Course 2BA1, 2008–09
Section 6: Vectors

David R. Wilkins

Contents

6 Vectors

Vector quantities are objects that have attributes of magnitude and direction. Many physical quantities, such as velocity, acceleration, force, electric field and magnetic field are examples of vector quantities. Displacements between points of space may also be represented using vectors.

Quantities that do not have a sense of direction associated with them are known as *scalar quantities*. Such physical quantities as temperature and energy are scalar quantities. Scalar quantities are usually represented by real numbers.

6.1 Displacement Vectors

Displacements measure the distance and direction necessary to get from one point of space to some other point. Consider the relative locations of the offices of Dr. Smith and Professors Jones and Robinson, which are all to be found within some university building. Let the office of Dr. Smith be on the 2nd floor of the building, and that of Prof. Jones on the 3rd floor. To get from the office of Dr. Smith to that of Prof. Jones it is necessary to walk 40 meters eastwards along a corridor, then up a flight of steps to the floor 4 meters above, and then walk 10 meters westwards, turn a corner, and walk 40 meters northwards. The office of Prof. Jones is therefore situated 30 meters to the east of, 40 meters to the north of, and 4 meters above that of Dr. Smith; and the displacement between the two offices may be represented (in appropriate units), by the ordered triple $(30, 40, 4)$ of real numbers. If the office of Prof. Robinson is located directly beneath that of Prof. Jones, on the 1st floor, then the displacement from the office of Prof. Jones to that of Prof. Robinson is represented by the ordered triple $(0, 0, -8)$ (assuming that the floors of the building are 4 meters apart), and the displacement from the office of Dr. Smith to that of Prof. Robinson is represented by the ordered triple $(30, 40, -4)$.

What is the distance from the office of Dr. Smith to that of Prof. Jones? Let d denote this distance, in metres. And let d' denote the distance, in meters, from Dr. Smith' office to a point on the 2nd floor directly below Prof. Jones's office. Then d' is the length of the hypotenuse of a right-angled triangle whose other sides are of lengths 30 and 40 meters. It follows from Pythagoras's Theorem that $d'^2 = 30^2 + 40^2$, and therefore $d' = 50$. Similarly $d^2 = d'^2 + 4^2 = 2516$, and therefore the direct distance d between the offices of Dr. Smith and Prof. Jones is approximately 50.16 metres.

The ordered triple $(30, 50, 4)$ is said to represent a *displacement vector* of length 50.16 (to two decimal places), and measures the *displacement* between the offices of Dr. Smith and Prof. Jones.

Now let us approach the notion of *displacement vector* more formally. Points of three-dimensional space may be represented, in a Cartesian coordinate system, by ordered triples (x, y, z) of real numbers. Two ordered triples (x_1, y_1, z_1) and (x_2, y_2, z_2) of real numbers represent the same point of three-dimensional space if and only if $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2$. The point whose Cartesian coordinates are given by the ordered triple $(0, 0, 0)$ is referred to as the *origin* of the Cartesian coordinate system.

It is usual to employ a Coordinate system such that the points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are situated at a unit distance from the origin $(0, 0, 0)$, and so that the three lines that join the origin to these points are mutually perpendicular. Moreover it is customary to require that if the thumb of your right hand points in the direction from the origin to the point $(1, 0, 0)$, and if the first finger of that hand points in the direction from the origin to the point $(0, 1, 0)$, and if the second finger of that hand points in a direction perpendicular to the directions of the thumb and first finger, then that second finger points in the direction from the origin to the point $(0, 0, 1)$. (Thus if, at a point on the surface of the earth, away from the north and south pole, the point $(1, 0, 0)$ is located to the east of the origin, and the point $(0, 1, 0)$ is located to the north of the origin, then the point $(0, 0, 1)$ will be located above the origin.

Let P_1, P_2, P_3 and P_4 denote four points of three-dimensional space, represented in a Cartesian coordinate system by ordered triples as follows:

$$P_1 = (x_1, y_1, z_1), \quad P_2 = (x_2, y_2, z_2), \quad P_3 = (x_3, y_3, z_3), \quad P_4 = (x_4, y_4, z_4).$$

The *displacement vector* $\overrightarrow{P_1, P_2}$ from the point P_1 to the point P_2 measures the distance and the direction in which one would have to travel in order to get from P_1 to P_2. This displacement vector may be represented by an ordered triple as follows:

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1).$$

The displacement vector $\overrightarrow{P_3P_4}$ is *equal* to the displacement vector $\overrightarrow{P_1P_2}$ if and only if

$$x_2 - x_1 = x_4 - x_3, \quad y_2 - y_1 = y_4 - y_3, \quad z_2 - z_1 = z_4 - z_3,$$

in which case we represent the fact that these two displacement vectors are equal by writing

$$\overrightarrow{P_1P_2} = \overrightarrow{P_3P_4}.$$

Geometrically, these two displacement vectors are equal if and only if P_1, P_2, P_4 and P_3 are the vertices of a parallelogram in three-dimensional space, in which case

$$x_3 - x_1 = x_4 - x_2, \quad y_3 - y_1 = y_4 - y_2, \quad z_3 - z_1 = z_4 - z_2,$$

and thus

$$\overrightarrow{P_1P_3} = \overrightarrow{P_2P_4}.$$

These displacement vectors may be regarded as objects in their own right, and denoted by symbols of their own: we use a symbol such as $\vec{a}$ to denote the displacement vector $\overrightarrow{P_1P_2}$ from the point P_1 to the point P_2, and we write $\vec{a} = (a_x, a_y, a_z)$ where $a_x = x_2 - x_1$, $a_y = y_2 - y_1$ and $a_z = z_2 - z_1$.

