Course 2BA1: Michaelmas Term 2006
Section 4: Introduction to Fourier Methods

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Contents

4 Introduction to Fourier Methods

4.1 Representation of Periodic Sequences

Definition A *doubly-infinite* sequence $(z_n : n \in \mathbb{Z})$ of complex numbers associates to every integer n a corresponding complex number z_n.

Definition We say that doubly-infinite sequence $(z_n : n \in \mathbb{Z})$ of complex numbers is *m-periodic* if $z_{n+m} = z_n$ for all integers n.

Lemma 4.1 *Let m be a positive integer, and let $\omega_m = e^{2\pi i/m}$. Then the value of $\sum_{k=0}^{m-1} \omega_m^{kn}$ is determined, for any integer n, as follows:*

$$\sum_{k=0}^{m-1} \omega_m^{kn} = \begin{cases} m & \text{if } n \text{ is divisible by } m; \\ 0 & \text{if } n \text{ is not divisible by } m. \end{cases}$$

Proof The complex number ω_m has the property that $\omega_m^m = 1$. Also

$$(1 - z)(1 + z + z^2 + \cdots + z^{m-1}) = 1 - z^m$$

for any complex number z. It follows that

$$(1 - \omega_m^n) \sum_{k=0}^{m-1} \omega_m^{kn} = 1 - \omega_m^{mn} = 0$$

for all integers n, and therefore

$$\sum_{k=0}^{m-1} \omega_m^{kn} = 0 \quad \text{provided that} \quad \omega_m^n \neq 1.$$

Now $\omega_m^n = 1$ if and only if the integer n is divisible by m. We can therefore conclude that

$$\sum_{k=0}^{m-1} \omega_m^{kn} = \begin{cases} m & \text{if } n \text{ is divisible by } m, \\ 0 & \text{if } n \text{ is not divisible by } m, \end{cases}$$

as required. ∎

Theorem 4.2 *Let $(z_n : n \in \mathbb{Z})$ be a doubly-infinite sequence of complex numbers which is m-periodic . Then*

$$z_n = \sum_{k=0}^{m-1} c_k \omega_m^{kn},$$

for all integers n, where $\omega_m = e^{2\pi i/m}$ and

$$c_k = \frac{1}{m} \sum_{j=0}^{m-1} z_j \omega_m^{-kj}.$$

Proof It follows from the definition of the numbers c_k that

$$\sum_{k=0}^{m-1} c_k \omega_m^{kn} = \frac{1}{m} \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} z_j \omega_m^{-kj} \omega_m^{kn} = \frac{1}{m} \sum_{j=0}^{m-1} \left(z_j \sum_{k=0}^{m-1} \omega_m^{(n-j)k} \right),$$

for all integers n. Now it follows from Lemma 4.1 that

$$\sum_{k=0}^{m-1} \omega_m^{(n-j)k} = 0$$

unless $n-j$ is divisible by m, in which case

$$\sum_{k=0}^{m-1} \omega_m^{(n-j)k} = m.$$

Moreover, given any integer n, there is a unique integer r between 0 and $m-1$ for which $n-r$ is divisible by m. It follows that

$$\sum_{k=0}^{m-1} c_k \omega_m^{kn} = z_r \quad \text{where } 0 \le r < m \text{ and } r \equiv n \pmod{m}.$$

Moreover $z_r = z_n$, because the sequence $(z_n : n \in \mathbb{Z})$ is m-periodic. Thus

$$\sum_{k=0}^{m-1} c_k \omega_m^{kn} = z_n$$

for all integers n, as required. ∎

Example Let $(z_n : n \in \mathbb{Z})$ be an 3-periodic sequence with $z_0 = 2$, $z_1 = 4$, $z_2 = 5$. Let $\omega = \omega_3 = e^{2\pi i/3}$. It follows from Theorem 4.2 that

$$z_n = c_0 + c_1\omega^n + c_2\omega^{2n}$$

for all integers n, where $\omega_m = e^{2\pi i/m}$ and

$$c_k = \frac{1}{3}\left(z_0 + z_1\omega^{-k} + z_2\omega^{-2k}\right).$$

for $k = 0, 1, 2$. Now $\omega^{-1} = \omega^2$ and $\omega^{-2} = \omega$, because $\omega^3 = 1$. Therefore

$$c_k = \frac{1}{3}\left(z_0 + z_1\omega^{2k} + z_2\omega^{k}\right),$$

and thus

$$\begin{aligned} c_0 &= \tfrac{1}{3}(2+4+5) = \tfrac{11}{3}, \\ c_1 &= \tfrac{1}{3}(2+4\omega^2+5\omega), \\ c_2 &= \tfrac{1}{3}(2+4\omega+5\omega^2). \end{aligned}$$

