C_{i-1}$ for each $i \in \mathbb{Z}$, such that $\partial_i \circ \partial_{i+1} = 0$ for all integers i.

The ith *homology group* $H_i(C_*)$ of the complex C_* is defined to be the quotient group $Z_i(C_*)/B_i(C_*)$, where $Z_i(C_*)$ is the kernel of $\partial_i \colon C_i \to C_{i-1}$ and $B_i(C_*)$ is the image of $\partial_{i+1} \colon C_{i+1} \to C_i$.

Definition Let C_* and D_* be chain complexes. A *chain map* $f \colon C_* \to D_*$ is a sequence $f_i \colon C_i \to D_i$ of homomorphisms which satisfy the commutativity condition $\partial_i \circ f_i = f_{i-1} \circ \partial_i$ for all $i \in \mathbb{Z}$.

Note that a collection of homomorphisms $f_i\colon C_i \to D_i$ defines a chain map $f_*\colon C_* \to D_*$ if and only if the diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & C_{i+1} & \xrightarrow{\partial_{i+1}} & C_i & \xrightarrow{\partial_i} & C_{i-1} & \longrightarrow \cdots \\
& \Big\downarrow f_{i+1} & & \Big\downarrow f_i & & \Big\downarrow f_{i-1} & \\
\cdots \longrightarrow & D_{i+1} & \xrightarrow{\partial_{i+1}} & D_i & \xrightarrow{\partial_i} & D_{i-1} & \longrightarrow \cdots
\end{array}$$

is commutative.

Let C_* and D_* be chain complexes, and let $f_*\colon C_* \to D_*$ be a chain map. Then $f_i(Z_i(C_*)) \subset Z_i(D_*)$ and $f_i(B_i(C_*)) \subset B_i(D_*)$ for all i. It follows from this that $f_i\colon C_i \to D_i$ induces a homomorphism $f_*\colon H_i(C_*) \to H_i(D_*)$ of homology groups sending $[z]$ to $[f_i(z)]$ for all $z \in Z_i(C_*)$, where $[z] = z + B_i(C_*)$, and $[f_i(z)] = f_i(z) + B_i(D_*)$.

Definition A *short exact sequence* $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ *of chain complexes* consists of chain complexes A_*, B_* and C_* and chain maps $p_*\colon A_* \to B_*$ and $q_*\colon B_* \to C_*$ such that the sequence

$$0 \longrightarrow A_i \xrightarrow{p_i} B_i \xrightarrow{q_i} C_i \longrightarrow 0$$

is exact for each integer i.

We see that $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ is a short exact sequence of chain complexes if and only if the diagram

$$\begin{array}{ccccccccc}
& & \vdots & & \vdots & & \vdots & & \\
& & \Big\downarrow \partial_{i+2} & & \Big\downarrow \partial_{i+2} & & \Big\downarrow \partial_{i+2} & & \\
0 & \longrightarrow & A_{i+1} & \xrightarrow{p_{i+1}} & B_{i+1} & \xrightarrow{q_{i+1}} & C_{i+1} & \longrightarrow & 0 \\
& & \Big\downarrow \partial_{i+1} & & \Big\downarrow \partial_{i+1} & & \Big\downarrow \partial_{i+1} & & \\
0 & \longrightarrow & A_i & \xrightarrow{p_i} & B_i & \xrightarrow{q_i} & C_i & \longrightarrow & 0\,. \\
& & \Big\downarrow \partial_i & & \Big\downarrow \partial_i & & \Big\downarrow \partial_i & & \\
0 & \longrightarrow & A_{i-1} & \xrightarrow{p_{i-1}} & B_{i-1} & \xrightarrow{q_{i-1}} & C_{i-1} & \longrightarrow & 0 \\
& & \Big\downarrow \partial_{i-1} & & \Big\downarrow \partial_{i-1} & & \Big\downarrow \partial_{i-1} & & \\
& & \vdots & & \vdots & & \vdots & &
\end{array}$$

is a commutative diagram whose rows are exact sequences and whose columns are chain complexes.

Lemma 6.4 *Given any short exact sequence* $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ *of chain complexes, there is a well-defined homomorphism*

$$\alpha_i \colon H_i(C_*) \to H_{i-1}(A_*)$$

which sends the homology class $[z]$ *of* $z \in Z_i(C_*)$ *to the homology class* $[w]$ *of any element* w *of* $Z_{i-1}(A_*)$ *with the property that* $p_{i-1}(w) = \partial_i(b)$ *for some* $b \in B_i$ *satisfying* $q_i(b) = z$.

Proof Let $z \in Z_i(C_*)$. Then there exists $b \in B_i$ satisfying $q_i(b) = z$, since $q_i \colon B_i \to C_i$ is surjective. Moreover

$$q_{i-1}(\partial_i(b)) = \partial_i(q_i(b)) = \partial_i(z) = 0.$$

But $p_{i-1} \colon A_{i-1} \to B_{i-1}$ is injective and $p_{i-1}(A_{i-1}) = \ker q_{i-1}$, since the sequence

$$0 \longrightarrow A_{i-1} \xrightarrow{p_{i-1}} B_{i-1} \xrightarrow{q_{i-1}} C_{i-1}$$

is exact. Therefore there exists a unique element w of A_{i-1} such that $\partial_i(b) = p_{i-1}(w)$. Moreover

$$p_{i-2}(\partial_{i-1}(w)) = \partial_{i-1}(p_{i-1}(w)) = \partial_{i-1}(\partial_i(b)) = 0$$

(since $\partial_{i-1} \circ \partial_i = 0$), and therefore $\partial_{i-1}(w) = 0$ (since $p_{i-2} \colon A_{i-2} \to B_{i-2}$ is injective). Thus $w \in Z_{i-1}(A_*)$.

