Course 2BA1, 2008–09
Section 9: Differential Equations

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Contents

9 Differential Equations

9.1 Examples of Differential Equations

A *differential equation* is an equation that relates a function y of a variable x to its derivatives. Such a differential equation can usually be written in the form

$$F\left(\frac{d^p y}{dx^p}, \frac{d^{p-1}y}{dx^{p-1}}, \ldots \frac{dy}{dx}, y, x\right) = 0,$$

where p is a positive integer and F is a real-valued (or complex-valued) function with $p+2$ arguments. If the differential equation can be expressed in the above form for some positive integer p, but cannot be expressed in this form with p replaced by any smaller integer, then the differential equation is said to be of *order* p.

The following are typical examples of differential equations:

$$\frac{dy}{dx} + 2y = 0; \quad (1)$$

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0; \quad (2)$$

$$\frac{dy}{dx} - 2xy = 0; \quad (3)$$

$$\left(\frac{dy}{dx}\right)^2 + y^2 - 1 = 0. \quad (4)$$

Equation (2) is a 2nd order differential equation. The other three equations are first order differential equations.

The function $y = e^{-2x}$ is the solution to the differential equation (1), since

$$\frac{d}{dx}e^{-2x} + 2e^{-2x} = -2e^{-2x} + 2e^{-2x} = 0.$$

It follow easily from this that the function $y = Ae^{-2x}$ solves this differential equation for any constant A.

The function $y = e^{2x}$ solves the differential equation (2), since

$$\frac{d^2}{d^2x}e^{2x} - 4\frac{d}{dx}e^{2x} + 4e^{2x} = 4e^{2x} - 8e^{2x} + 4e^{2x} = 0.$$

The function $y = xe^{2x}$ also solves this differential equation, since

$$\begin{aligned}
&\frac{d^2}{d^2x}(xe^{2x}) - 4\frac{d}{dx}(xe^{2x}) + 4xe^{2x} \\
&\qquad = \frac{d}{dx}((2x+1)e^{2x}) - 4(2x+1)e^{2x} + 4xe^{2x} \\
&\qquad = (4x+4)e^{2x} - 4(2x+1)e^{2x} + 4xe^{2x} = 0
\end{aligned}$$

Now if $y = (Ax + B)e^{2x}$ then $y = Au + Bv$, where $u = xe^{2x}$ and $v = e^{2x}$, and therefore

$$\frac{d^2y}{d^2x} - 4\frac{dy}{dx} + 4y = A\left(\frac{d^2u}{d^2x} - 4\frac{du}{dx} + 4u\right) + B\left(\frac{d^2v}{d^2x} - 4\frac{dv}{dx} + 4v\right) = 0.$$

We conclude that, for any given values of the constants A and B, the function $(Ax + B)e^{2x}$ solves the differential equation (2).

The function $y = e^{x^2}$ is a solution of the differential equation (3). And the functions $y = \sin x$ and $y = \cos x$ are solutions of the differential equation (4).

9.2 Real-Analytic Functions and Power Series

We shall solve certain important types differential equation by representing the solutions that we are seeking as a *power series*, and then determining the constraints on the coefficients of the power series.

Many familiar functions of mathematics may be represented through power series. Let $f: D \to \mathbb{R}$ be a function whose domain D is a subset of the real numbers, and whose values are real numbers, and let $s \in D$. The function f is said to be *real-analytic* at s if there exists some positive real number δ and real numbers $a_0, a_1, a_2, a_3, \ldots$ such that $(s - \delta, s + \delta) \subset D$ and

$$f(s + h) = \sum_{n=0}^{+\infty} a_n h^n$$

for all real numbers h satisfying $-\delta < h < \delta$. The above equation represents the value of $f(s + h)$ as a *power series* in the variable h (for values of h sufficiently close to zero.) The constants $a_0, a_1, a_2, \ldots$ that determine this power series are referred to as the *coefficients* of the power series. The Nth *partial sum* $\sum_{n=0}^{N-1} a_n h^n$ of the power series provides a good approximation to $f(s + h)$ for sufficiently large values of N, where

$$\sum_{n=0}^{N-1} a_n h^n = a_0 + a_1 h + a_2 h^2 + a_3 h^3 + \cdots + a_{N-1} h^{N-1},$$

and the value of this approximation converges on $f(s + h)$ as the value of N increases so that

$$f(s + h) = \lim_{N \to +\infty} \sum_{n=0}^{N-1} a_n h^n$$

for all real numbers h satisfying $-\delta < h < \delta$.