6.2 The Parallelogram Law of Vector Addition

Let P_1, P_2, P_3 and P_4 denote four points of three-dimensional space, located such that $\overrightarrow{P_1P_2} = \overrightarrow{P_3P_4}$. Then (as we have seen) $\overrightarrow{P_1P_3} = \overrightarrow{P_2P_4}$ and the geometrical figure $P_1P_2P_4P_3$ is a parallelogram. Let

$$\vec{a} = \overrightarrow{P_1P_2} = \overrightarrow{P_3P_4}, \quad \vec{b} = \overrightarrow{P_1P_3} = \overrightarrow{P_2P_4}.$$

Let

$$P_1 = (x_1, y_1, z_1), \quad P_2 = (x_2, y_2, z_2), \quad P_3 = (x_3, y_3, z_3), \quad P_4 = (x_4, y_4, z_4).$$

Then $\vec{a} = (a_x, a_y, a_z)$ and $\vec{b} = (b_x, b_y, b_z)$, where

$$a_x = x_2 - x_1 = x_4 - x_3, \quad a_y = y_2 - y_1 = y_4 - y_3, \quad a_z = z_2 - z_1 = z_4 - z_3,$$

$$b_x = x_3 - x_1 = x_4 - x_2, \quad b_y = y_3 - y_1 = y_4 - y_2, \quad b_z = z_3 - z_1 = z_4 - z_2,$$

Let $\vec{e} = \overrightarrow{P_1P_4}$. Then $\vec{e} = (e_x, e_y, e_z)$, where

$$e_x = x_4 - x_1 = a_x + b_x, \quad e_y = y_4 - y_1 = a_y + e_y, \quad e_z = z_4 - z_1 = a_z + e_z,$$

We say that the vector $\vec{e}$ is the *sum* of the vectors $\vec{a}$ and $\vec{b}$, and denote this fact by writing

$$\vec{e} = \vec{a} + \vec{b}.$$

This rule for addition of vectors is known as the *parallelogram rule*, due to its association with the geometry of parallelograms. Note that vectors are added, by adding together the corresponding components of the two vectors. For example,

$$(0, 3, 2) + (4, 8, -5) = (4, 11, -3).$$

Note that

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

for all points A, B and C of space. Also

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

and

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

for all vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ in three-dimensional space. Thus addition of vectors satisfies the Commutative Law and the Associative Law.

The *zero vector* $\vec{0}$ is the vector $(0, 0, 0)$ that represents the displacement from any point in space to itself. The zero vector $\vec{0}$ has the property that

$$\vec{a} + \vec{0} = \vec{a}$$

for all vectors $\vec{a}$. Moroever, given any vector $\vec{a}$, there exists a vector, denoted by $-\vec{a}$, characterized by the property that

$$\vec{a} + (-\vec{a}) = \vec{0}.$$

If $\vec{a} = (a_x, a_y, a_z)$, then $-\vec{a} = (-a_x, -a_y, -a_z)$.

6.3 The Length of Vectors

Let P_1 and P_2 be points in space, and let $\vec{a}$ denote the displacement vector $\overrightarrow{P_1P_2}$ from the point P_1 to the point P_2. If $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ then $\vec{a} = (a_x, a_y, a_z)$ where $a_x = x_2 - x_1$, $a_y = y_2 - y_1$ and $a_z = z_2 - z_1$.

The *length* (or *magnitude*) of the vector $\vec{a}$ is defined to be the distance from the point P_1 to the point P_2. This distance may be calculated using Pythagoras's Theorem. Let $Q = (x_2, y_2, z_1)$ and $R = (x_2, y_1, z_1)$. If the points P_1 and P_2 are distinct, and if $z_1 \neq z_2$, then the triangle P_1QP_2 is a right-angled triangle with hypotenuse P_1P_2, and it follows from Pythagoras's Theorem that

$$P_1P_2^2 = P_1Q^2 + QP_2^2 = P_1Q^2 + (z_2 - z_1)^2.$$

This identity also holds when $P_1 = P_2$, and when $z_1 = z_2$, and therefore holds wherever the points P_1 and P_2 are located. Similarly

$$P_1Q^2 = P_1R^2 + RQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

(since P_1RQ is a right-angled triangle with hypotenuse P_1Q whenever the points P_1, R and Q are distinct), and therefore the length $|\vec{a}|$ of the displacement vector $\vec{a}$ from the point P_1 to the point P_2 satisfies the equation

$$|\vec{a}|^2 = P_1P_2^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = a_x^2 + a_y^2 + a_z^2.$$

In general we define the *length*, or *magnitude*, $|\vec{v}|$ of any vector quantity $\vec{v}$ by the formula

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2},$$

where $\vec{v} = (v_x, v_y, v_z)$. This ensures that the length of any displacement vector is equal to the distance between the two points that determine the displacement.

Example The vector $(3, 4, 12)$ is of length 13, since

$$3^2 + 4^2 + 12^2 = 5^2 + 12^2 = 13^2.$$

A vector whose length is equal to one is said to be a *unit vector*.

6.4 Scalar Multiples of Vectors

Let $\vec{v}$ be a vector, represented by the ordered triple (v_x, v_y, v_z), and let t be a real number. We define $t\vec{v}$ to be the vector represented by the ordered triple (tv_x, tv_y, tv_z). Thus $t\vec{v}$ is the vector obtained on multiplying each of the components of $\vec{v}$ by the real number t.

Note that if $t > 0$ then $t\vec{v}$ is a vector, pointing in the same direction as $\vec{v}$, whose length is obtained on multiplying the length of $\vec{v}$ by the positive real number t.

Similarly if $t < 0$ then $t\vec{v}$ is a vector, pointing in the opposite direction to $\vec{v}$, whose length is obtained on multiplying the length of $\vec{v}$ by the positive real number $|t|$.

Note that

$$(s + t)\vec{a} = s\vec{a} + t\vec{a}, \quad t(\vec{a} + \vec{b}) = t\vec{a} + t\vec{b}, \text{ and } s(t\vec{a}) = (st)\vec{a},$$

for all vectors $\vec{a}$ and $\vec{b}$ and real numbers s and t.