Now let $b, b' \in B_i$ satisfy $q_i(b) = q_i(b') = z$, and let $w, w' \in Z_{i-1}(A_*)$ satisfy $p_{i-1}(w) = \partial_i(b)$ and $p_{i-1}(w') = \partial_i(b')$. Then $q_i(b - b') = 0$, and hence $b' - b = p_i(a)$ for some $a \in A_i$, by exactness. But then

$$p_{i-1}(w + \partial_i(a)) = p_{i-1}(w) + \partial_i(p_i(a)) = \partial_i(b) + \partial_i(b' - b) = \partial_i(b') = p_{i-1}(w'),$$

and $p_{i-1} \colon A_{i-1} \to B_{i-1}$ is injective. Therefore $w + \partial_i(a) = w'$, and hence $[w] = [w']$ in $H_{i-1}(A_*)$. Thus there is a well-defined function $\tilde{\alpha}_i \colon Z_i(C_*) \to H_{i-1}(A_*)$ which sends $z \in Z_i(C_*)$ to $[w] \in H_{i-1}(A_*)$, where $w \in Z_{i-1}(A_*)$ is chosen such that $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$ satisfying $q_i(b) = z$. This function $\tilde{\alpha}_i$ is clearly a homomorphism from $Z_i(C_*)$ to $H_{i-1}(A_*)$.

Suppose that elements z and z' of $Z_i(C_*)$ represent the same homology class in $H_i(C_*)$. Then $z' = z + \partial_{i+1}c$ for some $c \in C_{i+1}$. Moreover $c = q_{i+1}(d)$ for some $d \in B_{i+1}$, since $q_{i+1} \colon B_{i+1} \to C_{i+1}$ is surjective. Choose $b \in B_i$ such that $q_i(b) = z$, and let $b' = b + \partial_{i+1}(d)$. Then

$$q_i(b') = z + q_i(\partial_{i+1}(d)) = z + \partial_{i+1}(q_{i+1}(d)) = z + \partial_{i+1}(c) = z'.$$

Moreover $\partial_i(b') = \partial_i(b + \partial_{i+1}(d)) = \partial_i(b)$ (since $\partial_i \circ \partial_{i+1} = 0$). Therefore $\tilde{\alpha}_i(z) = \tilde{\alpha}_i(z')$. It follows that the homomorphism $\tilde{\alpha}_i \colon Z_i(C_*) \to H_{i-1}(A_*)$ induces a well-defined homomorphism $\alpha_i \colon H_i(C_*) \to H_{i-1}(A_*)$, as required. ∎

Let $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ and $0 \longrightarrow A'_* \xrightarrow{p'_*} B'_* \xrightarrow{q'_*} C'_* \longrightarrow 0$ be short exact sequences of chain complexes, and let $\lambda_*: A_* \to A'_*$, $\mu_*: B_* \to B'_*$ and $\nu_*: C_* \to C'_*$ be chain maps. For each integer i, let $\alpha_i: H_i(C_*) \to H_{i-1}(A_*)$ and $\alpha'_i: H_i(C'_*) \to H_{i-1}(A'_*)$ be the homomorphisms defined as described in Lemma 6.4. Suppose that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \xrightarrow{p_*} & B_* & \xrightarrow{q_*} & C_* & \longrightarrow & 0 \\ & & \big\downarrow \lambda_* & & \big\downarrow \mu_* & & \big\downarrow \nu_* & & \\ 0 & \longrightarrow & A'_* & \xrightarrow{p'_*} & B'_* & \xrightarrow{q'_*} & C'_* & \longrightarrow & 0 \end{array}$$

commutes (i.e., $p'_i \circ \lambda_i = \mu_i \circ p_i$ and $q'_i \circ \mu_i = \nu_i \circ q_i$ for all i). Then the square

$$\begin{array}{ccc} H_i(C_*) & \xrightarrow{\alpha_i} & H_{i-1}(A_*) \\ \big\downarrow \nu_* & & \big\downarrow \lambda_* \\ H_i(C'_*) & \xrightarrow{\alpha'_i} & H_{i-1}(A'_*) \end{array}$$

commutes for all $i \in \mathbb{Z}$ (i.e., $\lambda_* \circ \alpha_i = \alpha'_i \circ \nu_*$).

Proposition 6.5 *Let $0 \longrightarrow A_* \xrightarrow{p_*} B_* \xrightarrow{q_*} C_* \longrightarrow 0$ be a short exact sequence of chain complexes. Then the (infinite) sequence*

$$\cdots \xrightarrow{\alpha_{i+1}} H_i(A_*) \xrightarrow{p_*} H_i(B_*) \xrightarrow{q_*} H_i(C_*) \xrightarrow{\alpha_i} H_{i-1}(A_*) \xrightarrow{p_*} H_{i-1}(B_*) \xrightarrow{q_*} \cdots$$

of homology groups is exact, where $\alpha_i: H_i(C_*) \to H_{i-1}(A_*)$ is the well-defined homomorphism that sends the homology class $[z]$ of $z \in Z_i(C_*)$ to the homology class $[w]$ of any element w of $Z_{i-1}(A_*)$ with the property that $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$ satisfying $q_i(b) = z$.