Polynomial functions are real-analytic. Also trigonometrical functions such as sin and cos are real-analytic everywhere, as is the exponential function. Other functions such as the logarithm function are real-analytic over their domains.

A power series representation of a real-analytic function may be differentiated term by term. Thus if f is a real-analytic function, and if

$$f(s+h) = \sum_{n=0}^{+\infty} a_n h^n$$

for all real numbers h satisfying $-\delta < h < \delta$, where the coefficients

$$a_0, a_1, a_2, \ldots$$

are real numbers, then the derivative f' of the function f satisfies

$$f'(s+h) = \frac{d}{dh} f(s+h) = \sum_{n=0}^{+\infty} \frac{d}{dh}(a_n h^n) = \sum_{n=1}^{+\infty} n a_n h^{n-1}.$$

Repetition of this process yields the power series representation of the kth derivative $f^{(k)}(s+h)$ of the function f at $s+h$:

$$f^{(k)}(s+h) = \sum_{n=k}^{+\infty} n(n-1)\cdots(n-k+1) a_n h^{n-k} = \sum_{n=k}^{+\infty} \frac{n!}{(n-k)!} a_n h^{n-k}.$$

(Note that $0! = 1$ by definition. This ensures that $!n = (n-1)!n$ for all positive integers n.) In particular, we may set $h = 0$ in the above identity. Now if $h = 0$ then $h^0 = 1$, and $h^{n-k} = 0$ whenever $n > k$. It follows that all terms of the power series for $f^{(k)}(s+h)$ after the first term are zero when $h = 0$, and therefore

$$f^{(k)}(s) = \frac{k!}{0!} a_k h^0 = k! a_k$$

for all positive integers k. We see from this that the real coefficients

$$a_0, a_2, a_3, \ldots$$

are determined by the derivatives of the function f at s. Specifically $a_n = \dfrac{f^{(n)}}{n!}$ for all non-negative integers n. (Note that $f^{(n)}(s) = f(s)$ and $n! = 0$ when $n = 0$.) It follows that

$$f(s+h) = \sum_{n=0}^{+\infty} \frac{h^n}{n!} f^{(n)}(s) = f(s) + hf'(s) + \frac{h^2}{2!} f''(s) + \frac{h^3}{3!} f'''(s) + \cdots$$

for all real numbers h satisfying $-\delta < h < \delta$. This power series representation of the values of f around s is referred to as the *Taylor series* of the real-analytic function f.

One can show that a number of important functions are real-analytic using a theorem of calculus known as *Taylor's Theorem*. We now state this theorem without proof.

Theorem 9.1 (Taylor's Theorem) *Let s and h be real numbers, and let f be a k times differentiable real-valued function defined on some open interval containing s and $s+h$. Then*

$$f(s+h) = f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + \frac{h^k}{k!} f^{(k)}(s+\theta h)$$

for some real number θ satisfying $0 < \theta < 1$.

Example Consider the exponential function exp, where $\exp x = e^x$ for all real numbers x. This function has the property that

$$\frac{d}{dx} \exp x = \exp x$$

for all real numbers x. Also $\exp 0 = 1$. Therefore, on applying Taylor's Theorem (setting $s = 0$ and $h = x$ in the identity above in the statement of that theorem), we find that, given any real number x, and given any positive integer k, there exists some real number θ satisfying $0 < \theta < 1$ such that

$$\exp x = \sum_{n=0}^{k-1} \frac{x^n}{n!} + \frac{x^k}{k!} \exp(\theta x).$$

The quantity

$$\frac{x^k}{k!} \exp(\theta x)$$

then represents the remainder, or error, that results when the exponential function is approximated by the first k terms of its Taylor series about zero. Now

$$\left| \frac{x^k}{k!} \exp(\theta x) \right| \leq b_k(x)$$

whenever $0 < \theta < 1$, where

$$b_k(x) = \frac{|x|^k}{k!} \exp(|x|)$$

for all real numbers x. Now $b_{k+1}(x) = |x| b_k(x)/(k+1)$. Therefore $b_{k+1}(x) \leq \frac{1}{2} b_k(x)$ when $k > 2|x|$. It follows that $\lim_{k \to +\infty} b_k(x) = 0$. It follows that

$$\begin{aligned} \exp x &= \sum_{n=0}^{+\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots \end{aligned}$$

for all real numbers x.