Now

$$\begin{aligned} \omega &= \cos\tfrac{2\pi}{3} + i\sin\tfrac{2\pi}{3} = \tfrac{1}{2}(-1+\sqrt{3}\,i), \\ \omega^2 &= \cos\tfrac{4\pi}{3} + i\sin\tfrac{4\pi}{3} = \tfrac{1}{2}(-1-\sqrt{3}\,i). \end{aligned}$$

It follows that

$$c_1 = \tfrac{1}{6}(-5+\sqrt{3}\,i), \quad c_2 = \tfrac{1}{6}(-5-\sqrt{3}\,i).$$

Example Let $(z_n : n \in \mathbb{Z})$ be an 4-periodic sequence with $z_0 = 2$, $z_1 = 4$, $z_2 = 5$, $z_3 = 1$. Now if ω_4 is defined as in the statement of Theorem 4.2 then $\omega_4 = e^{2\pi i/4} = i$. It follows from Theorem 4.2 that

$$z_n = c_0 + c_1 i^n + c_2(-1)^n + c_3(-i)^n$$

for all integers n, where $\omega_m = e^{2\pi i/m}$ and

$$\begin{aligned} c_k &= \frac{1}{4}\left(z_0 + z_1 i^{-k} + z_2 i^{-2k} + z_3 i^{-3k}\right) \\ &= \frac{1}{4}\left(2 + 4 \times (-i)^k + 5 \times (-1)^k + i^k\right). \end{aligned}$$

Thus

$$c_0 = 3, \quad c_1 = -\tfrac{3}{4} - \tfrac{3}{4}i, \quad c_2 = \tfrac{1}{2}, \quad c_3 = -\tfrac{3}{4} + \tfrac{3}{4}i.$$

4.2 Periodic Sequences of Real Numbers

Theorem 4.3 *Let $(x_n : n \in \mathbb{Z})$ be a doubly-infinite sequence of real numbers which is m-periodic . Then*

$$x_n = \sum_{k=0}^{m-1} \left(p_k \cos \frac{2\pi kn}{m} + q_k \sin \frac{2\pi kn}{m} \right),$$

for all integers n, where $\omega_m = e^{2\pi i/m}$ and

$$p_k = \frac{1}{m} \sum_{j=0}^{m-1} x_j \cos \frac{2\pi kj}{m}, \quad q_k = \frac{1}{m} \sum_{j=0}^{m-1} x_j \sin \frac{2\pi kj}{m}.$$

Proof It follows from Theorem 4.2 that

$$x_n = \sum_{k=0}^{m-1} c_k \omega_m^{kn},$$

for all integers n, where $\omega_m = e^{2\pi i/m}$ and

$$c_k = \frac{1}{m} \sum_{j=0}^{m-1} x_j \omega_m^{-kj}.$$

Now

$$\begin{aligned} \omega_m^n &= \cos \frac{2n\pi}{m} + i \sin \frac{2n\pi}{m} \\ \omega_m^{-n} &= \cos \frac{2n\pi}{m} - i \sin \frac{2n\pi}{m} \end{aligned}$$

for all integers n. Now $c_k = p_k - q_k i$ for $k = 0, 1, \ldots, m-1$, where

$$p_k = \frac{1}{m} \sum_{j=0}^{m-1} x_j \cos \frac{2\pi kj}{m}, \quad q_k = \frac{1}{m} \sum_{j=0}^{m-1} x_j \sin \frac{2\pi kj}{m}.$$

(Note that p_k and q_k are real numbers for all k. It follows that

$$x_n = \operatorname{Re}\left(\sum_{k=0}^{m-1} c_k \omega_m^{kn}\right) = \sum_{k=0}^{m-1} \left(p_k \cos \frac{2\pi kn}{m} + q_k \sin \frac{2\pi kn}{m}\right),$$

where $\operatorname{Re}\left(\sum_{k=0}^{m-1} c_k \omega_m^{kn}\right)$ denotes the real part of $\sum_{k=0}^{m-1} c_k \omega_m^{kn}$. ∎

4.3 Periodic Functions and Fourier Series

Definition A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *periodic* if there exists some positive real number l such that $f(x + l) = f(x)$ for all real numbers x. The smallest real number l with this property is the *period* of the periodic function f.

