

Course 2BA1, 2008–09
Section 7: Quaternions and Rotations

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7 Quaternions and Rotations

7.1 Quaternions

A *quaternion* may be defined to be an expression of the form $w + xi + yj + zk$, where w, x, y and z are real numbers. There are operations of addition, subtraction and multiplication defined on the set \mathbb{H} of quaternions. These are binary operations on that set.

Quaternions were introduced into mathematics in 1843 by William Rowan Hamilton (1805–1865).

The definitions of addition and subtraction are straightforward. The *sum* and *difference* of two quaternions $w + xi + yj + zk$ and $w' + x'i + y'j + z'k$ are given by the formulae

$$\begin{aligned}(w + xi + yj + zk) + (w' + x'i + y'j + z'k) \\ &= (w + w') + (x + x')i + (y + y')j + (z + z')k; \\ (w + xi + yj + zk) - (w' + x'i + y'j + z'k) \\ &= (w - w') + (x - x')i + (y - y')j + (z - z')k.\end{aligned}$$

If the quaternions $w + xi + yj + zk$ and $w' + x'i + y'j + z'k$ are denoted by q and q' respectively, then we may denote the sum and the difference of these quaternions by $q + q'$ and $q - q'$.

These operations of addition and subtraction of quaternions are binary operations on the set \mathbb{H} of quaternions. It is easy to see that the operation of addition is commutative and associative, and that the *zero quaternion* $0 + 0i + 0j + 0k$ is an identity element for the operation of addition.

The operation of subtraction of quaternions is neither commutative nor associative. This results directly from the fact that the operation of subtraction on the set of real numbers is neither commutative nor associative.

Let q be a quaternion. Then $q = w + xi + yj + zk$ for some real numbers w, x, y and z , and there is a corresponding quaternion $-q$, with $-q = (-w) + (-x)i + (-y)j + (-z)k$. Then $q + (-q) = (-q) + q = 0$, where 0 here denotes the zero quaternion $0 + 0i + 0j + 0k$. Thus, to every quaternion q there corresponds a quaternion $-q$ that is the inverse of q with respect to the operation of addition.

These properties of quaternions ensure that the quaternions constitute a group with respect to the operation of addition.

The definition of quaternion multiplication is somewhat more complicated than the definitions of addition and subtraction. The *product* of two

quaternions $w + xi + yj + zk$ and $w' + x'i + y'j + z'k$ is given by the formula

$$\begin{aligned} (w + xi + yj + zk) \times (w' + x'i + y'j + z'k) \\ = (ww' - xx' - yy' - zz') + (wx' + xw' + yz' - zy')i \\ + (wy' + yw' + zx' - xz')j + (wz' + zw' + xy' - yx')k. \end{aligned}$$

We shall often denote the product $q \times q'$ of quaternions q and q' by qq' .

Given any real number w , let us denote the quaternion $w + 0i + 0j + 0k$ by w itself. Let us also denote the quaternions $0 + 1i + 0j + 0k$, $0 + 0i + 1j + 0k$ and $0 + 0i + 0j + 1k$ by i , j and k respectively. It follows directly from the above formula defining multiplication of quaternions that

$$i^2 = j^2 = k^2 = -1,$$

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j,$$

where $i^2 = i \times i$, $ij = i \times j$ etc. It follows directly from these identities that

$$ijk = -1,$$

where $ijk = i \times (j \times k) = (i \times j) \times k$.

Let q be a quaternion, given by the expression $w + xi + yj + zk$, where w , x , y and z are real numbers. One can easily verify that the quaternion q can be formed from the seven quaternions w , x , y , z , i , j and k according to the formula

$$q = w + (x \times i) + (y \times j) + (z \times k).$$

The operation of multiplication on the set \mathbb{H} of quaternions is not commutative. Indeed $i \times j = k$, but $j \times i = -k$.

One can however verify by a straightforward but somewhat tedious calculation that this operation of multiplication of quaternions is associative. Moreover the quaternion $1 + 0i + 0j + 0k$ is an identity element for this operation of multiplication.