Example We use Taylor's Theorem to derive power series representations of the sine and cosine functions. Now the derivatives of these functions are as follows:

$$\frac{d}{dx} \sin x = \cos x, \quad \frac{d}{dx} \cos x = -\sin x.$$

It follows that

$$\frac{d^{4m}}{dx^{4m}} \sin x = \sin x, \quad \frac{d^{4m+1}}{dx^{4m+1}} \sin x = \cos x,$$

$$\frac{d^{4m+2}}{dx^{4m+2}} \sin x = -\sin x, \quad \frac{d^{4m+3}}{dx^{4m+3}} \sin x = -\cos x,$$

for all non-negative integers m and real numbers x. Also

$$\frac{d^n}{dx^n} \cos x = \frac{d^{n-1}}{dx^{n-1}} \sin x$$

for all positive integers n and real numbers x. Thus, if we apply Taylor's Theorem to the sine function on the interval between zero and x, we see that given any real number x, there exists some real number θ satisfying $0 < \theta < 1$ such that

$$\sin(x) = \sum_{m=0}^{N-1} \frac{(-1)^m x^{2m+1}}{(2m+1)!} + \frac{(-1)^N x^{2N+1} \cos(\theta x)}{(2N+1)!}.$$

(Note that if $f(x) = \sin x$ for all real numbers x then $f^{(n}(0) = 0$ whenever n is even, and $f^{(2m+1)}(0) = (-1)^m$ for all non-negative integers m.) The expression

$$\frac{(-1)^N x^{2N+1} \cos(\theta x)}{(2N+1)!}$$

therefore represents the remainder, or error, that results when we approximate $\sin x$ by the sum of the first m non-zero terms of the Taylor series of the sine function about zero. Now the sine and cosine functions take values between -1 and $+1$. Therefore

$$\left|\frac{(-1)^N x^{2N+1}\cos(\theta x)}{(2N+1)!}\right| \leq \frac{|x|^{2N+1}}{(2N+1)!}.$$

whenever $0 < \theta < 1$. Moreover

$$\lim_{N\to+\infty} \frac{|x|^{2N+1}}{(2N+1)!} = 0.$$

Indeed let

$$b_N = \frac{|x|^{2N+1}}{(2N+1)!}$$

for all non-negative integers N. Then

$$b_{N+1} = \frac{|x|^2}{(2N+2)(2N+3)} b_N$$

for all non-negative integers N. It follows that $b_{N+1} \leq \frac{1}{4} b_N$ whenever $N > 2|x|$. This is sufficient to ensure that $b_N \to 0$ as $n \to +\infty$.

We conclude therefore that

$$\sin(x) = \sum_{m=0}^{+\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

for all real numbers x. Similarly

$$\cos(x) = \sum_{m=0}^{+\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

for all real numbers x.

Example Let

$$f(x) = \frac{1}{1-x}$$

for all real numbers x satisfying $x \neq 1$. A straightforward proof by induction on n, using standard rules such as the Quotient Rule for differentiation, shows that

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$$

for all non-negative integers n and real numbers x. In particular that $f^{(n)}(0) = n!$ for all non-negative integers n. One can then apply Taylor's Theorem to show that

$$\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

when $-1 < x < 1$. The power series on the right hand side of this inequality fails to converge when $x \geq 1$ and when $x \leq -1$.

