Course 311, Part III: Commutative Algebra Problems
Michaelmas Term 2005

1. Let R be a unital commutative ring (i.e., a commutative ring with a non-zero multiplicative identity element, denoted by 1, which satisfies $1x = x = x1$ for all $x \in R$). We say that an element x of R is a *unit* if and only if there exists some element x^{-1} of R satisfying $xx^{-1} = 1 = x^{-1}x$.

   (a) Show that the set of units of R is a group with respect to the operation of multiplication.

   (b) Let $x \in R$. Suppose that there exist $s, t \in R$ such that $sx = 1 = xt$. Prove that x is a unit of R.

   (c) Show that any proper ideal I of R cannot contain any units of R.

   (d) Let x be an element of R that is not a unit of R. Show that the set Rx of multiples of x is a proper ideal of R, and that $x \in Rx$.

   (e) Prove that a unital ring R has a exactly one maximal ideal if and only if the set

   $$\{x \in R : X \text{ is not a unit of } R\}$$

   is an ideal of R.

2. (a) Let R be a ring. Let $\hat{R}$ be the set of all infinite sequences

   $$(r_0, r_1, r_2, \ldots)$$

   with $r_i \in R$ for all i, and let operations of addition and multiplication be defined on $\hat{R}$ by the formulae

   $$\begin{aligned}(r_0, r_1, r_2, \ldots) + (s_0, s_1, s_2, \ldots) &= (r_0 + s_0, r_1 + s_1, r_2 + s_2, \ldots),\\ (r_0, r_1, r_2, \ldots)(s_0, s_1, s_2, \ldots) &= (t_0, t_1, t_2, \ldots),\end{aligned}$$

   where $t_0 = r_0 s_0$, $t_1 = r_0 s_1 + s_0 r_1$, and

   $$t_i = r_0 s_i + r_1 s_{i-1} + \cdots + r_{i-1} s_1 + r_i s_0.$$

   Show that $\hat{R}$, with these algebraic operations, is a ring.

(b) Explain why the polynomial ring $R[t]$ is isomorphic to the subring of $\hat{R}$ consisting of all sequences $(r_0, r_1, r_2, \ldots)$ in $\hat{R}$ with the property that $r_i \neq 0$ for at most finitely many values of i.

(c) Suppose that the ring R has a non-zero multiplicative identity element 1. Show that $(1, 0, 0, \ldots)$ is a multiplicative identity element for the ring $\hat{R}$. By examining the formula for the product of two elements of $\hat{R}$, or otherwise, show that an element of $(r_0, r_1, r_2, \ldots)$ of $\hat{R}$ is a unit of $\hat{R}$ if and only if r_0 is a unit of R.

(d) Suppose that that R is an integral domain. Prove that $\hat{R}$ is also an integral domain. [Hint: given non-zero elements $(r_0, r_1, r_2, \ldots)$ and $(s_0, s_1, s_2, \ldots)$ of $\hat{R}$ with product $(t_0, t_1, t_2, \ldots)$, consider t_{m+n}, where m and n are the smallest non-negative integers with the property that $r_m \neq 0$ and $s_n \neq 0$.]

(e) Suppose that R is a field. Prove that $\hat{R}$ has exactly one maximal ideal, and that this maximal ideal consists of all elements $(r_0, r_1, r_2, \ldots)$ of $\hat{R}$ satisfying $r_0 = 0$.

(We can think of an element $(r_0, r_1, r_2, \ldots)$ of the ring $\hat{R}$ as representing a *formal power series*

$$r_0 + r_1 t + r_2 t^2 + \cdots$$

with coefficients in the ring R. Such formal power series are added and multiplied in the obvious fashion. The ring $\hat{R}$ is therefore referred to as the ring of *formal power series* in the indeterminate t with coefficients in the ring R, and is customarily denoted by $R[[t]]$.)