6.5 Linear Combinations of Vectors

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be vectors in three-dimensional space. A vector $\vec{v}$ is said to be a *linear combination* of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ if there exist real numbers $t_1, t_2, \ldots, t_k$ such that

$$\vec{v} = t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_k\vec{v}_k.$$

Let O, P_1 and P_2 be distinct points of three-dimensional space that are not colinear (i.e., that do not all lie on any one line in that space). The displacement vector $\overrightarrow{OP}$ of a point P in three-dimensional space is a linear combination of the displacement vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ if and only if the point P lies in the unique plane that contains the points O, P_1 and P_2.

6.6 Linear Dependence and Independence

Vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are said to be be *linearly dependent* if there exist real numbers $t_1, t_2, \ldots, t_k$, not all zero, such that

$$t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + \vec{v}_k = \vec{0}.$$

If the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are not linearly dependent, then they are said to be *linearly independent*.

Note that if any of the vectors $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k$ is the zero vector, then those vectors are linearly dependent. Indeed if $\vec{v}_i = 0$ then these vectors satisfy a relation of the form

$$t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + \vec{v}_k = \vec{0}.$$

where $t_j = 0$ if $j \neq i$ and $t_i \neq 0$. We conclude that, in any list of linearly independent vectors, the vectors are all non-zero.

Also if any two of the vectors in the list $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k$ are colinear, then these vectors are linearly dependent. For example, if $\vec{v}_1$ and $\vec{v}_2$ are colinear, then they satisfy a relation of the form $t_1\vec{v_1} + t_2\vec{v_2} = \vec{0}$, where t_1 and t_2 are not both zero. If we then set $t_i = 0$ when $i > 2$, then $\sum_{i=1}^{k} t_i\vec{v}_i = \vec{0}$.

If a vector $\vec{v}$ is expressible as a linear combination of vectors $\vec{v_1}, \ldots, \vec{v}_k$ then the vectors $\vec{v_1}, \ldots, \vec{v}_k, \vec{v}$ are linearly dependent. For there exist real numbers $s_1, \ldots, s_k$ such that

$$\vec{v} = s_1\vec{v_1} + s_2\vec{v_2} + \cdots + s_k\vec{v_k},$$

and then

$$s_1\vec{v_1} + s_2\vec{v_2} + \cdots + s_k\vec{v_k} - \vec{v} = 0.$$

Theorem 6.1 *Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors in three-dimensional space which are linearly independent. Then, given any vector $\vec{v}$, there exist unique real numbers p, q and r such that*

$$\vec{v} = p\vec{a}_1 + q\vec{b} + r\vec{c}.$$

Proof First we note that the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are all non-zero, and no two of these vectors are colinear. Let O denote the origin of a Cartesian coordinate system, and let A, B, C and V denote the points of three-dimensional space whose displacement vectors from the origin O are $\vec{a}$, $\vec{b}$ $\vec{c}$ and $\vec{v}$ respectively. The points O, A, B and C are then all distinct, and there is a unique plane which contains the three points O, A and B. The point C does not lie in that plane, since otherwise the displacement vector $\vec{c}$ of this point from the origin would be expressible as a linear combination of the vectors $\vec{a}$ and $\vec{b}$, and the three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ would not then be linearly independent. The set of all points P for which the line PV is parallel to the line OC form a line in three-dimensional space, and this line cannot lie in the plane OAB, and must therefore intersect this plane in a single point Q. Now the displacement vector $\overrightarrow{QV}$ must be a scalar multiple of the vector $\vec{c}$, and therefore $\overrightarrow{QV} = r\vec{c}$ for some real number r. But then

$$\overrightarrow{OQ} = \overrightarrow{OV} + \overrightarrow{VQ} = \overrightarrow{OV} - \overrightarrow{QV} = \vec{v} - r\vec{c}.$$

However the point Q also lies in the plane OAB, and therefore the displacement vector $\overrightarrow{OQ}$ may be expressed as a linear combination of the vectors $\vec{a}$ and $\vec{b}$. Thus there exist real numbers p and q such that

$$\overrightarrow{OQ} = p\vec{a} + q\vec{b}.$$

But then

$$\vec{v} = p\vec{a} + q\vec{b} + r\vec{c}.$$

These real numbers p, q and r are uniquely determined by the vector $\vec{v}$, for if

$$\vec{v} = p_1\vec{a} + q_1\vec{b} + r_1\vec{c} = p_2\vec{a} + q_2\vec{b} + r_2\vec{c}$$

for some real numbers p_1, q_1, r_1, p_2, q_2 and r_2, then

$$(p_2 - p_1)\vec{a} + (q_2 - q_1)\vec{b} + (r_2 - r_1)\vec{c} = \vec{0}.$$

It then follows from the linear independence of the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ that

$$p_2 - p_1 = q_2 - q_1 = r_2 - r_1 = 0,$$

and therefore $p_1 = p_2$, $q_1 = q_2$ and $r_1 = r_2$. This shows the real numbers p, q and r are uniquely determined by the vector $\vec{v}$, as required. ∎

It follows from this theorem that no linearly independent list of vectors in three-dimensional space can contain more than three vectors, since were there a fourth vector in the list, then it would be expressible as a linear combination of the other three, and the vectors would not then be linearly independent.

6.7 The Scalar Product

Let $\vec{a}$ and $\vec{b}$ be vectors in three-dimensional space, represented in some Cartesian coordinate system by the ordered triples (a_1, a_2, a_3) and (b_1, b_2, b_3) respectively. The *scalar product* of the vectors $\vec{a}$ and $\vec{b}$ is defined to be the real number $\vec{a} \,.\, \vec{b}$ defined by the formula

$$\vec{a} \,.\, \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

In particular,

$$\vec{a} \,.\, \vec{a} = a_1^2 + a_2^2 + a_3^2 = |\vec{a}|^2,$$

for any vector $\vec{a}$, where $|\vec{a}|$ denotes the length of the vector $\vec{a}$.