A periodic function f with period l satisfies $f(x + ml) = f(x)$ for all real numbers x and integers m.

The period l of a periodic function f is said to *divide* some positive real number K if K/l is an integer. If the period of the function f divides a positive real number K then $f(x + mK) = f(x)$ for all real numbers x and integers m.

Mathematicians have proved that if $f\colon \mathbb{R} \to \mathbb{R}$ is any sufficiently well-behaved function from $\mathbb{R}$ to $\mathbb{R}$ with the property that $f(x + 2\pi) = f(x)$ for all real numbers x then f may be represented as an infinite series of the form

$$f(x) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx. \qquad (1)$$

In particular it follows from theorems proved by Dirichlet in 1829 that a function $f\colon \mathbb{R} \to \mathbb{R}$ which satisfies $f(x + 2\pi) = f(x)$ for all real numbers x can be represented as a trigonometrical series of this form if the function is bounded, with at most finitely many points of discontinuity, local maxima and local minima in the interval $[-\pi, \pi]$, and if

$$f(x) = \tfrac{1}{2}\left(\lim_{h \to 0+} f(x + h) + \lim_{h \to 0+} f(x - h)\right)$$

at each value x at which the function is discontinuous (where $\lim_{h\to 0+} f(x+h)$ and $\lim_{h\to 0+} f(x-h)$ denote the limits of $f(x+h)$ and $f(x-h)$ respectively as h tends to 0 from above).

Fourier in 1807 had observed that if a sufficiently well-behaved function could be expressed as the sum of a trigonometrical series of the above form, then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \tag{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \tag{3}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \tag{4}$$

for each positive integer n. These expressions for the coefficients a_n and b_n may readily be verified on substituting the trigonometic series for the function f (equation (1)) into the integrals on the right hand side of the equation, provided that one is permitted to interchange the operations of integration and summation in the resulting expressions.

Now it is not generally true that the integral of an infinite sum of functions is necessarily equal to the sum of the integrals of those functions. However if the function f is sufficiently well-behaved then the trigonometric series for the function f will converge sufficiently rapidly for this interchange of integration and summation to be valid, so that

$$\int_{-\pi}^{\pi}\left(\sum_{m=1}^{\infty} a_m \cos mx\right)dx = \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi}\cos mx\,dx \quad \text{etc.}$$

If we interchange summations and integrations in this fashion and make use of the trigonometric integrals provided by Theorem 2.1, we find that

$$\begin{aligned}\int_{-\pi}^{\pi} f(x)\,dx &= a_0\pi + \int_{-\pi}^{\pi}\left(\sum_{m=1}^{\infty} a_m\cos mx\right)dx + \int_{-\pi}^{\pi}\left(\sum_{m=1}^{\infty} b_m\sin mx\right)dx \\ &= a_0\pi + \sum_{m=1}^{\infty} a_m\int_{-\pi}^{\pi}\cos mx\,dx + \sum_{m=1}^{\infty} b_m\int_{-\pi}^{\pi}\sin mx\,dx \\ &= a_0\pi,\end{aligned}$$