A quaternion $w + xi + yj + zk$ is said to be *real* if $x = y = z = 0$. Such a quaternion may be identified with the real number w . In this way the set of real numbers may be regarded as a subset of the set of quaternions.

Although quaternion multiplication is not commutative, one can readily show that $a \times q = q \times a$ for all real numbers a and for all quaternions q . Indeed if $q = w + xi + yj + zk$, where w , x , y and z are real numbers, then the rules of quaternion multiplication ensure that

$$a \times q = q \times a = (aw) + (ax)i + (ay)j + (az)k.$$

Let q be a quaternion. Then $q = w + xi + yj + zk$ for some real numbers w, x, y and z . We define the *conjugate* \bar{q} of q to be the quaternion $\bar{q} = w - xi - yj - zk$. The definition of quaternion multiplication may then be used to show that

$$q \times \bar{q} = \bar{q} \times q = w^2 + x^2 + y^2 + z^2.$$

We define the *modulus* $|q|$ of the quaternion q by the formula

$$|q| = \sqrt{w^2 + x^2 + y^2 + z^2}.$$

Then $q\bar{q} = \bar{q}q = |q|^2$ for all quaternions q . Moreover $|q| = 0$ if and only if $q = 0$.

If q and r are quaternions, and if \bar{q} and \bar{r} denote the conjugates of q and r respectively, then the conjugate $\overline{q \times r}$ of the product $q \times r$ is given by the formula $\overline{q \times r} = \bar{r} \times \bar{q}$.

If q is a non-zero quaternion, and if the quaternion q^{-1} is defined by the formula $q^{-1} = |q|^{-2}\bar{q}$, then $qq^{-1} = q^{-1}q = 1$. We conclude therefore that, given any non-zero quaternion q , there exists a quaternion q^{-1} that is the inverse of q with respect to the property of multiplication.

These properties of quaternions ensure that the non-zero quaternions constitute a group with respect to the operation of multiplication.

7.2 Quaternions and Vectors

Let q be a quaternion. We can write

$$q = q_0 + q_1i + q_2j + q_3k,$$

where q_0, q_1, q_2 and q_3 are real numbers. We can then write

$$q = q_0 + \vec{q}$$

where

$$\vec{q} = q_1i + q_2j + q_3k.$$

Following Hamilton, we can refer to q_0 as the *scalar part* of the quaternion q , and we can refer to \vec{q} as the *vector part* of the quaternion q . Moreover \vec{q} may be identified with the vector (q_1, q_2, q_3) in three-dimensional space whose components (with respect to some fixed orthonormal basis) are q_1, q_2 and q_3 . Thus a quaternion may be regarded as, in some sense, a formal sum of a scalar and a vector.

In particular, we can regard vectors as a special type of quaternion: a quaternion $q_0 + q_1i + q_2j + q_3k$ represents a vector \vec{q} in three-dimensional

space if and only if $q_0 = 0$. Thus vectors are identified with those quaternions whose scalar part is zero.

Now let \vec{q} and \vec{r} be vectors, with Cartesian components (q_1, q_2, q_3) and (r_1, r_2, r_3) respectively. If we consider \vec{q} and \vec{r} to be quaternions (with zero scalar part), and multiply them together in accordance with the rules of quaternion multiplication, we find that

$$\vec{q}\vec{r} = -(\vec{q} \cdot \vec{r}) + (\vec{q} \wedge \vec{r}),$$

where $\vec{q} \cdot \vec{r}$ denotes the *scalar product* of the vectors \vec{q} and \vec{r} , and $\vec{q} \wedge \vec{r}$ denotes the *vector product* of these vectors. Thus the scalar part of the quaternion $\vec{q}\vec{r}$ is $-\vec{q} \cdot \vec{r}$, and the vector part is $\vec{q} \wedge \vec{r}$.