Proof First we prove exactness at $H_i(B_*)$. Now $q_i \circ p_i = 0$, and hence $q_* \circ p_* = 0$. Thus the image of $p_*: H_i(A_*) \to H_i(B_*)$ is contained in the kernel of $q_*: H_i(B_*) \to H_i(C_*)$. Let x be an element of $Z_i(B_*)$ for which $[x] \in \ker q_*$. Then $q_i(x) = \partial_{i+1}(c)$ for some $c \in C_{i+1}$. But $c = q_{i+1}(d)$ for some $d \in B_{i+1}$, since $q_{i+1}: B_{i+1} \to C_{i+1}$ is surjective. Then

$$q_i(x - \partial_{i+1}(d)) = q_i(x) - \partial_{i+1}(q_{i+1}(d)) = q_i(x) - \partial_{i+1}(c) = 0,$$

and hence $x - \partial_{i+1}(d) = p_i(a)$ for some $a \in A_i$, by exactness. Moreover

$$p_{i-1}(\partial_i(a)) = \partial_i(p_i(a)) = \partial_i(x - \partial_{i+1}(d)) = 0,$$

since $\partial_i(x) = 0$ and $\partial_i \circ \partial_{i+1} = 0$. But $p_{i-1}: A_{i-1} \to B_{i-1}$ is injective. Therefore $\partial_i(a) = 0$, and thus a represents some element $[a]$ of $H_i(A_*)$. We deduce that

$$[x] = [x - \partial_{i+1}(d)] = [p_i(a)] = p_*([a]).$$

We conclude that the sequence of homology groups is exact at $H_i(B_*)$.

Next we prove exactness at $H_i(C_*)$. Let $x \in Z_i(B_*)$. Now $\alpha_i(q_*[x]) = \alpha_i([q_i(x)]) = [w]$, where w is the unique element of $Z_i(A_*)$ satisfying $p_{i-1}(w) = \partial_i(x)$. But $\partial_i(x) = 0$, and hence $w = 0$. Thus $\alpha_i \circ q_* = 0$. Now let z be an element of $Z_i(C_*)$ for which $[z] \in \ker \alpha_i$. Choose $b \in B_i$ and $w \in Z_{i-1}(A_*)$ such that $q_i(b) = z$ and $p_{i-1}(w) = \partial_i(b)$. Then $w = \partial_i(a)$ for some $a \in A_i$, since $[w] = \alpha_i([z]) = 0$. But then $q_i(b - p_i(a)) = z$ and $\partial_i(b - p_i(a)) = 0$. Thus $b - p_i(a) \in Z_i(B_*)$ and $q_*([b - p_i(a)]) = [z]$. We conclude that the sequence of homology groups is exact at $H_i(C_*)$.

Finally we prove exactness at $H_{i-1}(A_*)$. Let $z \in Z_i(C_*)$. Then $\alpha_i([z]) = [w]$, where $w \in Z_{i-1}(A_*)$ satisfies $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$ satisfying $q_i(b) = z$. But then $p_*(\alpha_i([z])) = [p_{i-1}(w)] = [\partial_i(b)] = 0$. Thus $p_* \circ \alpha_i = 0$. Now let w be an element of $Z_{i-1}(A_*)$ for which $[w] \in \ker p_*$. Then $[p_{i-1}(w)] = 0$ in $H_{i-1}(B_*)$, and hence $p_{i-1}(w) = \partial_i(b)$ for some $b \in B_i$. But

$$\partial_i(q_i(b)) = q_{i-1}(\partial_i(b)) = q_{i-1}(p_{i-1}(w)) = 0.$$

Therefore $[w] = \alpha_i([z])$, where $z = q_i(b)$. We conclude that the sequence of homology groups is exact at $H_{i-1}(A_*)$, as required. ∎

6.3 The Mayer-Vietoris Sequence

Let K be a simplicial complex and let L and M be subcomplexes of K such that $K = L \cup M$. Let

$$i_q\colon C_q(L \cap M) \to C_q(L), \qquad j_q\colon C_q(L \cap M) \to C_q(M),$$

$$u_q\colon C_q(L) \to C_q(K), \qquad v_q\colon C_q(M) \to C_q(K)$$

be the inclusion homomorphisms induced by the inclusion maps $i\colon L \cap M \hookrightarrow L$, $j\colon L \cap M \hookrightarrow M$, $u\colon L \hookrightarrow K$ and $v\colon M \hookrightarrow K$. Then

$$0 \longrightarrow C_*(L \cap M) \xrightarrow{k_*} C_*(L) \oplus C_*(M) \xrightarrow{w_*} C_*(K) \longrightarrow 0$$

is a short exact sequence of chain complexes, where

$$\begin{aligned} k_q(c) &= (i_q(c), -j_q(c)), \\ w_q(c', c'') &= u_q(c') + v_q(c''), \\ \partial_q(c', c'') &= (\partial_q(c'), \partial_q(c'')) \end{aligned}$$

for all $c \in C_q(L \cap M)$, $c' \in C_q(L)$ and $c'' \in C_q(M)$. It follows from Lemma 6.4 that there is a well-defined homomorphism $\alpha_q\colon H_q(K) \to H_{q-1}(L \cap M)$ such that $\alpha_q([z]) = [\partial_q(c')] = -[\partial_q(c'')]$ for any $z \in Z_q(K)$, where c' and c''

are any q-chains of L and M respectively satisfying $z = c' + c''$. (Note that $\partial_q(c') \in Z_{q-1}(L \cap M)$ since $\partial_q(c') \in Z_{q-1}(L)$, $\partial_q(c'') \in Z_{q-1}(M)$ and $\partial_q(c') = -\partial_q(c'')$.) It now follows immediately from Proposition 6.5 that the infinite sequence

$$\cdots \xrightarrow{\alpha_{q+1}} H_q(L \cap M) \xrightarrow{k_*} H_q(L) \oplus H_q(M) \xrightarrow{w_*} H_q(K) \xrightarrow{\alpha_q} H_{q-1}(L \cap M) \xrightarrow{k_*} \cdots,$$

of homology groups is exact. This long exact sequence of homology groups is referred to as the *Mayer-Vietoris sequence* associated with the decomposition of K as the union of the subcomplexes L and M.

7 The Topological Invariance of Simplicial Homology Groups

7.1 Contiguous Simplicial Maps

Definition Two simplicial maps $s: K \to L$ and $t: K \to L$ between simplicial complexes K and L are said to be *contiguous* if, given any simplex σ of K, there exists a simplex τ of L such that $s(\mathbf{v})$ and $t(\mathbf{v})$ are vertices of τ for each vertex $\mathbf{v}$ of σ.

