Course 2BA1

Worked Solutions to Assignment V, 2002–03.

Question 1

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 58y = x \sin 2x.$$

The general solution y is of the form $y = y_P + y_C$ where y_P is a particular integral, and y_C is the complementary function.

The complementary function y_C satisfies the equation

$$\frac{d^2y_C}{dx^2} - 6\frac{dy_C}{dx} + 58y_C = 0$$

The auxiliary polynomial for this differential equation is $s^2 - 6s + 58$. This polynomial has roots $3 \pm 7i$. By standard results, the complementary function is then of the form

$$y_C = Ae^{3x} \sin 7x + Be^{3x} \cos 7x.$$

We now have to find a particular integral. We look for one of the form

$$y_P = (Ex + F) \sin 2x + (Gx + H) \cos 2x.$$

Setting $y = y_P$, we find

$$y' = (-2Gx + E - 2H) \sin 2x + (2Ex + 2F + G) \cos 2x.$$

$$y'' = (-4Ex - 4G - 4F) \sin 2x + (-4Gx + 4E - 4H) \cos 2x.$$

and therefore

$$\begin{aligned} y'' - 6y' + 58y &= ((54E + 12G)x + (-6E + 54F - 4G + 12H)) \sin 2x \\ &\quad + ((54G - 12E)x + (-6G + 54H + 4E - 12F)) \cos 2x. \end{aligned}$$

We require $y = x \sin 2x$. We must therefore solve the equations

$$\left.\begin{aligned} 54E + 12G &= 1, \\ 54G - 12E &= 0, \\ -6E + 54F - 4G + 12H &= 0, \\ -6G + 54H + 4E - 12F &= 0 \end{aligned}\right\}$$

for E, F, G, H. We solve the first two for E and G, and the second two for F and H.

Now $54G - 12E = 0 \Rightarrow 9G = 2E$. Then

$$54E + 12G = 1 \Rightarrow (9 \times 27 + 12)G = 1 \Rightarrow 255G = 1.$$

Thus $G = \frac{1}{255}$ and $E = \frac{3}{170}$.

We then find that

$$\begin{aligned} 54F + 12H &= 6E + 4G = \frac{31}{255} \\ 54H - 12F &= 6G - 4E = -\frac{4}{85} \end{aligned}$$

Solving these equations, we find

$$F = \frac{101}{43350}, \quad H = -\frac{23}{65025}.$$

The general solution of the differential equation is therefore

$$\begin{aligned} y &= Ae^{3x} \sin 7x + Be^{3x} \cos 7x + \left(\frac{3}{170} x + \frac{101}{43350} \right) \sin 2x \\ &\quad + \left(\frac{1}{255} x - \frac{23}{65025} \right) \cos 2x. \end{aligned}$$

Note on Question 1

What would motivate one to look for a particular integral of the form

$$y_P = (Ex + F) \sin 2x + (Gx + H) \cos 2x?$$

for appropriate values of E, F, G and H. This is essentially intelligent guesswork (following patterns exhibited in other examples.) Let L be the differential operator with $Ly = y'' - 6y' + 58y$. We need to solve the equation $Ly = x \sin 2x$. It is easy to see that $L(x \sin 2x)$ is a linear combination of the four functions $x \sin 2x$, $\sin 2x$, $x \cos 2x$ and $\cos 2x$. The same is true of $L(\sin 2x)$, $L(x \cos 2x)$ and $L(\cos 2x)$. It follows that that if we apply this differential operator L to a linear combination of $x \sin 2x$, $\cos 2x$, $x \cos 2x$ and $\cos 2x$, we obtain some other linear combination of these functions. Thus if we try to solve the equation

$$L(Ex \sin 2x + F \sin 2x + Gx \cos 2x + H \cos 2x) = x \sin 2x$$

we will end up with four simultaneous (inhomogeneous) linear equations in four unknowns E, F, G and H. In such cases one can usually solve the simulaneous equations for the unknowns.