Example The natural logarithm function log satisfies

$$\frac{d}{dx} \log x = \frac{1}{x}$$

for all positive real numbers x. It follows that

$$\frac{d^n}{dx^n} \log(1-x) = -\frac{d^{n-1}}{dx^{n-1}} \frac{1}{1-x} = -\frac{(n-1)!}{(1-x)^n}$$

for all positive integers n and for all real numbers x satisfying $x < 1$. One can then apply Taylor's Theorem to show that

$$\log(1-x) = -\sum_{n=1}^{+\infty} \frac{x^n}{n}$$

when $-1 < x < 1$.

9.3 The Differential Equation $\dfrac{dy}{dx} + ay = 0$

Let a be a non-zero real number, and let us seek solutions to the differential equation

$$\frac{dy}{dx} + ay = 0. \tag{5}$$

We suppose that our function y can be represented as a power series in x, of the form

$$y = \sum_{n=0}^{\infty} \frac{y_n}{n!} x^n,$$

where $y_0, y_1, y_2, y_3, \ldots$ are constants to be determined. Now

$$y = y_0 + \sum_{n=0}^{\infty} \frac{y_{n+1}}{(n+1)!} x^{n+1},$$

and

$$\frac{d}{dx}\left(\frac{y_{n+1}}{(n+1)!}x^{n+1}\right) = \frac{(n+1)y_{n+1}}{(n+1)!}x^n = \frac{y_{n+1}}{n!}x^n.$$

It follows that

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} \frac{y_{n+1}}{n!}x^n.$$

(Here we have differentiated the power series for the function y term by term. It can be proved that we are justified in doing so, but we do not attempt such a proof here.) Therefore

$$0 = \frac{dy}{dx} + ay = \sum_{n=0}^{\infty} \frac{y_{n+1} + ay_n}{n!}x^n.$$

Now if the right hand side is to be the zero function, then the coefficient of x^n must be zero for all non-negative integers n, and therefore $y_{n+1} + ay_n = 0$ for all non-negative integers n. Thus $y_n = C(-a)^n$ for all non-negative integers n, where $C = y_0$. But then

$$y = \sum_{n=0}^{\infty} \frac{C(-a)^n x^n}{n!} = C\sum_{n=0}^{\infty} \frac{(-ax)^n}{n!} = Ce^{-ax}.$$

We conclude, therefore, that any solution to the differential equation 5 that can be represented as a power series must be a function y of the variable x that is given by an equation of the form $y = Ce^{-ax}$ for some constant C. (There are no other solutions to this differential equation.)

9.4 The Differential Equation $\dfrac{d^2y}{dx^2} - k^2y = 0$

We now use the method of power series to find solutions to the equation

$$\frac{d^2y}{dx^2} - k^2y = 0, \qquad (6)$$

where k is a real number satisfing $k \neq 0$. Let

$$y = \sum_{n=0}^{\infty} \frac{y_n}{n!}x^n.$$

Then

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} \frac{y_{n+1}}{n!}x^n,$$

and hence

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} \frac{y_{n+2}}{n!} x^n,$$

It follows that the function y satisfies the differential equation 6 if and only if

$$\sum_{n=0}^{\infty} \frac{y_{n+2} - k^2 y_n}{n!} x^n = 0,$$

and thus if and only if

$$y_{n+2} - k^2 y_n = 0$$

for all non-negative integers n. It is then easy to see that the values of $y_2, y_3, y_4, y_5, \ldots$ are determined by the values of y_0 and y_1. Now we can find constants A and B such that $y_0 = A+B$ and $y_1 = Ak - Bk$. (These constants are given by the formulae $A = (ky_0 + y_1)/(2k)$ and $B = (ky_0 - y_1)/(2k)$.) One then readily verify that $y_n = Ak^n + B(-k)^n$ for all non-negative integers n. Therefore

$$y = A \sum_{n=0}^{\infty} \frac{(kx)^n}{n!} + B \sum_{n=0}^{\infty} \frac{(-kx)^n}{n!} = Ae^{kx} + Be^{-kx}.$$

One can readily verify that any function of this form satisfies the differential equation. There are no other solutions.