Lemma 7.1 *Let K and L be simplicial complexes, and let $s: K \to L$ and $t: K \to L$ be simplicial approximations to some continuous map $f: |K| \to |L|$. Then the simplicial maps s and t are contiguous.*

Proof Let $\mathbf{x}$ be a point in the interior of some simplex σ of K. Then $f(\mathbf{x})$ belongs to the interior of a unique simplex τ of L, and moreover $s(\mathbf{x}) \in \tau$ and $t(\mathbf{x}) \in \tau$, since s and t are simplicial approximations to the map f. But $s(\mathbf{x})$ and $t(\mathbf{x})$ are contained in the interior of the simplices $s(\sigma)$ and $t(\sigma)$ of L. It follows that $s(\sigma)$ and $t(\sigma)$ are faces of τ, and hence $s(\mathbf{v})$ and $t(\mathbf{v})$ are vertices of τ for each vertex $\mathbf{v}$ of σ, as required. ∎

Proposition 7.2 *Let $s: K \to L$ and $t: K \to L$ be simplicial maps between simplicial complexes K and L. Suppose that s and t are contiguous. Then the homomorphisms $s_*: H_q(K) \to H_q(L)$ and $t_*: H_q(K) \to H_q(L)$ coincide for all q.*

Proof Choose an ordering of the vertices of K. Then there are well-defined homomorphisms $D_q: C_q(K) \to C_{q+1}(L)$ characterized by the property that

$$D_q(\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q \rangle) = \sum_{j=0}^{q} (-1)^j \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q) \rangle.$$

whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots \mathbf{v}_q$ are the vertices of a q-simplex of K listed in increasing order (with respect to the chosen ordering of the vertices of K). Then

$$\partial_1(D_0(\langle \mathbf{v} \rangle)) = \partial_1(\langle s(\mathbf{v}), t(\mathbf{v}) \rangle) = \langle t(\mathbf{v}) \rangle - \langle s(\mathbf{v}) \rangle,$$

and thus $\partial_1 \circ D_0 = t_0 - s_0$. Also

$$\begin{aligned}
&D_{q-1}(\partial_q(\langle \mathbf{v}_0, \ldots, \mathbf{v}_q \rangle)) \\
&\quad = \sum_{i=0}^{q} (-1)^i D_{q-1}(\langle \mathbf{v}_0, \ldots, \hat{\mathbf{v}}_i, \ldots, \mathbf{v}_q \rangle) \\
&\quad = \sum_{i=0}^{q} \sum_{j=0}^{i-1} (-1)^{i+j} \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, \widehat{t(\mathbf{v}_i)}, \ldots, t(\mathbf{v}_q) \rangle \\
&\qquad + \sum_{i=0}^{q} \sum_{j=i+1}^{q} (-1)^{i+j-1} \langle s(\mathbf{v}_0), \ldots, \widehat{s(\mathbf{v}_i)}, \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q) \rangle
\end{aligned}$$

and

$$\begin{aligned}
&\partial_{q+1}(D_q(\langle \mathbf{v}_0, \ldots \mathbf{v}_q \rangle)) \\
&\quad = \sum_{j=0}^{q} (-1)^j \partial_{q+1}(\langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q) \rangle) \\
&\quad = \sum_{j=0}^{q} \sum_{i=0}^{j-1} (-1)^{i+j} \langle s(\mathbf{v}_0), \ldots, \widehat{s(\mathbf{v}_i)}, \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q) \rangle \\
&\qquad + \langle t(\mathbf{v}_0), \ldots, t(\mathbf{v}_q) \rangle + \sum_{j=1}^{q} \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_{j-1}), t(\mathbf{v}_j), \ldots, t(\mathbf{v}_q) \rangle \\
&\qquad - \sum_{j=0}^{q-1} \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_{j+1}), \ldots, t(\mathbf{v}_q) \rangle - \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_q) \rangle \\
&\qquad + \sum_{j=0}^{q} \sum_{i=j+1}^{q} (-1)^{i+j+1} \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_j), t(\mathbf{v}_j), \ldots, \widehat{t(\mathbf{v}_i)}, \ldots, t(\mathbf{v}_q) \rangle \\
&\quad = -D_{q-1}(\partial_q(\langle \mathbf{v}_0, \ldots, \mathbf{v}_q \rangle)) + \langle t(\mathbf{v}_0), \ldots, t(\mathbf{v}_q) \rangle - \langle s(\mathbf{v}_0), \ldots, s(\mathbf{v}_q) \rangle
\end{aligned}$$

and thus

$$\partial_{q+1} \circ D_q + D_{q-1} \circ \partial_q = t_q - s_q$$

for all $q > 0$. It follows that $t_q(z) - s_q(z) = \partial_{q+1}(D_q(z))$ for any q-cycle z of K, and therefore $s_*([z]) = t_*([z])$. Thus $s_* = t_*$ as homomorphisms from $H_q(K)$ to $H_q(L)$, as required. ∎

7.2 The Homology of Barycentric Subdivisions

We shall show that the homology groups of a simplicial complex are isomorphic to those of its first barycentric subdivision.

We recall that the vertices of the first barycentric subdivision K' of a simplicial complex K are the barycentres $\hat{\sigma}$ of the simplices σ of K, and that K' consists of the simplices spanned by $\hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_q$, where $\sigma_0, \sigma_1, \ldots, \sigma_q \in K$ and σ_{j-1} is a proper face of σ_j for $j = 1, 2, \ldots, q$.

