Course 311: Group Theory Problems
Academic Year 2007–8

1. Let G be a group. An *automorphism* of G is an isomorphism sending G onto itself. Show that the set $\mathrm{Aut}(G)$ of automorphisms of G is a group with respect to the operation of composition of automorphisms.

2. Let G be a group. The *centre* $Z(G)$ of G is defined by

$$Z(G) = \{z \in G : gz = zg \text{ for all } g \in G\}.$$

Prove that the centre $Z(G)$ of a group G is a normal subgroup of G. [In particular, you should show that $Z(G)$ is a subgroup of G.]

3. Let H be a subgroup of a group G. The *normalizer* $N(H)$ of H in G is defined by $N(H) = \{g \in G : gHg^{-1} = H\}$. Verify that $N(H)$ is a subgroup of G and H is a normal subgroup of $N(H)$.

4. What are the normal subgroups of the alternating group A_4 (which is the group of all even permutations of a set with four elements)?

5. (a) Show that the elements of the alternating group A_5 fall into five conjugacy classes, and calculate the number of elements in each conjugacy class. Verify that the sum of the numbers obtained equals the order of A_5.

(b) Any normal subgroup of A_5 is a union of conjugacy classes. Show how information on the sizes of the conjugacy classes of A_5 can be combined with Lagrange's Theorem to show that the group A_5 is simple.