9.5 The Differential Equation $\dfrac{d^2y}{dx^2} + k^2 y = 0$

Let y be a solution to the differential equation

$$\frac{d^2y}{dx^2} + k^2 y = 0, \tag{7}$$

where k is a real number satisfing $k \neq 0$, and let

$$y = \sum_{n=0}^{\infty} \frac{y_n}{n!} x^n.$$

Then

$$y_{n+2} + k^2 y_n = 0$$

for all non-negative integers n. It is then easy to see that the values of $y_2, y_3, y_4, y_5, \ldots$ are determined by the values of y_0 and y_1. Let $A = y_0$ and

$B = y_1/k$. Then $y_{2m} = (-1)^m A k^{2m}$ and $y_{2m+1} = (-1)^m B k^{2m+1}$ for all non-negative integers m. On referring to the Taylor series for the sine and cosine functions, we find easily that

$$y = A \sum_{m=0}^{\infty} \frac{(-)^m (kx)^{2m}}{(2m)!} + B \sum_{m=0}^{\infty} \frac{(-1)^m (kx)^{2m+1}}{(2m+1)!} = A \cos kx + B \sin kx.$$

It is then easy to verify that the function $A \cos kx + B \sin kx$ does indeed satisfy the differential equation for any values of the constants A and B. There are no other solutions.

9.6 The Differential Equation $\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0$

Let y be a solution to the differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0, \tag{8}$$

and let $u = e^{\frac{bx}{2}} y$. Then $y = e^{-\frac{bx}{2}} u$, and therefore

$$\begin{aligned} \frac{dy}{dx} &= e^{-\frac{bx}{2}} \frac{du}{dx} - \tfrac{1}{2} b e^{-\frac{bx}{2}} u, \\ \frac{d^2y}{dx^2} &= e^{-\frac{bx}{2}} \frac{d^2u}{dx^2} - b e^{-\frac{bx}{2}} \frac{du}{dx} + \tfrac{1}{4} b^2 e^{-\frac{bx}{2}} u. \end{aligned}$$

On substituting these values into the differential equation, we find that

$$e^{-\frac{bx}{2}} \left(\frac{d^2u}{dx^2} - \tfrac{1}{4} b^2 u + cu \right) = 0.$$

Thus the function u is a solution to the differential equation

$$\frac{d^2u}{dx^2} - \tfrac{1}{4}(b^2 - 4c)u = 0.$$

If $b^2 - 4c > 0$, then our previous results show that $u = Ae^{kx} + Be^{-kx}$, where $k = \frac{1}{2}\sqrt{b^2 - 4c}$. It follows that

$$y = Ae^{px} + Be^{qx}$$

where

$$p = \frac{1}{2}(-b + \sqrt{b^2 - 4c}), \quad q = \frac{1}{2}(-b - \sqrt{b^2 - 4c}).$$

Note that p and q are roots of the quadratic polynomial $s^2 + bs + c$.

If $b^2 - 4c = 0$, then the second derivative of the function u vanishes, and therefore $u = Ax + B$. But then

$$y = (Ax + B)e^{-\frac{bx}{2}}.$$

In this case $-\frac{1}{2}b$ is a repeated root of the quadratic polynomial $s^2 + bs + c$.

If $b^2 - 4c < 0$, then $u = A \cos kx + B \sin kx$, where $k = \frac{1}{2}\sqrt{4c - b^2}$. It follows that

$$y = e^{-\frac{bx}{2}}(A \cos kx + B \sin kx) \qquad (k = \frac{1}{2}\sqrt{4c - b^2})$$

In this case $-\frac{1}{2}b \pm ik$ are the roots of the quadratic polynomial $s^2 + bs + c$.

From these observations, we see that the solutions of the differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

can be found from the roots of the associated *auxiliary polynomial* s^2+bs+c, as described in the following theorem.