Also

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$
\begin{aligned}
&= \tfrac{1}{2}a_0 \int_{-\pi}^{\pi} \cos nx\,dx + \int_{-\pi}^{\pi} \left(\sum_{m=1}^{\infty} a_m \cos mx\ \cos nx\right) dx \\
&\qquad + \int_{-\pi}^{\pi} \left(\sum_{m=1}^{\infty} b_m \sin mx\ \cos nx\right) dx \\
&= \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi} \cos mx\ \cos nx\,dx + \sum_{m=1}^{\infty} b_m \int_{-\pi}^{\pi} \sin mx\ \cos nx\,dx \\
&= a_n\pi,
\end{aligned}
$$

$$
\begin{aligned}
\int_{-\pi}^{\pi} & f(x)\sin nx\,dx \\
&= \tfrac{1}{2}a_0 \int_{-\pi}^{\pi} \sin nx\,dx + \int_{-\pi}^{\pi} \left(\sum_{m=1}^{\infty} a_m \cos mx\ \sin nx\right) dx \\
&\qquad + \int_{-\pi}^{\pi} \left(\sum_{m=1}^{\infty} b_m \sin mx\ \sin nx\right) dx \\
&= \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi} \cos mx\ \sin nx\,dx + \sum_{m=1}^{\infty} b_m \int_{-\pi}^{\pi} \sin mx\ \sin nx\,dx \\
&= b_n\pi.
\end{aligned}
$$

A trigonometric series of the form (1) with coefficients a_n and b_n given by the integrals (2), (3) and (4) is referred to the *Fourier series* for the function f. The coefficients defined by the integrals (2), (3) and (4) are referred to as the *Fourier coefficients* of the function f.

Example Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$
f(x) = \begin{cases} \frac{1}{2} & \text{if } x = m\pi \text{ for some integer } m; \\ 1 & \text{if } 2m\pi < x < (2m+1)\pi \text{ for some integer } m; \\ 0 & \text{if } (2m-1)\pi < x < 2m\pi \text{ for some integer } m. \end{cases}
$$

This function f has the property that $f(x) = f(x + 2m\pi)$ for all real numbers x and integers m, and can be represented by a Fourier series. The coefficients a_n and b_n of the Fourier series are given by the formulae

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad (n > 0), \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad (n > 0),
\end{aligned}
$$

Now $f(x) = 0$ if $-\pi < x < 0$, and $f(x) = 1$ if $0 < x < \pi$. Therefore $a_0 = 1$, and

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^\pi \cos nx\,dx = \frac{1}{n\pi}\left[\sin nx\right]_0^\pi \\
&= 0 \qquad (n > 0), \\
b_n &= \frac{1}{\pi}\int_0^\pi \sin nx\,dx = \frac{1}{n\pi}\left[-\cos nx\right]_0^\pi \\
&= \frac{1}{n\pi}(1 - \cos n\pi) = \frac{1}{n\pi}(1 - (-1)^n) \\
&= \begin{cases} \dfrac{2}{n\pi} & \text{if } n \text{ is odd and } n > 0, \\ 0 & \text{if } n \text{ is even and } n > 0, \end{cases}
\end{aligned}$$

(We have here made use of the fact that $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$ for all integers n.) Thus

$$\begin{aligned}
f(x) &= \frac{1}{2} + \sum_{\substack{n \text{ odd} \\ n>0}} \frac{2}{n\pi}\sin nx \\
&= \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi}\sin\left((2k-1)x\right)
\end{aligned}$$

4.4 Fourier Series of Even and Odd Functions

Definition A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *even* if $f(x) = f(-x)$ for all real numbers x. A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *odd* if $f(x) = -f(-x)$ for all real numbers x.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be an integrable function. Then

$$\int_{-\pi}^{0} f(x)\,dx = \int_0^\pi f(-x)\,dx, \tag{5}$$

$$\int_{-\pi}^{0} f(x)\cos nx\,dx = \int_0^\pi f(-x)\cos nx\,dx, \tag{6}$$

$$\int_{-\pi}^{0} f(x)\sin nx\,dx = -\int_0^\pi f(-x)\sin nx\,dx \tag{7}$$

(The first of these identities may be verified by making the substitution $x \mapsto -x$ and then interchanging the two limits of integration. The second and the third follow from the first on replacing $f(x)$ by $f(x)\cos nx$ and

$f(x)\sin nx$ and noting that $\cos(-nx) = \cos nx$ and $\sin(-nx) = -\sin nx$.) It follows that the Fourier coefficients of f are given by the following formulae:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} (f(x)+f(-x))\,dx, \tag{8}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} (f(x)+f(-x))\cos nx\,dx, \tag{9}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi} (f(x)-f(-x))\sin nx\,dx \tag{10}$$

for all positive integers n.