Note that $\vec{q}\vec{r}$ is itself a vector if and only if the vectors \vec{q} and \vec{r} are orthogonal.

More generally, let q and r be quaternions with scalar parts q_0 and r_0 and with vector parts \vec{q} and \vec{r} , so that

$$q = q_0 + \vec{q}, \quad r = r_0 + \vec{r}.$$

Then

$$qr = q_0r_0 - \vec{q} \cdot \vec{r} + q_0\vec{r} + r_0\vec{q} + \vec{q} \wedge \vec{r},$$

and thus the scalar part of the quaternion qr is

$$q_0r_0 - \vec{q} \cdot \vec{r},$$

and the vector part of the quaternion qr is

$$q_0\vec{r} + r_0\vec{q} + \vec{q} \wedge \vec{r}.$$

Now let \vec{u} , \vec{v} and \vec{w} be an orthonormal triad of vectors in three dimensional space, with

$$|\vec{u}| = |\vec{v}| = |\vec{w}| = 1,$$

$$\begin{aligned} \vec{u} \wedge \vec{v} &= -\vec{v} \wedge \vec{u} = \vec{w}, \\ \vec{v} \wedge \vec{w} &= -\vec{w} \wedge \vec{v} = \vec{u}, \\ \vec{w} \wedge \vec{u} &= -\vec{u} \wedge \vec{w} = \vec{v}, \end{aligned}$$

If we multiply these with one another in accordance with the rules for quaternion multiplication, we find that

$$\vec{u}^2 = \vec{v}^2 = \vec{w}^2 = -1,$$

$$\begin{aligned}\vec{u}\vec{v} &= -\vec{v}\vec{u} = \vec{w}, \\ \vec{v}\vec{w} &= -\vec{w}\vec{v} = \vec{u}, \\ \vec{w}\vec{u} &= -\vec{u}\vec{w} = \vec{v},\end{aligned}$$

(Note that the rules for multiplying \vec{u} , \vec{v} and \vec{w} with one another correspond to Hamilton's rules for multiplying the basic quaternions i , j and k with one another, whenever \vec{u} , \vec{v} and \vec{w} constitute a positively oriented basis of three-dimensional space.)

7.3 Quaternions and Rotations

Let us consider the effect of a rotation through an angle θ about an axis in three-dimensional space passing through the origin. Let l , m and n be the cosines of the angles between the axis of the rotation and the three coordinate axes. In Cartesian coordinates, the axis of rotation is then in the direction of the vector (l, m, n) , where $l^2 + m^2 + n^2 = 1$. The angle θ and the direction cosines l , m , n of the axis of the rotation together determine a quaternion q , with

$$q = \cos \frac{\theta}{2} + \sin \frac{\theta}{2}(li + mj + nk).$$

Let \bar{q} be the conjugate of q , given by the formula

$$\bar{q} = \cos \frac{\theta}{2} - \sin \frac{\theta}{2}(li + mj + nk).$$

Let (x, y, z) and (x', y', z') be the Cartesian coordinates of two points in three-dimensional space, and let r and r' be the quaternions r and r' be defined by

$$r = xi + jy + zk \text{ and } r' = x'i + y'j + z'k.$$

We shall show that if $r' = qr\bar{q}$ then a rotation about the axis (l, m, n) through an angle θ will send the point (x, y, z) to the point (x', y', z') . (The effect of a rotation through an angle θ in the opposite sense can be calculated by replacing θ by $-\theta$ in the definition of the quaternion q .)

In this way the algebra of quaternions may be used in areas of application such as computer-aided design and the programming of computer games, in order to calculate the results of rotations applied to points in three-dimensional space.

