# Course 2BA1
# Supplement concerning
# Integration by Parts

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## The Rule for Integration by Parts

*Let $u$ and $v$ be continuously differentiable real-valued functions on the interval $[a,b]$. Then*

$$\int_a^b u\,\frac{dv}{dx}\,dx = [uv]_a^b - \int_a^b v\,\frac{du}{dx}\,dx,$$

*where*

$$[uv]_a^b = u(b)v(b) - u(a)v(a).$$

(A function is said to be *continuously differentiable* if it is differentiable and its derivative is a continuous function.)

## Derivation of the Rule for Integration by Parts

The rule for Integration by Parts is a consequence of the Fundamental Theorem of Calculus and the Product Rule for Differentiation.

We begin with the Fundamental Theorem of Calculus.

**Fundamental Theorem of Calculus.** *Let $f$ be a continuous real-valued function on the interval $[a,b]$. Then*

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

*for all $x$ satisfying $a < x < b$.*

Another way of expressing the Fundamental Theorem of Calculus is the following:

$$\frac{d}{dx}\left(\int f(x)\,dx + C\right) = f(x),$$

where $\int f(x)\,dx$ is an indefinite integral of the function $f$ and $C$ is an arbitrary real constant (often referred to as a 'constant of integration').

From the Fundamental Theorem of Calculus we can derive the following basic result:

**Corollary.** *Let $f$ be a continuously differentiable real-valued function on the interval $[a, b]$. Then*

$$\int_a^b \frac{df(x)}{dx}\,dx = f(b) - f(a).$$

In order to derive this result from the Fundamental Theorem of Calculus, define a function $G$ on the interval $[a, b]$ as follows:

$$G(s) = f(s) - f(a) - \int_a^s \frac{df(x)}{dx}\,dx.$$

Then $G(a) = 0$, and

$$\frac{d}{ds}G(s) = \frac{df(s)}{ds} - \frac{d}{ds}\int_a^s \frac{df(x)}{dx}\,dx = \frac{df(s)}{ds} - \frac{df(s)}{ds} = 0$$

for all real numbers $s$ satisfying $a < s < b$. The fact that the derivative of the function $G$ is zero everywhere in the interior of the interval $[a, b]$ is sufficient to ensure that the function is constant on the interval, and thus

$$0 = G(a) = G(b) = f(b) - f(a) - \int_a^b \frac{df(x)}{dx}\,dx,$$

and this identity may be rearranged to yield the required result.

In order to derive the rule for Integration by Parts, we apply the corollary derived above in the case where the function $f$ is the product of two continuously differentiable real-valued functions $u$ and $v$. We find that

$$\int_a^b \frac{d}{dx}\left(u(x)v(x)\right)\,dx = \left[u(x)v(x)\right]_a^b = u(b)v(b) - u(a)v(a).$$

But the Product Rule for differentiation yields

$$\frac{d}{dx}\left(u(x)v(x)\right) = u(x)\frac{dv(x)}{dx} + v(x)\frac{du(x)}{dx}.$$

Therefore

$$\int_a^b u(x)\frac{dv(x)}{dx}\,dx + \int_a^b v(x)\frac{du(x)}{dx}\,dx = u(b)v(b) - u(a)v(a),$$

and thus

$$\int_a^b u(x)\frac{dv(x)}{dx}\,dx = u(b)v(b) - u(a)v(a) - \int_a^b v(x)\frac{du(x)}{dx}\,dx,$$

which is the rule for Integration by Parts.

## Examples of the Use of the Rule for Integration by Parts

**Example** We evaluate the integral $\displaystyle\int_0^1 x(x+2)^3\,dx$ using Integration by Parts. We define $u$ and $v$ so that

$$u(x) = x, \quad \frac{dv(x)}{dx} = (x+2)^3.$$

We may therefore take

$$v(x) = \frac{1}{4}(x+2)^4,$$

and note that

$$\frac{du(x)}{dx} = 1.$$

On substituting in these functions into the formula for Integration by Parts, we find that

$$\begin{aligned}
\int_0^1 x(x+2)^3\,dx &= \left[\frac{1}{4}x(x+2)^4\right]_0^1 - \frac{1}{4}\int_0^1 (x+2)^4\,dx \\
&= \frac{1}{4}\left(1\times 3^4 - 0\times 1^4\right) - \frac{1}{20}\left[(x+2)^5\right]_0^1 \\
&= \frac{81}{4} - \frac{243-32}{20} = \frac{405-211}{20} = \frac{194}{20} = \frac{97}{10}.
\end{aligned}$$