Note that $\vec{a} \,.\, \vec{b} = \vec{b} \,.\, \vec{a}$ for all vectors $\vec{a}$ and $\vec{b}$. Also

$$(s\vec{a} + t\vec{b}) \,.\, \vec{c} = s\vec{a} \,.\, \vec{c} + t\vec{b} \,.\, \vec{c}, \quad \vec{a} \,.\, (s\vec{b} + t\vec{c}) = s\vec{a} \,.\, \vec{b} + t\vec{a} \,.\, \vec{c}$$

for all vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ and real numbers s and t.

Lemma 6.2 *Let $\vec{a}$ and $\vec{b}$ be non-zero vectors in three-dimensional space that are perpendicular to one another. Then $\vec{a} \,.\, \vec{b} = 0$.*

Proof Let O, A and C be the points of three-dimensional space with Cartesian coordinates

$$0 = (0, 0, 0), \quad A = (a_1, a_2, a_3), \quad C = (a_1 + b_1, a_2 + b_2, a_3 + b_3).$$

Then $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{AC} = \vec{b}$ and $\overrightarrow{OC} = \vec{a} + \vec{b}$. The directions of the sides OA and AC of the triangle OAC are those of the vectors $\vec{a}$ and $\vec{b}$, and therefore the triangle OAC is a right-angled triangle (with the right angle located at the vectex A). It follows from well-known geometry (Pythagoras' Theorem) that

$$OC^2 = OA^2 + AC^2.$$

But $OA^2 = |\vec{a}|^2$, $AC^2 = |\vec{b}|^2$ and $OC^2 = |\vec{a} + \vec{b}|^2$. It follows that

$$|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2$$

whenever the vectors $\vec{a}$ and $\vec{b}$ are perpendicular to one another.

However the lengths $|\vec{a}|$, $|\vec{b}|$ and $|\vec{a}+\vec{b}|$ of the vectors $\vec{a}$, $\vec{b}$ and $\vec{a}+\vec{b}$ satisfy the equations

$$|\vec{a}|^2 = a_1^2 + a_2^2 + a_3^2, \quad |\vec{b}|^2 = b_1^2 + b_2^2 + b_3^2,$$

and

$$\begin{aligned} |\vec{a}+\vec{b}|^2 &= (a_1+b_1)^2 + (a_2+b_2)^2 + (a_3+b_3)^2, \\ &= a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 + 2a_1b_1 + 2a_2b_2 + 2a_3b_3 \\ &= |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\,.\,\vec{b} \end{aligned}$$

Thus if the vectors $\vec{a}$ and $\vec{b}$ are perpendicular, then $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2$, and therefore the scalar product of these vectors must satisfy the equation $\vec{a}\,.\,\vec{b} = 0$.

Lemma 6.3 *Let O, A and B be points in three-dimensional space, where neither A nor B coincides with the point O, and let D be the point, lying on the line passing through the points O and A, that is the closest point on that line to the point B. Let $\vec{a}$, $\vec{b}$ and $\vec{d}$ denote the displacement vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OD}$ of the points A, B and D respectively from the point O. Then*

$$\vec{d} = \left(\frac{|\vec{b}|}{|\vec{a}|}\cos\theta\right)\vec{a},$$

where θ denotes the angle between the vectors $\vec{a}$ and $\vec{b}$. Also $\vec{a}\,.\,\vec{b} = \vec{a}\,.\,\vec{d}$.

Proof If the vector $\vec{b}$ points in the same direction as the vector $\vec{a}$ then $\vec{d} = \vec{b}$, $\cos\theta = 1$ and $\vec{b} = t\vec{a}$, where $t = |\vec{b}|/|\vec{a}|$, and the formula for $\vec{d}$ holds.

If the vectors $\vec{a}$ and $\vec{b}$ point in opposite directions then $\vec{d} = \vec{b}$, $\cos\theta = -1$ and $\vec{b} = t\vec{a}$, where $t = -|\vec{b}|/|\vec{a}|$, and the formula for $\vec{d}$ holds.

If the vectors $\vec{a}$ and $\vec{b}$ are perpendicular then the point D is located at the point O (since in this case the point O is closer to B than any other point on the line passing through O and A), and thus $\vec{d} = 0$. Also $\cos\theta = 0$. Both sides of the formula for $\vec{d}$ are equal to the zero vector in this case, and are therefore equal to one another.

It remains to consider the case when the vector $\vec{b}$ is neither parallel nor perpendicular to the vector $\vec{a}$. In this case the points O, D and B are distinct, and are the vertices of a triangle. Moreover this triangle is a right-angled triangle, with a right angle at the vertex D. (Indeed if this angle were not a right angle, then one could construct a right-angled triangle BED with hypotenuse BD, and with a vertex E situated on the line that passes through the points O, A and D. The point E would then be a point on that line which was closer to B than the point D, and this is impossible since D is the closest point to B on that line.) If follows from basic trigonometry that the lengths of the sides OB and OD of the right-angled triangle ODB satisfy the relation $OD = OB|\cos\theta|$, from which it follows that $|\vec{d}| = |\vec{b}||\cos\theta|$.

If $\cos\theta > 0$ then the angle θ is less than a right angle, the points D and A lie on the same side of point O (on the line that passes through the points O, A and D), and

$$\vec{d} == \frac{|\vec{d}|}{|\vec{a}|}\,\vec{a} = \left(\frac{|\vec{b}|}{|\vec{a}|}\,\cos\theta\right)\vec{a}.$$

If $\cos\theta < 0$ then the angle θ is more than a right angle, the points D and A lie on opposite sides of point O, and

$$\vec{d} == -\frac{|\vec{d}|}{|\vec{a}|}\,\vec{a} = -\left(\frac{|\vec{b}|}{|\vec{a}|}\,|\cos\theta|\right)\vec{a} = \left(\frac{|\vec{b}|}{|\vec{a}|}\,\cos\theta\right)\vec{a}.$$

We have therefore verified our formula for the vector $\vec{d}$ in all cases.