Question 2

Let $f\colon \mathbb{R} \to \mathbb{R}$ *such that*

$$f(x) = 4(x - m) \text{ if } m \le x \le m + \tfrac{1}{2} \text{ for some integer } m;$$

$$f(x) = 4(m + 1 - x) \text{ if } m + \tfrac{1}{2} \le x \le m + 1 \text{ for some integer } m.$$

Express the function f *as a Fourier series of the form*

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos 2\pi n x.$$

The function f is periodic with period 1 (so that $f(x + 1) = f(x)$ for all x. (Note that m is the greatest integer satisfying $m \le x$, so that if x is replaced by $x + 1$, then m is similarly replaced by $m + 1$.)

The function f is also an even function. Therefore

$$f = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos 2\pi n x,$$

where

$$a_0 = 2\int_0^1 f(x)\, dx = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\, dx = 4\int_0^{\frac{1}{2}} f(x)\, dx,$$

and

$$\begin{aligned} a_n &= 2\int_0^1 f(x)\cos 2\pi n x\, dx = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\cos 2\pi n x\, dx \\ &= 4\int_0^{\frac{1}{2}} f(x)\cos 2\pi n x\, dx, \end{aligned}$$

for $n > 0$. Now $f(x) = 4x$ if $0 \le x \le \frac{1}{2}$. Therefore

$$a_0 = 16\int_0^{\frac{1}{2}} x\, dx = 2,$$

and if $n > 0$ then

$$\begin{aligned} a_n &= 4\int_0^{\frac{1}{2}} f(x)\cos 2\pi n x\, dx = 16\int_0^{\frac{1}{2}} x\cos 2\pi n x\, dx \\ &= \frac{8}{\pi n}\int_0^{\frac{1}{2}} x\frac{d}{dx}(\sin 2\pi n x)\, dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{8}{\pi n}\left[x \sin 2\pi n x\right]_0^{\frac{1}{2}} - \frac{8}{\pi n}\int_0^{\frac{1}{2}} \sin 2\pi n x \, dx \\
&= -\frac{8}{\pi n}\int_0^{\frac{1}{2}} \sin 2\pi n x \, dx = \frac{4}{\pi^2 n^2}\left[\cos 2\pi n x\right]_0^{\frac{1}{2}} \\
&= \frac{4}{\pi^2 n^2}((-1)^n - 1) = \begin{cases} 0 & \text{if } n \text{ is even;} \\ -\frac{8}{\pi^2 n^2} & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}
$$

Thus

$$
f(x) = 1 - \sum_{\substack{n \text{ odd} \\ n>0}} \frac{8}{\pi^2 n^2} \cos 2\pi n x = 1 - \sum_{k=1}^{\infty} \frac{8}{\pi^2 (2k-1)^2} \cos 2\pi (2k-1) x.
$$

Note on Question 2

The algebraic expression defining the function f changes from interval to interval, the intervals being of length $\frac{1}{2}$ and either starting or ending at an integer value. In particular, $f(x)$ is given by the formula $f(x) = 4x$ if $x \in [0, \frac{1}{2}]$, and is given by the formula $f(x) = 4(1-x)$ if $x \in [\frac{1}{2}, 1]$. It follows that one evaluates the integral such as $\int_0^1 f(x)\, dx$ by summing up the integrals over the relevant subintervals. In this instance

$$
\begin{aligned}
\int_0^1 f(x)\, dx &= \int_0^{\frac{1}{2}} f(x)\, dx + \int_{\frac{1}{2}}^1 f(x)\, dx = \int_0^{\frac{1}{2}} 4x \, dx + \int_{\frac{1}{2}}^1 4(1-x)\, dx \\
&= \frac{1}{2} + \frac{1}{2} = 1.
\end{aligned}
$$

Thus in each of the subintervals, we substitute in the formula that gives the value of $f(x)$ over that subinterval.