Theorem 9.2 *Let b and c be real numbers. Then the solutions of the differential equation*

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0,$$

are determined by the roots of the auxiliary polynomial

$$s^2 + bs + c$$

as follows:—

(i) *if $b^2 > 4c$ then the auxiliary polynomial $s^2 + bs + c$ has two real roots r_1 and r_2, and the general solution of the differential equation is given by*

$$y = Ae^{r_1 x} + Be^{r_2 x},$$

where A and B are constants;

(ii) *if $b^2 = 4c$ then the auxiliary polynomial s^2+bs+c has a repeated root r, and the general solution of the differential equation is given by*

$$y = (Ax + B)e^{rx},$$

where A and B are constants;

(iii) *if* $b^2 < 4c$ *then the auxiliary polynomial* $s^2 + bs + c$ *has two non-real roots* $p+iq$ *and* $p-iq$ *(where* p *and* q *are real numbers), and the general solution of the differential equation is given by*

$$y = e^{px}\left(A \sin qx + B \cos qx\right),$$

where A *and* B *are constants.*

Example Consider the differential equation

$$\frac{d^2y}{dx^2} - 11\frac{dy}{dx} + 24y = 0.$$

The auxiliary polynomial associated to this equation is the quadratic polynomial $s^2 - 11s + 24$. This polynomial has two real roots with values 3 and 8. The general solution of this differential equation is therefore of the form

$$y = Ae^{3x} + Be^{8x},$$

where A and B are arbitrary real constants.

Example Consider the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0.$$

The auxiliary polynomial associated to this equation is the quadratic polynomial $s^2 + 4s + 4$. This polynomial has a repeated real root with value -2. The general solution of this differential equation is therefore of the form

$$y = (Ax + B)e^{-2x},$$

where A and B are arbitrary real constants.

Example Consider the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0.$$

The auxiliary polynomial associated to this equation is the quadratic polynomial $s^2 - 4s + 5$. This polynomial has a a pair of non-real roots with values $2 + i$ and $2 - i$. The general solution of this differential equation is therefore of the form

$$y = Ae^{2x} \sin x + Be^{2x} \cos x,$$

where A and B are arbitrary real constants.

9.7 Inhomogeneous Linear Differential Equations of the Second Order with Constant Coefficients

We now discuss the general solution of an *inhomogenous linear differential equation of the second order with constant coefficients.* Such a differential equation is of the form

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x),$$

where b and c are real numbers.

Suppose that y_P is some function of the variable x which satisfies this differential equation. Let y be any twice-differentiable function of the variable x, and let $y_C = y - y_P$. Then

$$\begin{aligned} a\frac{d^2y_C}{dx^2} + b\frac{dy_C}{dx} + cy_C &= a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy - a\frac{d^2y_P}{dx^2} - b\frac{dy_P}{dx} - cy_P \\ &= a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy - f(x). \end{aligned}$$

It follows that the function y satisfies the inhomogeneous differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x),$$

if and only if y_C satisfies the corresponding homogeneous differential equation

$$\frac{d^2y_C}{dx^2} + b\frac{dy_C}{dx} + cy_C = 0,$$

We see therefore that, once a particular solution y_P of the inhomogeneous differential equation has been found, any other solution of the inhomogeneous differential equation may be obtained by adding to y_P a solution y_C of the corresponding homogeneous differential equation. The function y_P is referred to as a *particular integral* of the inhomogeneous differential equation, and the function y_C is referred to as the *complementary function.* Any solution y of the given inhomogeneous differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x),$$

is the sum of the particular integral y_P, which satisfies the same differential equation, and a complementary function y_C, which satisfies the corresponding homogeneous linear differential equation

$$\frac{d^2y_C}{dx^2} + b\frac{dy_C}{dx} + cy_C = 0.$$

Example Let us find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 10y = x^2.$$

We first find a particular integral of this equation. Examination of this equation shows that it might be sensible to look for a particular integral which is a quadratic polynomial in x of the form $px^2 + qx + r$, where the coefficients p, q and r are chosen appropriately. Now if $y = px^2 + qx + r$ then

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 10y = 10px^2 + (10q + 14p)x + 10r + 7q + 2p.$$

If the right hand side of this equation is to equal x^2, then p, q and r must be chosen so as to satisfy the equations

$$10p = 1, \quad 10q + 14p = 0, \quad 10r + 7q + 2p = 0.$$

The solution of these equations is given by

$$p = \frac{1}{10}, \quad q = -\frac{7}{50}, \quad r = -\frac{39}{500}.$$

We conclude that a particular integral y_P of the differential equation is given by

$$y_P = \frac{1}{10}x^2 - \frac{7}{50}x - \frac{39}{500}.$$

The complementary function y_C must satisfy the differential equation

$$\frac{d^2y_C}{dx^2} + 7\frac{dy_C}{dx} + 10y_C = 0.$$

The roots of auxiliary polynomial $s^2 + 7s + 10$ associated to this differential equation are -2 and -5. The complementary function y_C is then of the form

$$y_C = Ae^{-2x} + Be^{-5x}.$$

where A and B are arbitrary real constants. The general solution of the differential equation