3. Let R be a unital commutative ring.

(a) Let I, J and K be ideals of R. Verify that

$$I + J = J + I, \qquad IJ = JI, \qquad (I + J) + K = I + (J + K),$$

$$(IJ)K = I(JK), \qquad (I + J)K = IK + JK, \qquad I(J + K) = IJ + IK.$$

(Here $I+J$ denotes the ideal of R consisting of all elements of R that are of the form $i+j$ for some $i \in I$ and $j \in J$, and IJ denotes the ideal of R consisting of all elements of R that are of the form $i_1 j_1 + i_2 j_2 + \cdots + i_k j_k$ for some elements $i_1, i_2, \ldots, i_k$ of I and $j_1, j_2, \ldots, j_k$ of J.] Explain why the set of ideals of a ring R is *not* itself a unital commutative ring with respect to these operations of addition and multiplication.

(b) Let I and J be ideals of R satisfying $I + J = R$. Show that $(I + J)^n \subset I + J^n$ for all natural numbers n and hence prove that $I + J^n = R$ for all n. Thus show that $I^m + J^n = R$ for all natural numbers m and n. (The ideal J^n is by definition the set of all elements of R that can be expressed as a finite sum of elements of R of the form $a_1 a_2 \cdots a_n$ with $a_i \in J$ for $i = 1, 2, \ldots, n$.)

(c) Let I and J be ideals of R satisfying $I + J = R$. By considering the ideal $(I \cap J)(I + J)$, or otherwise, show that $IJ = I \cap J$.

4. Let R be a unital commutative ring, and let I be a finitely generated ideal of R. Show that there exists some natural number m such that $I^m \subset \sqrt{I}$, where $\sqrt{I}$ is the radical of I. [Hint: let $\{x_1, x_2, \ldots, x_k\}$ be a finite set that generates the ideal I and let $m = m_1 + m_2 + \cdots + m_k$, where $m_1, m_2, \ldots, m_k$ are chosen such that $x_i^{m_i} \in \sqrt{I}$ for $i = 1, 2, \ldots, k$.]

5. (a) Show that the cubic curve $\{(t, t^2, t^3) \in \mathbb{A}^3(\mathbb{R}) : t \in \mathbb{R}\}$ is an algebraic set.

   (b) Show that the cone $\{(s \cos t, s \sin t, s) \in \mathbb{A}^3(\mathbb{R}) : s, t \in \mathbb{R}\}$ is an algebraic set.

   (c) Show that the unit sphere $\{(z, w) \in \mathbb{A}^2(\mathbb{C}) : |z|^2 + |w|^2 = 1\}$ in $\mathbb{A}^2(\mathbb{C})$ is not an algebraic set.

   (d) Show that the curve $\{(t \cos t, t \sin t, t) \in \mathbb{A}^3(\mathbb{R}) : t \in \mathbb{R}\}$ is not an algebraic set.

6. Let K be a field, and let $\mathbb{A}^n$ denote n-dimensional affine space over the field K.

   Let V and W be algebraic sets in $\mathbb{A}^m$ and $\mathbb{A}^n$ respectively. Show that the Cartesian product $V \times W$ of V and W is an algebraic set in $\mathbb{A}^{m+n}$, where

$$\begin{aligned} V \times W = \{&(x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n) \in \mathbb{A}^{m+n} : \\ &(x_1, x_2, \ldots, x_m) \in V \text{ and } (y_1, y_2, \ldots, y_n) \in W\}. \end{aligned}$$

7. Give an example of a proper ideal I in $\mathbb{R}[X]$ with the property that $V[I] = \emptyset$. [Hint: consider quadratic polynomials in X.]

8. Show that the ideal I of $K[X, Y, Z]$ generated by the polynomials $X^2 + Y^2 + Z^2$ and $XY + YZ + ZX$ is not a radical ideal.

9. Prove that a topological space Z is irreducible if and only if every non-empty open set in Z is connected.

10. Let K be a field, and let $\mathbb{A}^n$ denote n-dimensional affine space over the field K.

    (a) Consider the algebraic set

    $$\{(x, y, z) \in \mathbb{A}^3 : xy = yz = zx = 0\}.$$

    Is this set irreducible? Is it connected (with respect to the Zariski topology)?

    (b) Consider the algebraic set

    $$\{(x, y) \in \mathbb{A}^2(K) : (y - x)(y - x^2) = 0\},$$

    where K is a field with at least 3 elements. Is this set irreducible? Is it connected (with respect to the Zariski topology)?