Lemma 7.3 *Let K' be the first barycentric subdivision of a simplicial complex K. Then a function $\zeta\colon \operatorname{Vert} K' \to \operatorname{Vert} K$ from the vertices of K' to those of K represents a simplicial approximation to the identity map of $|K|$ if and only if it sends the barycentre of any simplex of K to some vertex of that simplex.*

Proof If ζ represents a simplicial approximation to the identity map of $|K|$ then $\zeta(\hat{\sigma}) \in \sigma$ for any $\sigma \in K$, and hence $\zeta(\hat{\sigma})$ is a vertex of σ.

Conversely suppose that the function ζ sends the barycentre of any simplex of K to a vertex of that simplex. Let τ be a simplex of K'. Then it follows from the definition of K' that the interior of τ is contained in the interior of some simplex σ of K, and the vertices of τ are barycentres of faces of σ. Then ζ must map the vertices of τ to vertices of σ, and hence ζ represents a simplicial map from K' to K. Moreover this simplicial map is a simplicial approximation to the identity map, since the interior of τ is contained in σ and ζ maps the interior of τ into σ. ∎

It follows from Lemma 7.3 that there exist simplicial approximations $\zeta\colon K' \to K$ to the identity map of $|K|$: such a simplicial approximation can be obtained by choosing, for each $\sigma \in K$, a vertex $\mathbf{v}_\sigma$ of σ, and defining $\zeta(\hat{\sigma}) = \mathbf{v}_\sigma$.

Suppose that $\zeta\colon K' \to K$ and $\theta\colon K' \to K$ are both simplicial approximations to the identity map of $|K|$. Then ζ and θ are contiguous (Lemma 7.1), and therefore the homomorphisms ζ_* and θ_* of homology groups induced by ζ and θ must coincide. It follows that there is a well-defined natural homomorphism $\nu_K\colon H_q(K') \to H_q(K)$ of homology groups which coincides with ζ_* for any simplicial approximation $\zeta\colon K' \to K$ to the identity map of $|K|$.

Theorem 7.4 *The natural homomorphism $\nu_K\colon H_q(K') \to H_q(K)$ is an isomorphism for any simplicial complex K.*

Proof Let M be the simplicial complex consisting of some simplex σ together with all of its faces. Then $H_0(M) \cong \mathbb{Z}$, $H_0(M') \cong \mathbb{Z}$, and $H_q(M) =$

$0 = H_q(M')$ for all $q > 0$ (see Proposition 5.4 and the following example). Let $\mathbf{v}$ be a vertex of M. If $\theta\colon M' \to M$ is any simplicial approximation to the identity map of $|M|$ then $\theta(\mathbf{v}) = \mathbf{v}$. But the homology class of $\langle \mathbf{v} \rangle$ generates both $H_0(M)$ and $H_0(M')$. It follows that $\theta_*\colon H_0(M') \to H_0(M)$ is an isomorphism, and thus $\nu_M\colon H_q(M') \to H_q(M)$ is an isomorphism for all q.

We now use induction on the number of simplices in K to prove the theorem in the general case. It therefore suffices to prove that the required result holds for a simplicial complex K under the additional assumption that the result is valid for all proper subcomplexes of K.

Let σ be a simplex of K whose dimension equals the dimension of K. Then σ is not a face of any other simplex of K, and therefore $K \setminus \{\sigma\}$ is a subcomplex of K. Let M be the subcomplex of K consisting of the simplex σ, together with all of its faces. We have already proved the result in the special case when $K = M$. Thus we only need to verify the result in the case when M is a proper subcomplex of K. In that case $K = L \cup M$, where $L = K \setminus \{\sigma\}$.

Let $\zeta\colon K' \to K$ be a simplicial approximation to the identity map of $|K|$. Then the restrictions $\zeta|L'$, $\zeta|M'$ and $\zeta|L' \cap M'$ of ζ to L', M' and $L' \cap M'$ are simplicial approximations to the identity maps of $|L|$, $|M|$ and $|L| \cap |M|$ respectively. Therefore the diagram

$$\begin{tikzcd} 0 \arrow[r] & C_q(L'\cap M') \arrow[r] \arrow[d, "\zeta|L'\cap M'"] & C_q(L')\oplus C_q(M') \arrow[r] \arrow[d, "(\zeta|L')\oplus(\zeta|M')"] & C_q(K') \arrow[r] \arrow[d, "\zeta"] & 0 \\ 0 \arrow[r] & C_q(L\cap M) \arrow[r] & C_q(L)\oplus C_q(M) \arrow[r] & C_q(K) \arrow[r] & 0 \end{tikzcd}$$

commutes, and its rows are short exact sequences. But the restrictions $\zeta|L'$, $\zeta|M'$ and $\zeta|L' \cap M'$ of ζ to L', M' and $L' \cap M'$ are simplicial approximations to the identity maps of $|L|$, $|M|$ and $|L| \cap |M|$ respectively, and therefore induce the natural homomorphisms ν_L, ν_M and $\nu_{L\cap M}$. We therefore obtain a commutative diagram

$$\begin{tikzcd} H_q(L'\cap M') \arrow[r] \arrow[d, "\nu_{L\cap M}"] & H_q(L')\oplus H_q(M') \arrow[r] \arrow[d, "\nu_L\oplus\nu_M"] & H_q(K') \arrow[r, "\alpha_q"] \arrow[d, "\nu_K"] & H_{q-1}(L'\cap M') \arrow[r] \arrow[d, "\nu_{L\cap M}"] & H_{q-1}(L')\oplus H_{q-1}(M') \arrow[d, "\nu_L\oplus\nu_M"] \\ H_q(L\cap M) \arrow[r] & H_q(L)\oplus H_q(M) \arrow[r] & H_q(K) \arrow[r, "\alpha_q"] & H_{q-1}(L\cap M) \arrow[r] & H_{q-1}(L)\oplus H_{q-1}(M) \end{tikzcd}$$

in which the rows are exact sequences, and are the Mayer-Vietoris sequences corresponding to the decompositions $K = L \cup M$ and $K' = L' \cup M'$ of K and K'. But the induction hypothesis ensures that the homomorphisms ν_L, ν_M and $\nu_{L\cap M}$ are isomorphisms, since L, M and $L \cap M$ are all proper subcomplexes of K. It now follows directly from the Five-Lemma (Lemma 6.3) that $\nu_K\colon H_q(K') \to H_q(K)$ is also an isomorphism, as required. ∎

We refer to the isomorphism $\nu_K\colon H_q(K') \to H_q(K)$ as the *canonical isomorphism* from the qth homology group of K' to that of K.