One can easily check that if any two of the points O, B and D coincide then either $B = D$ or else $\vec{a}\,.\,\vec{b} = \vec{a}\,.\,\vec{d} = 0$. Thus $\vec{a}\,.\,\vec{b} = \vec{a}\,.\,\vec{d}$ in all cases where any two of the points O, B and D coincide. It only remains to verify the identity $\vec{a}\,.\,\vec{b} = \vec{a}\,.\,\vec{d}$ in the case when the points O, B and D are distinct. In that case the triangle OBD is a right-angled triangle, and its sides OD and DB are perpendicular. But the points O, A and D are colinear. It follows that the displacement vectors $\overrightarrow{OA}$ and $\overrightarrow{DB}$ are perpendicular. Now $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{DB} = \vec{b} - \vec{d}$. It follows from Lemma 6.2 that $\vec{a}\,.\,(\vec{b} - \vec{d}) = 0$. But $\vec{a}\,.\,(\vec{b} - \vec{d}) = \vec{a}\,.\,\vec{b} - \vec{a}\,.\,\vec{d}$. Therefore $\vec{a}\,.\,\vec{b} = \vec{a}\,.\,\vec{d}$, as required. ∎

Proposition 6.4 *Let $\vec{a}$ and $\vec{b}$ be vectors in three-dimensional space $\mathbb{R}^3$. Then their scalar product $\vec{a}\,.\,\vec{b}$ is given by the formula*

$$\vec{a}\,.\,\vec{b} = |\vec{a}|\,|\vec{b}|\,\cos\theta,$$

where θ denotes the angle between the vectors $\vec{a}$ and $\vec{b}$.

Proof We suppose that the vectors $\vec{a}$ and $\vec{b}$ are both non-zero (since if either is the zero vector then both sides of the identity to be proved have the value zero, and the result follows immediately). Let A and B denote the points in three-dimensional space whose displacement vectors from the origin O are given by the vectors $\vec{a}$ and $\vec{b}$. (Thus $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ where a_1, a_2 and a_3 are the Cartesian components of the vector $\vec{a}$, and b_1, b_2 and b_3 are the Cartesian components of the vector $\vec{b}$.) Let D denote the point, situated on the line through the origin and the point A, that is the closest point on that line to the point B, and let $\vec{d} = \overrightarrow{OD}$. Then it follows from Lemma 6.3 that

$$\vec{d} = \left(\frac{|\vec{b}|}{|\vec{a}|} \cos\theta \right) \vec{a}$$

and $\vec{a} \,.\, \vec{b} = \vec{a} \,.\, \vec{d}$. But then

$$\vec{a} \,.\, \vec{b} = \vec{a} \,.\, \vec{d} = \left(\frac{|\vec{b}|}{|\vec{a}|} \cos\theta \right) (\vec{a} \,.\, \vec{a}) = \left(\frac{|\vec{b}|}{|\vec{a}|} \cos\theta \right) |\vec{a}|^2 = |\vec{a}|\, |\vec{b}| \, \cos\theta,$$

as required. ∎

Corollary 6.5 *Two non-zero vectors $\vec{a}$ and $\vec{b}$ in three-dimensional space are perpendicular if and only if $\vec{a} \,.\, \vec{b} = 0$.*

Proof It follows directly from Proposition 6.4 that $\vec{a} \,.\, \vec{b} = 0$ if and only if $\cos\theta = 0$, where θ denotes the angle between the vectors $\vec{a}$ and $\vec{b}$. This is the case if and only if the vectors $\vec{a}$ and $\vec{b}$ are perpendicular.

Example We can use the scalar product to calculate the angle θ between the vectors $(2, 2, 0)$ and $(0, 3, 3)$ in three-dimensional space. Let $\vec{u} = (2, 2, 0)$ and $\vec{v} = (3, 3, 0)$. Then $|\vec{u}|^2 = 2^2 + 2^2 = 8$ and $|\vec{v}|^2 = 3^2 + 3^2 = 18$. It follows that $(|\vec{u}|\, |\vec{v}|)^2 = 8 \times 18 = 144$, and thus $|\vec{u}|\, |\vec{v}| = 12$. Now $\vec{u} \,.\, \vec{v} = 6$. It follows that

$$6 = |\vec{u}|\, |\vec{v}| \, \cos\theta = 12 \cos\theta.$$

Therefore $\cos\theta = \frac{1}{2}$, and thus $\theta = \frac{1}{3}\pi$.

We can use the scalar product to find the distance between points on a sphere. Now the Cartesian coordinates of a point P on the unit sphere about the origin O in three-dimensional space may be expressed in terms of angles θ and φ as follows:

$$P = (\sin\theta \, \cos\varphi, \, \sin\theta \, \sin\varphi, \, \cos\theta).$$

The angle θ is that between the displacement vector $\overrightarrow{OP}$ and the vectical vector $(0, 0, 1)$. Thus the angle $\frac{1}{2}\pi - \theta$ represents the 'latitude' of the point P, when we regard the point $(0, 0, 1)$ as the 'north pole' of the sphere. The angle φ measures the 'longitude' of the point P.