Of course $f(x) + f(-x) = 2f(x)$ and $f(x) - f(-x) = 0$ for all real numbers x if the function $f\colon \mathbb{R} \to \mathbb{R}$ is even, and $f(x) + f(-x) = 0$ and $f(x) - f(-x) = 2f(x)$ if the function $f\colon \mathbb{R} \to \mathbb{R}$ is odd. The following results follow immediately,

Theorem 4.4 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be an even periodic function whose period divides 2π. Suppose that the function f may be represented by a Fourier series. Then the Fourier series of f is of the form*

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx,$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

and

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

for all positive integers n.

Theorem 4.5 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be an odd periodic function whose period divides 2π. Suppose that the function f may be represented by a Fourier series. Then the Fourier series of f is of the form*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx,$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

for all positive integers n.

4.5 Fourier Series for General Periodic Functions

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a periodic function whose period divides l, where l is some positive real number. Then $f(x + l) = f(x)$ for all real numbers x. Let

$$g(x) = f\left(\frac{lx}{2\pi}\right) \quad \text{so that} \quad f(x) = g\left(\frac{2\pi x}{l}\right).$$

Then $g\colon \mathbb{R} \to \mathbb{R}$ is a periodic function, and $g(x + 2\pi) = g(x)$ for all real numbers x. If the function f is sufficiently well-behaved (and, in particular, if the function f is bounded, with only finitely many local maxima and minima and points of discontinuity in any finite interval, and if $f(x)$ at each point of discontinuity is the average of the limits of $f(x + h)$ and $f(x - h)$ as h tends to zero from above) then the function g may be represented by a Fourier series of the form

$$g(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

The coefficients of this Fourier series are then given by the formulae

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\, du, \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\cos nu\, du, \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(u)\sin nu\, du
\end{aligned}$$

for each positive integer n. If we make the substitution $u = \dfrac{2\pi x}{l}$ in these integrals, we find that

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{l}\right), \tag{11}$$

where

$$\begin{aligned}
a_0 &= \frac{2}{l}\int_{-\frac{1}{2}l}^{\frac{1}{2}l} g\left(\frac{2\pi x}{l}\right) dx \\
&= \frac{2}{l}\int_{-\frac{1}{2}l}^{\frac{1}{2}l} f(x)\, dx, \\
a_n &= \frac{2}{l}\int_{-\frac{1}{2}l}^{\frac{1}{2}l} g\left(\frac{2\pi x}{l}\right)\cos\left(\frac{2n\pi x}{l}\right) dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{l}\int_{-\frac{1}{2}l}^{\frac{1}{2}l} f(x)\cos\left(\frac{2n\pi x}{l}\right)\,dx,\\
b_n &= \frac{2}{l}\int_{-\frac{1}{2}l}^{\frac{1}{2}l} g\left(\frac{2\pi x}{l}\right)\sin\left(\frac{2n\pi x}{l}\right)\,dx\\
&= \frac{2}{l}\int_{-\frac{1}{2}l}^{\frac{1}{2}l} f(x)\sin\left(\frac{2n\pi x}{l}\right)\,dx
\end{aligned}
$$

for all positive integers n. Note that these integrals are taken over a single period of the function, from $-\frac{1}{2}l$ to $+\frac{1}{2}l$. It follows from the periodicity of the integrand that these integrals may be replaced by integrals from c to $c+l$ for any real number c, and thus

$$
a_0 = \frac{2}{l}\int_c^{c+l} f(x)\,dx, \tag{12}
$$

$$
a_n = \frac{2}{l}\int_c^{c+l} f(x)\cos\left(\frac{2n\pi x}{l}\right)\,dx, \tag{13}
$$

$$
b_n = \frac{2}{l}\int_c^{c+l} f(x)\sin\left(\frac{2n\pi x}{l}\right)\,dx \tag{14}
$$

for all positive integers n. (Indeed, if $h\colon \mathbb{R} \to \mathbb{R}$ is any integrable function with the property that $h(x+l) = h(x)$ for all real numbers x, and if p and q are real numbers with $p \le q \le p+l$ then

$$
\begin{aligned}
\int_p^{p+l} h(x)\,dx &= \int_p^q h(x)\,dx + \int_q^{p+l} h(x)\,dx\\
&= \int_{p+l}^{q+l} h(x)\,dx + \int_q^{p+l} h(x)\,dx = \int_q^{q+l} h(x)\,dx.
\end{aligned}
$$

Repeated applications of this identity show that

$$
\int_p^{p+l} h(x)\,dx = \int_q^{q+l} h(x)\,dx
$$

for all real numbers p and q.)

