Course 2BA1: Michaelmas Term 2006.

Sample Problems concerning Sets and Functions

1. (a) Prove that $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$ for all sets A, B and C.

   (b) Prove that $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$ for all sets A, B and C.

   (c) Prove that $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$ for all sets A, B and C.

   (d) Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ for all sets A, B and C.

2. For each of the following relations, determine whether or not that relation is reflexive, symmetric, transitive, anti-symmetric, an equivalence relation, and/or a partial order, giving appropriate reasons for your answers:—

   (a) the relation P on the set $\mathbb{N}$ of natural numbers, where natural numbers m and n satisfy mPn if and only if $m + n$ is divisible by 2;

   (b) the relation Q on the set $\mathbb{N}$ of natural numbers, where natural numbers m and n satisfy mQn if and only if $m + n$ is divisible by 3.

   (c) the relation R on the set $\mathbb{N}$ of natural numbers, where natural numbers m and n satisfy mRn if and only if $n = 2^k m$ for some integer k (which may be positive, zero or negative);

   (d) the relation S on the set $\mathbb{Z}$ of integers, where integers x and y satisfy xSy if and only if $x^2 \leq y^2$;

   (e) the relation Q on the set $\mathbb{Z}$ of integers, where integers x and y satisfy xQy if and only if $x - y = k^3$ for some integer k;

   (f) the relation S on the set $\mathbb{Z}$ of integers, where integers x and y satisfy xSy if and only if xy is even.

   (g) the relation P on the set $\mathbb{Z}$ of integers, where integers x and y satisfy xPy if and only if xy is odd.

(h) the relation Q on the set $\mathbb{R}$ of real numbers, where real numbers x and y satisfy xQy if and only if $y^3 = x^3 - x + y$.

(i) the relation P on the set $\mathbb{R}$ of real numbers, where real numbers x and y satisfy xPy if and only if $x^3 - y^3 + x - y \geq 0$.

3. For each of the following functions, determine whether or not that function is injective and/or surjective, and whether or not it has a well-defined inverse, giving appropriate reasons for your answers:—

(a) the function $f\colon \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ with $f(1) = 2$, $f(2) = 3$, $f(3) = 2$ and $f(4) = 4$;

(b) the function $g\colon \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ with $f(1) = 2$, $f(2) = 3$, $f(3) = 1$ and $f(4) = 4$;

(c) the function $h\colon [1, 2] \to [0, \frac{1}{2}]$ with

$$h(x) = \frac{x-1}{x},$$

where $[1, 2] = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$ and $[0, \frac{1}{2}] = \{x \in \mathbb{R} : 0 \leq x \leq \frac{1}{2}\}$.

(d) the function $f\colon [-1, 1] \to [-2, 2]$ with $f(x) = x^3 - x$ for all $x \in [-1, 1]$.

(e) the function $g\colon [1, 2] \to [0, 6]$ with $g(x) = x^3 - x$ for all $x \in [1, 3]$.

(f) the function $h\colon [0, 1] \to [-2, 2]$ with $h(x) = x^3 - x$ for all $x \in [0, 1]$.

(g) the function $f\colon [-1, 1] \to [-2, 2]$ with $f(x) = x^3 + x$ for all $x \in [-1, 1]$.

(h) the function $g\colon (-1, 1) \to \mathbb{R}$ with $g(x) = \dfrac{1}{1 - x^2}$ for all $x \in (-1, 1)$.