For each $j > 0$, we define the canonical isomorphism $\nu_{K,j}\colon H_q(K^{(j)}) \to H_q(K)$ from the homology groups of the jth barycentric subdivision $K^{(j)}$ of K to those of K itself to be the composition of the natural isomorphisms

$$H_q(K^{(j)}) \to H_q(K^{(j-1)}) \to \cdots \to H_q(K') \to H_q(K)$$

induced by appropriate simplicial approximations to the identity map of $|K|$. Note that if $i \leq j$ then $\nu_{K,i}^{-1} \circ \nu_{K,j}$ is induced by a composition of simplicial approximations to the identity map of $|K|$. But any composition of simplicial approximations to the identity map is itself a simplicial approximation to the identity map (Corollary 4.10). We deduce the following result.

Lemma 7.5 *Let K be a simplicial complex, let i and j be positive integers satisfying $i \leq j$. Then $\nu_{K,j} = \nu_{K,i} \circ \zeta_*$ for some simplicial approximation $\zeta\colon K^{(j)} \to K^{(i)}$ to the identity map of $|K|$.*

7.3 Continuous Maps and Induced Homomorphisms

Proposition 7.6 *Any continuous map $f\colon |K| \to |L|$ between the polyhedra of simplicial complexes K and L induces a well-defined homomorphism $f_*\colon H_q(K) \to H_q(L)$ of homology groups such that $f_* = s_* \circ \nu_{K,i}^{-1}$ for any simplicial approximation $s\colon K^{(i)} \to L$ to the map f, where $s_*\colon H_q(K^{(i)}) \to H_q(L)$ is the homomorphism induced by the simplicial map s and $\nu_{K,i}\colon H_q(K^{(i)}) \to H_q(K)$ is the canonical isomorphism.*

Proof The Simplicial Approximation Theorem (Theorem 4.11) guarantees the existence of a simplicial approximation $s\colon K^{(i)} \to L$ to the map f defined on the ith barycentric subdivision $K^{(i)}$ of K for some sufficiently large i. Thus it only remains to verify that if $s\colon K^{(i)} \to L$ and $t\colon K^{(j)} \to L$ are both simplicial approximations to the map f then $s_* \circ \nu_{K,i}^{-1} = t_* \circ \nu_{K,j}^{-1}$.

Suppose that $i \leq j$. Then $\nu_{K,i}^{-1}\nu_{K,j} = \zeta_*$ for some simplicial approximation $\zeta\colon K^{(j)} \to K^{(i)}$ to the identity map of $|K|$ (Lemma 7.5). Thus $s_* \circ \nu_{K,i}^{-1} = s_* \circ \zeta_* \circ \nu_{K,j}^{-1} = (s \circ \zeta)_* \circ \nu_{K,j}^{-1}$. Moreover $\zeta\colon K^{(j)} \to K^{(i)}$ and $s\colon K^{(i)} \to L$ are simplicial approximations to the identity map of $|K|$ and to $f\colon |K| \to |L|$ respectively, and therefore $s \circ \zeta\colon K^{(j)} \to L$ is a simplicial approximation to $f\colon |K| \to |L|$ (Corollary 4.10). But then $s \circ \zeta$ and t are simplicial approximations to the same continuous map, and thus are contiguous simplicial maps from $K^{(j)}$ to L (Lemma 7.1). It follows that $(s \circ \zeta)_*$ and t_* coincide as homomorphisms from $H_q(K^{(j)})$ to $H_q(L)$ (Lemma 7.2), and therefore $s_* \circ \nu_{K,i}^{-1} = t_* \circ \nu_{K,j}^{-1}$, as required. ∎

Proposition 7.7 *Let K, L and M be simplicial complexes and let $f\colon |K| \to |L|$ and $g\colon |L| \to |M|$ be continuous maps. Then the homomorphisms f_*, g_* and $(g \circ f)_*$ of homology groups induced by the maps f, g and $g \circ f$ satisfy $(g \circ f)_* = g_* \circ f_*$.*

Proof Let $t\colon L^{(m)} \to M$ be a simplicial approximation to g and let $s\colon K^{(j)} \to L^{(m)}$ be a simplicial approximation to f. Now the canonical isomorphism $\nu_{L,m}$ from $H_q(L^{(m)})$ to $H_q(L)$ is induced by some simplicial approximation to the identity map of $|L|$. It follows that $\nu_{L,m} \circ s_*$ is induced by some simplicial approximation to f (Corollary 4.10), and therefore $f_* = \nu_{L,m} \circ s_* \circ \nu_{K,j}^{-1}$. Also $g_* = t_* \circ \nu_{L,m}^{-1}$. It follows that $g_* \circ f_* = t_* \circ s_* \circ \nu_{K,j}^{-1} = (t \circ s)_* \circ \nu_{K,j}^{-1}$. But $t \circ s\colon K^{(j)} \to M$ is a simplicial approximation to $g \circ f$ (Corollary 4.10). Thus $(g \circ f)_* = g_* \circ f_*$, as required. ∎

Corollary 7.8 *If the polyhedra $|K|$ and $|L|$ of simplicial complexes K and L are homeomorphic then the homology groups of K and L are isomorphic.*