Example Let k be a positive real number. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$
f(x) = \begin{cases} e^{k(x-m)} & \text{if } m < x < m+1 \text{ for some integer } m;\\ \frac{1}{2}(e^k+1) & \text{if } x \text{ is an integer.}\end{cases}
$$

This function is periodic, with period 1, and may be expanded as a Fourier series

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\pi x + \sum_{n=1}^{\infty} b_n \sin 2n\pi x,$$

where

$$\begin{aligned} a_0 &= 2\int_0^1 f(x)\,dx, \\ a_n &= 2\int_0^1 f(x)\cos 2n\pi x\,dx \qquad (n > 0), \\ b_n &= 2\int_0^1 f(x)\sin 2n\pi x\,dx \qquad (n > 0). \end{aligned}$$

Note that $f(x) = e^{kx}$ if $0 < x < 1$. We see therefore that

$$a_0 = 2\int_0^1 e^{kx}\,dx = \frac{2}{k}\left[e^{kx}\right]_0^1 = \frac{2}{k}(e^k - 1).$$

Now if k and ω a positive real numbers then

$$\begin{aligned} \int e^{kx}\cos\omega x\,dx &= \frac{k}{k^2+\omega^2}e^{kx}\cos\omega x + \frac{\omega}{k^2+\omega^2}e^{kx}\sin\omega x + C, \\ \int e^{kx}\sin\omega x\,dx &= \frac{k}{k^2+\omega^2}e^{kx}\sin\omega x - \frac{\omega}{k^2+\omega^2}e^{kx}\cos\omega x + C, \end{aligned}$$

where C is a constant of integration. (These identities may be verified by differentiating the expressions on the right hand side.) We find therefore that

$$\begin{aligned} a_n &= 2\int_0^1 e^{kx}\cos 2n\pi x\,dx \\ &= \left[\frac{2k}{k^2+4n^2\pi^2}e^{kx}\cos 2n\pi x + \frac{4n\pi}{k^2+4n^2\pi^2}e^{kx}\sin 2n\pi x\right]_0^1 \\ &= \frac{2k}{k^2+4n^2\pi^2}(e^k - 1) \\ b_n &= 2\int_0^1 e^{kx}\sin 2n\pi x\,dx, \\ &= \left[\frac{2k}{k^2+4n^2\pi^2}e^{kx}\sin 2n\pi x - \frac{4n\pi}{k^2+4n^2\pi^2}e^{kx}\cos 2n\pi x\right]_0^1 \\ &= -\frac{4n\pi}{k^2+4n^2\pi^2}(e^k - 1), \end{aligned}$$

for each positive integer n, since $\cos 2n\pi = 1$ and $\sin 2n\pi = 0$ when n is an integer. Thus

$$\begin{aligned} e^{kx} &= \frac{1}{k}(e^k - 1) + \sum_{n=1}^{\infty} \frac{2k}{k^2 + 4n^2\pi^2}(e^k - 1)\cos 2n\pi x \\ &\quad - \sum_{n=1}^{\infty} \frac{4n\pi}{k^2 + 4n^2\pi^2}(e^k - 1)\sin 2n\pi x \end{aligned}$$

for all real numbers x satisfying $0 < x < 1$.

4.6 Sine Series

Let $f\colon [0, l] \to \mathbb{R}$ be a function defined on the interval $[0, l]$, where $[0, l] = \{x \in \mathbb{R} : 0 \leq x \leq l\}$. Suppose that $f(0) = f(l) = 0$. Let $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ be the function defined such that

$$\tilde{f}(x) = f(x - 2nl) \text{ if } 2nl \leq x \leq (2n + 1)l \text{ for some integer } n$$

and

$$\tilde{f}(x) = -f((2n + 2)l - x) \text{ if } (2n + 1)l \leq x \leq (2n + 2)l \text{ for some integer } n.$$

The function $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ is an odd function with the property that $\tilde{f}(x + 2l) = \tilde{f}(x)$ for all real numbers x. Indeed it is easily seen that $\tilde{f}\colon \mathbb{R} \to \mathbb{R}$ is the unique odd function with this property which agrees with the function f on the interval $[0, l]$.