Now let P_1 and P_2 be points on the unit sphere, where

$$\begin{aligned} P_1 &= (\sin\theta_1\,\cos\varphi_1,\, \sin\theta_1\,\sin\varphi_1,\, \cos\theta_1), \\ P_2 &= (\sin\theta_2\,\cos\varphi_2,\, \sin\theta_2\,\sin\varphi_2,\, \cos\theta_2). \end{aligned}$$

We wish to find the angle ψ between the displacement vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ of the points P_1 and P_2 from the origin. Now $|\overrightarrow{OP_1}| = 1$ and $|\overrightarrow{OP_2}| = 1$. On applying Proposition 6.4, we see that

$$\begin{aligned} \cos\psi &= \overrightarrow{OP_1} \,.\, \overrightarrow{OP_2} \\ &= \sin\theta_1\,\sin\theta_2\,\cos\varphi_1\,\cos\varphi_2 + \sin\theta_1\,\sin\theta_2\,\sin\varphi_1\,\sin\varphi_2 \\ &\qquad + \cos\theta_1\,\cos\theta_2 \\ &= \sin\theta_1\,\sin\theta_2\,(\cos\varphi_1\,\cos\varphi_2 + \sin\varphi_1\,\sin\varphi_2) + \cos\theta_1\,\cos\theta_2 \\ &= \sin\theta_1\,\sin\theta_2\,\cos(\varphi_1 - \varphi_2) + \cos\theta_1\,\cos\theta_2. \end{aligned}$$

6.8 The Vector Product

Definition Let $\vec{a}$ and $\vec{b}$ be vectors in three-dimensional space, with Cartesian components given by the formulae $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$. The *vector product* $\vec{a} \times \vec{b}$ of the vectors $\vec{a}$ and $\vec{b}$ is the vector defined by the formula

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2,\, a_3b_1 - a_1b_3,\, a_1b_2 - a_2b_1).$$

Note that $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ for all vectors $\vec{a}$ and $\vec{b}$. Also $\vec{a} \times \vec{a} = \vec{0}$ for all vectors $\vec{a}$. It follows easily from the definition of the vector product that

$$(s\vec{a} + t\vec{b}) \times \vec{c} = s\vec{a} \times \vec{c} + t\vec{b} \times \vec{c}, \quad \vec{a} \times (s\vec{b} + t\vec{c}) = s\vec{a} \times \vec{b} + t\vec{a} \times \vec{c}$$

for all vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ and real numbers s and t.

Proposition 6.6 *Let $\vec{a}$ and $\vec{b}$ be vectors in three-dimensional space $\mathbb{R}^3$. Then their vector product $\vec{a} \times \vec{b}$ is a vector of length $|\vec{a}|\,|\vec{b}|\,|\sin\theta|$, where θ denotes the angle between the vectors $\vec{a}$ and $\vec{b}$. Moreover the vector $\vec{a} \times \vec{b}$ is perpendicular to the vectors $\vec{a}$ and $\vec{b}$.*

Proof Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, and let l denote the length $|\vec{a} \times \vec{b}|$ of the vector $\vec{a} \times \vec{b}$. Then

$$\begin{aligned} l^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\ &= a_2^2b_3^2 + a_3^2b_2^2 - 2a_2a_3b_2b_3 \end{aligned}$$

$$
\begin{aligned}
& + a_3^2 b_1^2 + a_1^2 b_3^2 - 2a_3 a_1 b_3 b_1 \\
& + a_1^2 b_2^2 + a_2^2 b_1^2 - 2a_1 a_2 b_1 b_2 \\
= \; & a_1^2(b_2^2 + b_3^2) + a_2^2(b_1^2 + b_3^2) + a_3^2(b_1^2 + b_2^2) \\
& - 2a_2 a_3 b_2 b_3 - 2a_3 a_1 b_3 b_1 - 2a_1 a_2 b_1 b_2 \\
= \; & (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \\
& - a_1^2 b_1^2 - a_2^2 b_2^2 - a_3^2 b_3^2 - 2a_2 b_2 a_3 b_3 - 2a_3 b_3 a_1 b_1 - 2a_1 b_1 a_2 b_2 \\
= \; & (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \\
= \; & |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \,.\, \vec{b})^2
\end{aligned}
$$

since

$$
|\vec{a}|^2 = a_1^2 + a_2^2 + a_3^2, \quad |\vec{b}|^2 = b_1^2 + b_2^2 + b_3^2, \quad \vec{a} \,.\, \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3
$$

But $\vec{a} \,.\, \vec{b} = |\vec{a}|\,|\vec{b}|\,\cos\theta$ (Proposition 6.4). Therefore

$$
l^2 = |\vec{a}|^2 |\vec{b}|^2 (1 - \cos^2\theta) = |\vec{a}|^2 |\vec{b}|^2 \sin^2\theta
$$

(since $\sin^2\theta + \cos^2\theta = 1$ for all angles θ) and thus $l = |\vec{a}|\,|\vec{b}|\,|\sin\theta|$. Also

$$
\vec{a} \,.\, (\vec{a} \times \vec{b}) = a_1(a_2 b_3 - a_3 b_2) + a_2(a_3 b_1 - a_1 b_3) + a_3(a_1 b_2 - a_2 b_1) = 0
$$

and

$$
\vec{b} \,.\, (\vec{a} \times \vec{b}) = b_1(a_2 b_3 - a_3 b_2) + b_2(a_3 b_1 - a_1 b_3) + b_3(a_1 b_2 - a_2 b_1) = 0
$$

and therefore the vector $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$ (Corollary 6.5), as required. ∎

Using elementary geometry, and the formula for the length of the vector product $\vec{a} \times \vec{b}$ given by Proposition 6.6 it is not difficult to show that the length of this vector product is equal to the area of a parallelogram in three-dimensional space whose sides are represented, in length and direction, by the vectors $\vec{a}$ and $\vec{b}$.

Remark Let $\vec{a}$ and $\vec{b}$ be non-zero vectors that are not colinear (i.e., so that they do not point in the same direction, or in opposite directions). The direction of $\vec{a} \times \vec{b}$ may be determined, using the thumb and first two fingers of your right hand, as follows. Orient your right hand such that the thumb points in the direction of the vector $\vec{a}$ and the first finger points in the direction of the vector $\vec{b}$, and let your second finger point outwards from the palm of your hand so that it is perpendicular to both the thumb and the first finger. Then the second finger points in the direction of the vector product $\vec{a} \times \vec{b}$.