Proof Let $h\colon |K| \to |L|$ be a homeomorphism. Then $h_*\colon H_q(K) \to H_q(L)$ is an isomorphism whose inverse is $(h^{-1})_*\colon H_q(L) \to H_q(K)$. ∎

One can make use of induced homomorphisms in homology theory in order to prove the Brouwer Fixed Point Theorem (Theorem 4.14) in all dimensions. The Brouwer Fixed Point Theorem is a consequence of the fact that there is no continuous map $r\colon \Delta \to \partial\Delta$ from an n-simplex Δ to its boundary $\partial\Delta$ with the property that $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial\Delta$ (Proposition 4.13). Such a continuous map would induce homomorphisms $r_*\colon H_q(\Delta) \to H_q(\partial\Delta)$ of homology groups for all non-negative integers q, and $r_* \circ i_*$ would be the identity automorphism of $H_q(\partial\Delta)$ for all q, where $i_*\colon H_q(\partial\Delta) \to H_q(\Delta)$ is induced by the inclusion map $i\colon \partial\Delta \hookrightarrow \Delta$. But this would imply that $r_*\colon H_q(\Delta) \to H_q(\partial\Delta)$ was surjective for all non-negative integers q, which is impossible, since $H_{n-1}(\Delta) = 0$ and $H_{n-1}(\partial\Delta) \cong \mathbb{Z}$ when $n \geq 2$ (and $H_{n-1}(\Delta) \cong \mathbb{Z}$ and $H_{n-1}(\partial\Delta) \cong \mathbb{Z} \oplus \mathbb{Z}$ when $n = 1$). We conclude therefore that there is no continuous map $r\colon \Delta \to \partial\Delta$ that fixes all points of $\partial\Delta$, and therefore the Brouwer Fixed Point Theorem is satisfied in all dimensions.

We next show that homotopic maps between the polyhedra of simplicial complexes induce the same homomorphisms of homology groups. For this we require the following result.

Lemma 7.9 *For any simplicial complex L there is some $\varepsilon > 0$ with the following property: given continuous maps $f\colon |K| \to |L|$ and $g\colon |K| \to |L|$ defined on the polyhedron of some simplicial complex K, where $f(\mathbf{x})$ is within* *a distance ε of $g(\mathbf{x})$ for all $\mathbf{x} \in |K|$, there exists a simplicial map defined on $K^{(i)}$ for some sufficiently large i which is a simplicial approximation to both f and g.*

Proof An application of the Lebesgue Lemma shows that there exists $\varepsilon > 0$ such that the open ball of radius 2ε about any point of $|L|$ is contained in $\mathrm{st}_L(\mathbf{b})$ for some vertex $\mathbf{b}$ of L. Let $f\colon |K| \to |L|$ and $g\colon |K| \to |L|$ be continuous maps. Suppose that $f(\mathbf{x})$ is within a distance ε of $g(\mathbf{x})$ for all $\mathbf{x} \in |K|$. Another application of the Lebesgue Lemma (to the open cover of $|K|$ by preimages of open balls of radius ε) shows that there exists $\delta > 0$ such that any subset S of $|K|$ whose diameter is less than δ is mapped by f into an open ball of radius ε about some point of $|L|$, and is therefore mapped by g into an open ball of of radius 2ε about that point. But then $f(S) \subset \mathrm{st}_L(\mathbf{b})$ and $g(S) \subset \mathrm{st}_L(\mathbf{b})$ for some vertex $\mathbf{b}$ of L. Now choose i such that $\mu(K^{(i)}) < \frac{1}{2}\delta$. As in the proof of the Simplicial Approximation Theorem (Theorem 4.11) we see that, for every vertex $\mathbf{a}$ of $K^{(i)}$, the diameter of $\mathrm{st}_{K^{(i)}}(\mathbf{a})$ is less than δ, and hence $f(\mathrm{st}_{K^{(i)}}(\mathbf{a})) \subset \mathrm{st}_L(s(\mathbf{a}))$ and $g(\mathrm{st}_{K^{(i)}}(\mathbf{a})) \subset \mathrm{st}_L(s(\mathbf{a}))$ for some vertex $s(\mathbf{a})$ of L. It then follows from Proposition 4.9 that the function $s\colon \mathrm{Vert}\, K^{(i)} \to \mathrm{Vert}\, L$ constructed in this manner is the required simplicial approximation to f and g. ∎

Theorem 7.10 *Let K and L be simplicial complexes and let $f\colon |K| \to |L|$ and $g\colon |K| \to |L|$ be continuous maps from $|K|$ to $|L|$. Suppose that f and g are homotopic. Then the induced homomorphisms f_* and g_* from $H_q(K)$ to $H_q(L)$ are equal for all q.*

Proof Let $F\colon |K| \times [0,1] \to |L|$ be a homotopy with $F(\mathbf{x}, 0) = f(\mathbf{x})$ and $F(\mathbf{x}, 1) = g(\mathbf{x})$, and let $\varepsilon > 0$ be given. Using the well-known fact that continuous functions defined on compact metric spaces are uniformly continuous (which is easily proved using the Lebesgue Lemma), we see that there exists some $\delta > 0$ such that if $|s - t| < \delta$ then the distance from $F(\mathbf{x}, s)$ to $F(\mathbf{x}, t)$ is less than ε. Let $f_i(\mathbf{x}) = F(\mathbf{x}, t_i)$ for $i = 0, 1, \ldots, r$, where $t_0, t_1, \ldots, t_r$ have been chosen such that $0 = t_0 < t_1 < \cdots < t_r = 1$ and $t_i - t_{i-1} < \delta$ for all i. Then $f_{i-1}(\mathbf{x})$ is within a distance ε of $f_i(\mathbf{x})$ for all $\mathbf{x} \in |K|$. Using Lemma 7.9, we see that the maps f_{i-1} and f_i from $|K|$ to $|L|$ have a common simplicial approximation, and thus f_{i-1} and f_i induce the same homomorphisms of homology groups, provided that $\varepsilon > 0$ has been chosen sufficiently small. It follows that the maps f and g induce the same homomorphisms of homology groups, as required. ∎

7.4 Homotopy Equivalence

Definition Let X and Y be topological spaces. A continuous map $f\colon X \to Y$ is said to be a *homotopy equivalence* if there exists a continuous map $g\colon Y \to X$ such that $g \circ f$ is homotopic to the identity map of X and $f \circ g$ is homotopic to the identity map of Y. The spaces X and Y are said to be *homotopy equivalent* if there exists a homotopy equivalence from X to Y.