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 10y = x^2$$

is then

$$y = \frac{1}{10}x^2 - \frac{7}{50}x - \frac{39}{500} + Ae^{-2x} + Be^{-5x}.$$

Remark Suppose that one is seeking a particular integral of an inhomogeneous differential equation of the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x),$$

where $f(x)$ is a polynomial in x, and $c \neq 0$. There will exist a particular integral y_P of the form $y_P = g(x)$, where $g(x)$ is a polynomial in x of the same degree as $f(x)$. Let

$$f(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n, \quad g(x) = q_0 + q_1 x + q_2 x^2 + \cdots + q_n x^n,$$

If we equate coefficients of powers of x on both sides of the differential equation

$$a\frac{d^2}{dx^2}g(x) + b\frac{d}{dx}g(x) + cg(x) = f(x),$$

we obtain a system of simultaneous linear equations which determine the coefficients $q_0, q_1, \ldots, q_n$ of the polynomial $g(x)$ in terms of the coefficients $p_0, p_1, \ldots, p_n$ of the polynomial $f(x)$. This enables us to find a particular integral of the differential equation.

Example Let us find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \sin x.$$

First we seek a particular integral of this equation. Now

$$\text{if } y = \sin x \text{ then } y'' - 6y' + 9y = 8\sin x - 6\cos x,$$

$$\text{if } y = \cos x \text{ then } y'' - 6y' + 9y = 8\cos x + 6\sin x.$$

Thus if

$$y_P = \frac{1}{50}(4\sin x + 3\cos x)$$

then $y_P'' - 6y_P' + 9y_P = \sin x$, and thus y_P is a particular integral of the inhomogeneous differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \sin x.$$

The complementary function y_C is then a solution of the corresponding homogeneous differential equation $y_C'' - 6y_C' + 9y = 0$. The associated auxiliary polynomial $s^2 - 6s + 9$ has a repeated root, whose value is 3. The complementary function y_C is then given by $y_C = (Ax + B)e^{3x}$, where A and B are real constants. The general solution of the differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \sin x$$

is then given by

$$y = \frac{1}{50}\,(4\sin x + 3\cos x) + (Ax + B)e^{3x}.$$

Example Let us find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = xe^{3x}.$$

Examination of this differential equation suggests that it might be sensible to look for a particular integral of the form $y_P = (p + qx)e^{3x}$, where p and q are appropriately chosen real constants. Now if $y_P = (p + qx)e^{3x}$ then

$$y'_P = (3p + q + 3qx)e^{3x}, \quad y''_P = (9p + 6q + 9qx)e^{3x},$$

and thus

$$y''_P - 2y'_P + 5y_P = (8p + 4q + 8qx)e^{3x}.$$

Thus $y''_P - 2y'_P + 5y_P = xe^{3x}$ if and only if $p = -\frac{1}{16}$ and $q = \frac{1}{8}$. A particular integral y_P of the differential equation is thus given by

$$y_P = \frac{1}{16}(2x - 1)e^{3x}.$$

The complementary function y_C satisfies the differential equation $y''_C - 2y'_C + 5y_C = 0$. The roots of the associated auxiliary polynomial $s^2 - 2s + 5$ are $1 + 2i$ and $1 - 2i$. The complementary function y_C is therefore of the form

$$y_C = Ae^x \sin 2x + Be^x \cos 2x.$$

where A and B are arbitrary real constants. The general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = xe^{3x}$$

is thus given by

$$y = \frac{1}{16}(2x - 1)e^{3x} + Ae^x \sin 2x + Be^x \cos 2x.$$

9.8 Homogeneous and Inhomogeneous Linear Differential Equations of the First Order

We shall describe a method for solving differential equations of the form

$$\frac{dy}{dx} + p(x)y = r(x).$$

Such an equation is a homogeneous linear first order differential equation if $r(x) = 0$ for all x. It is inhomogeneous if the function r is not everywhere zero.