Indeed it is customary to describe points of three-dimensional space by Cartesian coordinates (x, y, z) oriented so that if the positive x-axis and positive y-axis are pointed in the directions of the thumb and first finger respectively of your right hand, then the positive z-axis is pointed in the direction of the second finger of that hand, when the thumb and first two fingers are mutually perpendicular. For example, if the positive x-axis points towards the East, and the positive y-axis points towards the North, then the positive z-axis is chosen so that it points upwards. Moreover if $\vec{\imath} = (1, 0, 0)$ and $\vec{\jmath} = (0, 1, 0)$ then these vectors $\vec{\imath}$ and $\vec{\jmath}$ are unit vectors pointed in the direction of the positive x-axis and positive y-axis respectively, and $\vec{\imath} \times \vec{\jmath} = \vec{k}$, where $\vec{k} = (0, 0, 1)$, and the vector $\vec{k}$ points in the direction of the positive z-axis. Thus the 'right-hand' rule for determining the direction of the vector product $\vec{a} \times \vec{b}$ using the fingers of your right hand is valid when $\vec{a} = \vec{\imath}$ and $\vec{b} = \vec{\jmath}$.

If the directions of the vectors $\vec{a}$ and $\vec{b}$ are allowed to vary continuously, in such a way that these vectors never point either in the same direction or in opposite directions, then their vector product $\vec{a} \times \vec{b}$ will always be a non-zero vector, whose direction will vary continuously with the directions of $\vec{a}$ and $\vec{b}$. It follows from this that if the 'right-hand rule' for determining the direction of $\vec{a} \times \vec{b}$ applies when $\vec{a} = \vec{\imath}$ and $\vec{b} = \vec{\jmath}$, then it will also apply whatever the directions of $\vec{a}$ and $\vec{b}$, since, if your right hand is moved around in such a way that the thumb and first finger never point in the same direction, and if the second finger is always perpendicular to the thumb and first finger, then the direction of the second finger will vary continuously, and will therefore always point in the direction of the vector product of two vectors pointed in the direction of the thumb and first finger respectively.

Example We shall find the area of the parallelogram $OACB$ in three-dimensional space, where

$$O = (0, 0, 0), \quad A = (1, 2, 0), \qquad B = (-4, 2, -5), \quad C = (-3, 4, -5).$$

Note that $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$. Let $\vec{a} = \overrightarrow{OA} = (1, 2, 0)$ and $\vec{b} = \overrightarrow{OB} = (-4, 2, -5)$. Then $\vec{a} \times \vec{b} = (-10, 5, 10)$. Now $(-10, 5, 10) = 5(-2, 1, 2)$, and $|(-2, 1, 2)| = \sqrt{9} = 3$. It follows that

$$\text{area}\, OACB = |\vec{a} \times \vec{b}| = 15.$$

Note also that the vector $(-2, 1, 2)$ is perpendicular to the parallelogram $OACB$.

Example We shall find the equation of the plane containing the points A, B and C where $A = (3, 4, 1)$, $B = (4, 6, 1)$ and $C = (3, 5, 3)$. Now if $\vec{u} = \overrightarrow{AB} = (1, 2, 0)$ and $\vec{v} = \overrightarrow{AC} = (0, 1, 2)$ then the vectors $\vec{u}$ and $\vec{v}$ are parallel to the plane. It follows that the vector $\vec{u} \times \vec{v}$ is perpendicular to this plane. Now $\vec{u} \times \vec{v} = (4, -2, 1)$, and therefore the displacement vector between any two points of the plane must be perpendicular to the vector $(4, -2, 1)$. It follows that the function mapping the point (x, y, z) to the quantity $4x - 2y + z$ must be constant throughout the plane. Thus the equation of the plane takes the form

$$4x - 2y + z = k,$$

for some constant k. We can calculate the value of k by substituting for x, y and z the coordinates of any chosen point of the plane. On taking this chosen point to be the point A, we find that $k = 4 \times 3 - 2 \times 4 + 1 = 5$. Thus the equation of the plane is the following:

$$4x - 2y + z = 5.$$

(We can check our result by verifying that the coordinates of the points A, B and C do indeed satisfy this equation.)

6.9 Scalar Triple Products

Given three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ in three-dimensional space, we can form the *scalar triple product* $\vec{a} \,.\, (\vec{b} \times \vec{c})$. This quantity can be expressed as the determinant of a 3×3 matrix whose rows contain the Cartesian components of the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$. Indeed

$$\vec{b} \times \vec{c} = (b_2c_3 - b_3c_2,\ b_3c_1 - b_1c_3,\ b_1c_2 - b_2c_1),$$

and thus

$$\vec{a} \,.\, (\vec{b} \times \vec{c}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1).$$

The quantity on the right hand side of this equality defines the determinant of the 3×3 matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

We have therefore obtained the following result.

Lemma 6.7 *Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be vectors in three-dimensional space. Then*

$$\vec{a} \,.\, (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Using basic properties of determinants, or by direct calculation, one can easily obtain the identities

$$\begin{aligned} \vec{a} \,.\, (\vec{b} \times \vec{c}) &= \vec{b} \,.\, (\vec{c} \times \vec{a}) = \vec{c} \,.\, (\vec{a} \times \vec{b}) \\ &= -\vec{a} \,.\, (\vec{c} \times \vec{b}) = -\vec{b} \,.\, (\vec{a} \times \vec{c}) = -\vec{c} \,.\, (\vec{b} \times \vec{a}) \end{aligned}$$

One can show that the absolute value of the scalar triple product $\vec{a} \,.\, (\vec{b} \times \vec{c})$ is the volume of the parallelepiped in three-dimensional space whose vertices are the points whose displacement vectors from some fixed point O are $\vec{0}$, $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{a} + \vec{b}$, $\vec{a} + \vec{c}$, $\vec{b} + \vec{c}$ and $\vec{a} + \vec{b} + \vec{c}$. (A *parallelepiped* is a solid like a brick, but whereas the faces of a brick are rectangles, the faces of the parallelepiped are parallelograms.)