**Example** We evaluate the integral $\displaystyle\int_0^c xe^{-sx}\,dx$ using Integration by Parts. We take

$$u(x) = x, \quad \frac{dv(x)}{dx} = e^{-sx},$$

$$\frac{du(x)}{dx} = 1, \quad v(x) = -\frac{1}{s}e^{-sx},$$

and apply the rule for Integration by Parts, which yields

$$\begin{aligned}
\int_0^c xe^{-sx}\,dx &= \left[-\frac{1}{s}xe^{-sx}\right]_0^c - \int_0^c \left(-\frac{1}{s}e^{-sx}\right)\,dx \\
&= -\frac{c}{s}e^{-sc} + \frac{1}{s}\int_0^c e^{-sx}\,dx \\
&= -\frac{c}{s}e^{-sc} + \frac{1}{s^2}(1 - e^{-sc})
\end{aligned}$$

If $s > 0$, we may take the limit as $c \to \infty$, which gives us

$$\int_0^\infty xe^{-sx}\,dx = \frac{1}{s^2}.$$

**Example** One can use the Principle of Mathematical Induction and the method of Integration by Parts to evaluate $\displaystyle\int_0^\infty x^n e^{-sx}\,dx$ for all positive integers $n$ and positive real numbers $s$. Let $n$ be a positive integer, and let $s > 0$ and $c > 0$. We take

$$u(x) = x^n, \qquad \frac{dv(x)}{dx} = e^{-sx},$$

$$\frac{du(x)}{dx} = nx^{n-1}, \quad v(x) = -\frac{1}{s}e^{-sx},$$

and apply the rule for Integration by Parts, which yields

$$\begin{aligned}
\int_0^c x^n e^{-sx}\,dx &= \left[-\frac{1}{s}x^n e^{-sx}\right]_0^c - \int_0^c \left(-\frac{n}{s}x^{n-1}e^{-sx}\right)\,dx \\
&= -\frac{c^n}{s}e^{-sc} + \frac{n}{s}\int_0^c x^{n-1}e^{-sx}\,dx
\end{aligned}$$

Now if $s > 0$ then $\lim\limits_{x\to\infty} x^n e^{-sx} \to 0$. Thus if we take the limit as $c \to \infty$, we find that

$$\int_0^\infty x^n e^{-sx}\,dx = \frac{n}{s}\int_0^\infty x^{n-1}e^{-sx}\,dx.$$

A simple proof by induction on $n$ now shows that

$$\int_0^\infty x^n e^{-sx}\,dx = \frac{n!}{s^{n+1}}$$

for all positive integers $n$, provided that $s > 0$.

**Example** We evaluate $\int_0^\pi \cos^n x\,dx$. Let us choose the functions $u$ and $v$ so that

$$u(x) = \cos^{n-1} x, \qquad \frac{dv(x)}{dx} = \cos x,$$

$$\frac{du(x)}{dx} = -(n-1)\cos^{n-2} x\,\sin x, \quad v(x) = \sin x.$$

Using the rule for Integration by Parts, we find that

$$\begin{aligned}\int_0^\pi \cos^n x\,dx &= \int_0^\pi \cos^{n-1} x\,\cos x\,dx \\ &= \left[\cos^{n-1} x \sin x\right]_0^\pi - \int_0^\pi \left(-(n-1)\cos^{n-2} x\,\sin^2 x\right)\,dx \\ &= (n-1)\int_0^\pi \cos^{n-2} x\,\sin^2 x\,dx\end{aligned}$$

since $\sin x = 0$ when $x = 0$ and when $x = \pi$. But $\sin^2 x = 1 - \cos^2 x$. Therefore

$$\int_0^\pi \cos^n x\,dx = (n-1)\int_0^\pi \cos^{n-2} x\,dx - (n-1)\int_0^\pi \cos^n x\,dx.$$

On rearranging this identity, we find that

$$\int_0^\pi \cos^n x\,dx = \frac{n-1}{n}\int_0^\pi \cos^{n-2} x\,dx$$

for all natural numbers $n$.

Now

$$\int_0^\pi \cos x\,dx = [\sin x]_0^\pi = 0.$$

It follows that

$$\int_0^\pi \cos^n x\,dx = 0 \quad \text{for all odd positive integers } n.$$

When $n$ is an even positive integer, we may set $n = 2m$ for some positive integer $m$. A straightforward proof by induction on $m$ then shows that

$$\int_0^\pi \cos^{2m} x\,dx = \frac{(2m)!\,\pi}{2^{2m}(m!)^2}$$

for all positive integers $m$.