Lemma 7.11 *A composition of homotopy equivalences is itself a homotopy equivalence.*

Proof Let X, Y and Z be topological spaces, and let $f\colon X \to Y$ and $h\colon Y \to Z$ be homotopy equivalences. Then there exist continuous maps $g\colon Y \to X$ and $k\colon Z \to Y$ such that $g \circ f \simeq i_X$, $f \circ g \simeq i_Y$, $k \circ h \simeq i_Y$ and $h \circ k \simeq i_Z$, where i_X, i_Y and i_Z denote the identity maps of the spaces X, Y, Z. Then $(g \circ k) \circ (h \circ f) = g \circ (k \circ h) \circ f \simeq g \circ i_Y \circ f = g \circ f \simeq i_X$ and $(h \circ f) \circ (g \circ k) = h \circ (f \circ g) \circ k \simeq h \circ i_Y \circ k = h \circ k \simeq i_Z$. Thus $h \circ f\colon X \to Z$ is a homotopy equivalence from X to Z. ∎

Lemma 7.12 *Let $f\colon |K| \to |L|$ be a homotopy equivalence between the polyhedra of simplicial complexes K and L. Then, for each non-negative integer q, the induced homomorphism $f_*\colon H_q(K) \to H_q(L)$ of homology groups is an isomorphism.*

Proof There exists a continuous map $g\colon |L| \to |K|$ such that $g \circ f$ is homotopic to the identity map of $|K|$ and $f \circ g$ is homotopic to the identity map of $|L|$. It follows that the induced homomorphisms $(g \circ f)_*\colon H_q(K) \to H_q(K)$ and $(f \circ g)_*\colon H_q(L) \to H_q(L)$ are the identity automorphisms of $H_q(K)$ and $H_q(L)$ for each q. But $(g \circ f)_* = g_* \circ f_*$ and $(f \circ g)_* = f_* \circ g_*$. It follows that $f_*\colon H_q(K) \to H_q(L)$ is an isomorphism with inverse $g_*\colon H_q(L) \to H_q(K)$. ∎

Definition A subset A of a topological space X is said to be a *deformation retract* of X if there exists a continuous map $H\colon X \times [0,1] \to X$ such that $H(x,0) = x$ and $H(x,1) \in A$ for all $x \in X$ and $H(a,1) = a$ for all $a \in A$.

Thus a subset A of a topological space X is a deformation retract of X if and only if there exists a function $r\colon X \to A$ such that $r(a) = a$ for all $a \in A$ and r is homotopic in X to the identity map of X.

Example The unit sphere S^{n-1} in $\mathbb{R}^n$ is a deformation retract of $\mathbb{R}^n \setminus \{\mathbf{0}\}$. For if $H(\mathbf{x}, t) = (1 - t + t/|\mathbf{x}|)\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $t \in [0,1]$ then $H(\mathbf{x}, 0) = \mathbf{x}$ and $H(\mathbf{x}, 1) \in S^{n-1}$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $H(\mathbf{x}, 1) = \mathbf{x}$ when $\mathbf{x} \in S^{n-1}$.

If A is a deformation retract of a topological space X then the inclusion map $i: A \hookrightarrow X$ is a homotopy equivalence.

Theorem 7.13 *The spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic if $m \neq n$.*

Proof Let S^{m-1} and S^{n-1} denote the unit spheres in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Then S^{m-1} and S^{n-1} are homeomorphic to the polyhedra of simplicial complexes K and L respectively. Let $i_m: S^{m-1} \to \mathbb{R}^m \setminus \{\mathbf{0}\}$ be the inclusion map and let $r_n: \mathbb{R}^n \setminus \{\mathbf{0}\} \to S^{n-1}$ be the map that sends $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ to $(1/|\mathbf{x}|)\mathbf{x}$. Then both $i_m: S^{m-1} \to \mathbb{R}^m \setminus \{\mathbf{0}\}$ and $r_n: \mathbb{R}^n \setminus \{\mathbf{0}\} \to S^{n-1}$ are homotopy equivalences.

Suppose that there were to exist a homeomorphism $h: \mathbb{R}^m \to \mathbb{R}^n$. Let $f(\mathbf{x}) = h(\mathbf{x}) - h(\mathbf{0})$ for all $\mathbf{x} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$. Then $f: \mathbb{R}^m \setminus \{\mathbf{0}\} \to \mathbb{R}^n \setminus \{\mathbf{0}\}$ would also be a homeomorphism, and therefore $r_n \circ f \circ i_m: S^{m-1} \to S^{n-1}$ would be a homotopy equivalence. Thus if $\mathbb{R}^m$ and $\mathbb{R}^n$ were homeomorphic then S^{m-1} and S^{n-1} would be homotopy equivalent, and therefore the homology groups of the simplicial complexes K and L would be isomorphic. But $H_q(K) \cong \mathbb{Z}$ when $q = 0$ and $q = m - 1$ and $H_q(K) = 0$ for all other values of q, whereas $H_q(L) \cong \mathbb{Z}$ when $q = 0$ and $q = n - 1$ and $H_q(L) = 0$ for all other values of q. Thus if $m \neq n$ then the homology groups of the simplicial complexes K and L are not isomorphic, and therefore $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic. ∎