Example We shall find the volume of the parallelepiped in 3-dimensional space with vertices at $(0, 0, 0)$, $(1, 2, 0)$, $(-4, 2, -5)$, $(0, 1, 1)$, $(-3, 4, -5)$, $(1, 3, 1)$, $(-4, 3, -4)$ and $(-3, 5, -4)$. The volume of this parallelepiped is the absolute value of the scalar triple product $\vec{a} \,.\, (\vec{b} \times \vec{c})$, where

$$\vec{a} = (1, 2, 0), \quad \vec{b} = (-4, 2, -5), \quad \vec{c} = (0, 1, 1).$$

Now

$$\begin{aligned} \vec{a} \,.\, (\vec{b} \times \vec{c}) &= (1, 2, 0) \,.\, (\,(-4, 2, -5) \times (0, 1, 1)\,) \\ &= (1, 2, 0) \,.\, (7, 4, -4) = 7 + 2 \times 4 = 15. \end{aligned}$$

Thus the volume of the paralellepiped is 15 units.

6.10 The Vector Triple Product Identity

Proposition 6.8 *Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be vectors in three-dimensional space. Then*

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \,.\, \vec{c})\, \vec{b} - (\vec{a} \,.\, \vec{b})\, \vec{c}.$$

Proof Let $\vec{d} = \vec{a} \times (\vec{b} \times \vec{c})$, and let $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$, $\vec{c} = (c_1, c_2, c_3)$, and $\vec{d} = (d_1, d_2, d_3)$. Then

$$\vec{b} \times \vec{c} = (b_2c_3 - b_3c_2,\, b_3c_1 - b_1c_3,\, b_1c_2 - b_2c_1).$$

and hence $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{d} = (d_1, d_2, d_3)$, where

$$\begin{aligned} d_1 &= a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3) \\ &= (a_2c_2 + a_3c_3)b_1 - (a_2b_2 + a_3b_3)c_1 \\ &= (a_1c_1 + a_2c_2 + a_3c_3)b_1 - (a_1b_1 + a_2b_2 + a_3b_3)c_1 \\ &= (\vec{a} \,.\, \vec{c})b_1 - (\vec{a} \,.\, \vec{b})c_1 \end{aligned}$$

Similarly

$$d_2 = (\vec{a} \,.\, \vec{c})b_2 - (\vec{a} \,.\, \vec{b})c_2$$

and

$$d_3 = (\vec{a} \,.\, \vec{c})b_2 - (\vec{a} \,.\, \vec{b})c_3$$

(In order to verify the formula for d_2 with an minimum of calculation, take the formulae above involving d_1, and cyclicly permute the subcripts 1, 2 and 3, replacing 1 by 2, 2 by 3, and 3 by 1. A further cyclic permutation of these subscripts yields the formula for d_3.) It follows that

$$\vec{d} = (\vec{a} \,.\, \vec{c})\,\vec{b} - (\vec{a} \,.\, \vec{b})\,\vec{c},$$

as required, since we have shown that the Cartesian components of the vectors on either side of this identity are equal. ∎

6.11 Orthonormal Triads of Unit Vectors

Let $\vec{u}$ and $\vec{v}$ be unit vectors (i.e., vectors of length one) that are perpendicular to each other, and let $\vec{w} = \vec{u} \times \vec{v}$. It follows immediately from Proposition 6.6 that $|\vec{w}| = |\vec{u}|\,|\vec{v}| = 1$, and that this unit vector $\vec{w}$ is perpendicular to both $\vec{u}$ and $\vec{v}$. Then

$$\vec{u} \,.\, \vec{u} = \vec{v} \,.\, \vec{v} = \vec{w} \,.\, \vec{w} = 1$$

and

$$\vec{u} \,.\, \vec{v} = \vec{v} \,.\, \vec{w} = \vec{w} \,.\, \vec{u} = 0.$$

On applying the Vector Triple Product Identity (Proposition 6.8) we find that

$$\vec{v} \times \vec{w} = \vec{v} \times (\vec{u} \times \vec{v}) = (\vec{v} \,.\, \vec{v})\,\vec{u} - (\vec{v} \,.\, \vec{u})\,\vec{v} = \vec{u},$$

and

$$\vec{w} \times \vec{u} = -\vec{u} \times \vec{w} = -\vec{u} \times (\vec{u} \times \vec{v}) = -(\vec{u} \,.\, \vec{v})\,\vec{u} + (\vec{u} \,.\, \vec{u})\,\vec{v} = \vec{v},$$

Therefore

$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u} = \vec{w}, \quad \vec{v} \times \vec{w} = -\vec{w} \times \vec{v} = \vec{u}, \quad \vec{w} \times \vec{u} = -\vec{u} \times \vec{w} = \vec{v},$$

Three unit vectors, such as the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ above, that are mutually perpendicular, are referred to as an *orthonormal triad* of vectors in three-dimensional space. The vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in any orthonormal triad are linearly independent. It follows directly from Theorem 6.1 that any vector in three-dimensional space may be expressed, uniquely, as a linear combination of the form

$$p\vec{u} + q\vec{v} + r\vec{w}.$$

Any Cartesian coordinate system on three-dimensional space determines an orthonormal triad $\vec{\imath}$, $\vec{\jmath}$ and $\vec{k}$, where

$$\vec{\imath} = (1, 0, 0), \quad \vec{\jmath} = (0, 1, 0), \quad \vec{k} = (0, 0, 1).$$

The scalar and vector products of these vectors satisfy the same relations as the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ above. A vector represented in these Cartesian components by an ordered triple (x, y, z) then satisfies the identity

$$(x, y, z) = x\vec{\imath} + y\vec{\jmath} + z\vec{k}.$$