Course 311: 2001–02. Supplementary Notes
The Resolvent Cubic of a Quartic Polynomial

Let α, β, γ and δ be the roots of the quartic polynomial

$$x^4 - px^2 - qx - r,$$

so that

$$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta) = x^4 - px^2 - qx - r.$$

Equating coefficients of x, we find that

$$\alpha + \beta + \gamma + \delta = 0,$$

and

$$\begin{aligned} p &= -(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta), \\ q &= \beta\gamma\delta + \alpha\gamma\delta + \alpha\beta\delta + \alpha\beta\gamma, \\ r &= -\alpha\beta\gamma\delta. \end{aligned}$$

Let

$$\begin{aligned} \lambda &= (\alpha+\beta)(\gamma+\delta) = -(\alpha+\beta)^2 = -(\gamma+\delta)^2, \\ \mu &= (\alpha+\gamma)(\beta+\delta) = -(\alpha+\gamma)^2 = -(\beta+\delta)^2, \\ \nu &= (\alpha+\delta)(\beta+\gamma) = -(\alpha+\delta)^2 = -(\beta+\gamma)^2. \end{aligned}$$

We wish to express $\lambda + \mu + \nu$, $\mu\nu + \lambda\nu + \lambda\mu$ and $\lambda\mu\nu$ in terms of p, q and r.

To do this we eliminate α from the above expressions using the identity $\alpha + \beta + \gamma + \delta = 0$. We find

$$\begin{aligned} p &= (\beta+\gamma+\delta)(\beta+\gamma+\delta) - \gamma\delta - \beta\delta - \beta\gamma \\ &= \beta^2 + \gamma^2 + \delta^2 + \gamma\delta + \beta\delta + \beta\gamma, \\ q &= \beta\gamma\delta - (\beta+\gamma+\delta)(\gamma\delta + \beta\delta + \beta\gamma) \\ &= -(\beta^2\gamma + \beta^2\delta + \gamma^2\beta + \gamma^2\delta + \delta^2\beta + \delta^2\gamma) - 2\beta\gamma\delta, \\ r &= \beta^2\gamma\delta + \gamma^2\beta\delta + \delta^2\beta\gamma. \end{aligned}$$

Then

$$\begin{aligned} \lambda + \mu + \nu &= -\Big((\gamma+\delta)^2 + (\beta+\delta)^2 + (\beta+\gamma)^2\Big) \\ &= -2\Big(\beta^2 + \gamma^2 + \delta^2 + \gamma\delta + \beta\delta + \beta\gamma\Big) \\ &= -2p, \end{aligned}$$

$$
\begin{aligned}
\lambda^2 + \mu^2 + \nu^2 &= (\gamma+\delta)^4 + (\beta+\delta)^4 + (\beta+\gamma)^4 \\
&= \gamma^4 + 4\gamma^3\delta + 6\gamma^2\delta^2 + 4\gamma\delta^3 + \delta^4 \\
&\quad + \beta^4 + 4\beta^3\delta + 6\beta^2\delta^2 + 4\beta\delta^3 + \delta^4 \\
&\quad + \beta^4 + 4\beta^3\gamma + 6\beta^2\gamma^2 + 4\beta\gamma^3 + \gamma^4 \\
&= 2(\beta^4 + \gamma^4 + \delta^4) + 4(\beta^3\gamma + \beta^3\delta + \gamma^3\beta + \gamma^3\delta + \delta^3\beta + \delta^3\gamma) \\
&\quad + 6(\gamma^2\delta^2 + \beta^2\delta^2 + \beta^2\gamma^2), \\
p^2 &= \beta^4 + \gamma^4 + \delta^4 + 3(\gamma^2\delta^2 + \beta^2\delta^2 + \beta^2\gamma^2) \\
&\quad + 4(\beta^2\gamma\delta + \gamma^2\beta\delta + \delta^2\beta\gamma) \\
&\quad + 2(\beta^3\gamma + \beta^3\delta + \gamma^3\beta + \gamma^3\delta + \delta^3\beta + \delta^3\gamma).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lambda^2 + \mu^2 + \nu^2 &= 2p^2 - 8(\beta^2\gamma\delta + \gamma^2\beta\delta + \delta^2\beta\gamma) \\
&= 2p^2 - 8r.
\end{aligned}
$$

But

$$4p^2 = (\lambda + \mu + \nu)^2 = \lambda^2 + \mu^2 + \nu^2 + 2(\mu\nu + \lambda\nu + \lambda\mu)$$

Therefore

$$
\begin{aligned}
\mu\nu + \lambda\nu + \lambda\mu &= 2p^2 - \tfrac{1}{2}(\lambda^2 + \mu^2 + \nu^2) \\
&= p^2 + 4r.
\end{aligned}
$$

Finally, we note that

$$\lambda\mu\nu = -\Big((\gamma+\delta)(\beta+\delta)(\beta+\gamma)\Big)^2.$$

Now

$$
\begin{aligned}
(\gamma+\delta)(\beta+\delta)(\beta+\gamma) &= \beta^2\gamma + \beta^2\delta + \gamma^2\beta + \gamma^2\delta + \delta^2\beta + \delta^2\gamma + 2\beta\gamma\delta \\
&= -q.
\end{aligned}
$$

Therefore

$$\lambda\mu\nu = -(-q)^2 = -q^2.$$

Thus λ, μ and ν are the roots of the *resolvent cubic*

$$x^3 + 2px^2 + (p^2 + 4r)x + q^2.$$