Consider the function $q(x)$ where

$$q(x) = \exp\left(\int p(x)\,dx\right).$$

(Here $\exp u = e^u$ for all real numbers u, and $\int p(x)\,dx$ denotes some indefinite integral of the function p.) On applying the Chain Rule and the Fundamental Theorem of Calculus, we find that

$$\frac{d}{dx}q(x) = \exp\left(\int p(x)\,dx\right)\frac{d}{dx}\int p(x)\,dx = q(x)p(x).$$

Thus

$$p(x) = \frac{q'(x)}{q(x)},$$

where

$$q'(x) = \frac{dq(x)}{dx}.$$

It follows that a function y of x is a solution of the differential equation

$$y'(x) + p(x)y(x) = r(x).$$

if and only if

$$q(x)y'(x) + q'(x)y(x) = q(x)r(x).$$

But

$$q(x)y'(x) + q'(x)y(x) = \frac{d}{dx}\left(q(x)y(x)\right).$$

It follows that the function y satisfies the differential equation

$$y'(x) + p(x)y(x) = r(x)$$

if and only if

$$q(x)y(x) = \int q(x)r(x)\,dx + C,$$

where C is a constant of integration. The general solution of the differential equation. On dividing this equation by $q(x)$, we obtain the following result:

Theorem 9.3 *The general solution of the differential equation*

$$\frac{dy}{dx} + p(x)y = r(x).$$

is thus given by

$$y(x) = \frac{1}{q(x)} \int q(x)r(x)\,dx + \frac{C}{q(x)},$$

where

$$q(x) = \exp\left(\int p(x)\,dx\right),$$

and where C *is some constant.*

The function q is referred to as an *integrating factor* for the differential equation.

Example Consider the differential equation

$$\frac{dy}{dx} + cy = x.$$

The general solution then has the form

$$y(x) = \frac{1}{q(x)} \int q(x)r(x)\,dx + \frac{C}{q(x)},$$

where

$$q(x) = \exp\left(\int c\,dx\right) = e^{cx}$$

and $r(x) = x$. Using the method of Integration by Parts, we find that

$$\begin{aligned} \int_0^x q(s)r(s)\,ds &= \int_0^x se^{cs}\,ds = \left[\frac{1}{c}se^{cs}\right]_0^x - \frac{1}{c}\int_0^x e^{cs}\,ds \\ &= \frac{x}{c}e^{cx} - \frac{1}{c^2}(e^{cx} - 1). \end{aligned}$$

Using this function as an indefinite integral of $q(x)r(x)$, we find that the general solution of the differential equation is given by

$$\begin{aligned} y(x) &= \frac{1}{e^{cx}}\left(\frac{x}{c}e^{cx} - \frac{1}{c^2}(e^{cx} - 1)\right) + \frac{C}{e^{cx}} \\ &= \frac{x}{c} - \frac{1}{c^2}(1 - e^{-cx}) + Ce^{-cx}. \end{aligned}$$

where C is an arbitrary constant. We may write this general solution in the simpler form

$$y(x) = \frac{x}{c} - \frac{1}{c^2} + Ae^{-cx},$$

where A is an arbitrary constant. The constants A and C in these two forms of the general solution are related by the equation

$$A = C + \frac{1}{c^2}.$$

Remark The solution to the differential equation

$$\frac{dy}{dx} + cy = x.$$

is of the form $y_P + y_C$, where y_P is a particular integral given by

$$y_P(x) = \frac{x}{c} - \frac{1}{c^2},$$

and y_C is the complementary function, given by $y_C = Ae^{-cx}$.

Example Consider the differential equation

$$\frac{dy}{dx} + 2xy = 0.$$

The integrating factor $q(x)$ is given by

$$q(x) = \exp\left(\int 2x\,dx\right) = e^{x^2}.$$

The solution to the differential equation therefore takes the form

$$y(x) = \frac{C}{q(x)} = Ce^{-x^2}.$$