Let \vec{u} , \vec{v} , \vec{w} be an orthonormal basis of three-dimensional space with $\vec{w} = \vec{u} \wedge \vec{v}$ (as above), and with \vec{u} directed along the axis of the rotation. Let θ be a real number, specifying the angle of rotation, and let q be the

quaternion

$$\begin{aligned} q &= \cos \frac{\theta}{2} + \left(\sin \frac{\theta}{2} \right) \vec{u} \\ &= \cos \frac{\theta}{2} + l \sin \frac{\theta}{2} i + m \sin \frac{\theta}{2} j + n \sin \frac{\theta}{2} k, \end{aligned}$$

where

$$\vec{u} = (l, m, n), \quad l^2 + m^2 + n^2 = 1.$$

Then

$$q^{-1} = \bar{q} = \cos \frac{\theta}{2} - \left(\sin \frac{\theta}{2} \right) \vec{u},$$

since

$$\begin{aligned} q\bar{q} &= \left(\cos \frac{\theta}{2} + \sin \frac{\theta}{2} \vec{u} \right) \left(\cos \frac{\theta}{2} - \sin \frac{\theta}{2} \vec{u} \right) \\ &= \cos^2 \frac{\theta}{2} + \left(\sin^2 \frac{\theta}{2} \right) \vec{u} \cdot \vec{u} \\ &\quad - \left(\sin^2 \frac{\theta}{2} \right) \vec{u} \wedge \vec{u} \\ &= \cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2} \\ &= 1. \end{aligned}$$

Also we find that

$$\begin{aligned} q^2 &= \left(\cos \frac{\theta}{2} + \sin \frac{\theta}{2} \vec{u} \right) \left(\cos \frac{\theta}{2} + \sin \frac{\theta}{2} \vec{u} \right) \\ &= \cos^2 \frac{\theta}{2} - \left(\sin^2 \frac{\theta}{2} \right) \vec{u} \cdot \vec{u} \\ &\quad + 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \vec{u} + \left(\sin^2 \frac{\theta}{2} \right) \vec{u} \wedge \vec{u} \\ &= \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} + 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \vec{u} \\ &= \cos \theta + \sin \theta \vec{u}. \end{aligned}$$

Let us now calculate the quaternion products $q\vec{u}\bar{q}$, $q\vec{v}\bar{q}$ and $q\vec{w}\bar{q}$. We first note that

$$\vec{u}\bar{q} = \bar{q}\vec{u}, \quad \vec{v}\bar{q} = q\vec{v}, \quad \vec{w}\bar{q} = q\vec{w}.$$

Therefore

$$\begin{aligned}
 q\vec{u}\bar{q} &= q\bar{q}\vec{u} = \vec{u}, \\
 q\vec{v}\bar{q} &= q^2\vec{v} = (\cos\theta + \sin\theta\vec{u})\vec{v} \\
 &= \cos\theta\vec{v} + \sin\theta\vec{w}, \\
 q\vec{w}\bar{q} &= q^2\vec{w} = (\cos\theta + \sin\theta\vec{u})\vec{w} \\
 &= \cos\theta\vec{w} - \sin\theta\vec{v}
 \end{aligned}$$

Thus if we define $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ to be the transformation that sends a vector \vec{r} to $q\vec{r}\bar{q}$, then T fixes the vector \vec{u} , rotates the vector \vec{v} about the direction of \vec{u} through an angle θ towards \vec{w} , and rotates \vec{w} about the direction of \vec{u} through an angle θ towards $-\vec{v}$. This transformation T is therefore a rotation about the direction of \vec{u} through an angle θ .

Problems

1. Calculate the products $q \times r$ and $r \times q$ of the quaternions q and r in each of the following cases:

(i) $q = 1 - i$ and $r = 2i - j - k$;

(ii) $q = 2j + k$ and $r = j + 2k$;

(iii) $q = 1 - i - j$ and $r = 2i + j - k$.

(This requires you to express $q \times r$ and $r \times q$ in the form $w + xi + yj + zk$ for appropriate real numbers w, x, y and z).

2. Let q and r be the quaternions defined by $q = \frac{1}{2} + \frac{1}{2}i + \frac{1}{2}j + \frac{1}{2}k$ and $r = i - j$. Calculate $qr\bar{q}$ (where \bar{q} denotes the conjugate of the quaternion q).