If the function f is sufficiently well-behaved (and, in particular, if the function f is bounded, with at most finitely many local maxima and minima and points of discontinuity, and has the property that $f(x)$ at each point of discontinuity is the average of the limits of $f(x + h)$ and $f(x - h)$ as h tends to zero from above) then the function $\tilde{f}$ may be represented as a Fourier series. This Fourier series is of the form

$$\tilde{f}(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right),$$

where

$$\begin{aligned} b_n &= \frac{1}{l}\int_{-l}^{l} \tilde{f}(x) \sin\left(\frac{n\pi x}{l}\right)\,dx \\ &= \frac{2}{l}\int_{0}^{l} \tilde{f}(x) \sin\left(\frac{n\pi x}{l}\right)\,dx \end{aligned}$$

for all positive integers n. (This follows from equations (11) and (14) on replacing l by $2l$, and then using the fact that $\tilde{f}(-x) = -\tilde{f}(x)$ for all real numbers x.)

Therefore every sufficiently well-behaved function $f\colon [0, l] \to \mathbb{R}$ which satisfies $f(0) = f(l) = 0$ may be represented in the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right), \tag{15}$$

where

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)\,dx \tag{16}$$

for each positive integer n.

Example Let l be a positive real numbers, and let $f\colon [0, l] \to \mathbb{R}$ be the function defined by $f(x) = x(l - x)$ (where $0 \leq x \leq l$). This function can be expanded in a sine series of the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right),$$

where

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$$

Using the method of integration by parts, and the result that $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$ for all integers n, we find then

$$\begin{aligned}
b_n &= \frac{2}{l}\int_0^l x(l-x)\sin\frac{n\pi x}{l}\,dx = -\frac{2}{n\pi}\int_0^l x(l-x)\frac{d}{dx}\left(\cos\frac{n\pi x}{l}\right)\,dx \\
&= -\frac{2}{n\pi}\left[x(l-x)\cos\frac{n\pi x}{l}\right]_0^l + \frac{2}{n\pi}\int_0^l (l-2x)\cos\frac{n\pi x}{l}\,dx \\
&= \frac{2}{n\pi}\int_0^l (l-2x)\cos\frac{n\pi x}{l}\,dx \\
&= \frac{2l}{n^2\pi^2}\int_0^l (l-2x)\frac{d}{dx}\left(\sin\frac{n\pi x}{l}\right)\,dx \\
&= \frac{2l}{n^2\pi^2}\left[(l-2x)\sin\frac{n\pi x}{l}\right]_0^l - \frac{2l}{n^2\pi^2}\int_0^l \left(-2\sin\frac{n\pi x}{l}\right)\,dx \\
&= \frac{4l}{n^2\pi^2}\int_0^l \sin\frac{n\pi x}{l}\,dx = -\frac{4l^2}{n^3\pi^3}\left[\cos\frac{n\pi x}{l}\right]_0^l
\end{aligned}$$

$$
\begin{aligned}
&= \frac{4l^2}{n^3\pi^3}(1 - \cos n\pi) = \frac{4l^2}{n^3\pi^3}(1 - (-1)^n) \\
&= \begin{cases} \dfrac{8l^2}{n^3\pi^3} & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even.} \end{cases}
\end{aligned}
$$

Thus

$$
f(x) = \sum_{\substack{n \text{ odd} \\ n>0}} \frac{8l^2}{n^3\pi^3} \sin \frac{n\pi x}{l}.
$$

or (setting $n = 2k - 1$ for each positive integer k),

$$
x(l - x) = \sum_{k=1}^{\infty} \frac{8l^2}{(2k-1)^3\pi^3} \sin \frac{(2k-1)\pi x}{l} \qquad (0 \leq x \leq l).
$$

4.7 Cosine Series

Let $f\colon [0, l] \to \mathbb{R}$ be a function defined on the interval $[0, l]$, where $[0, l] = \{x \in \mathbb{R} : 0 \leq x \leq l\}$. Let $\tilde{g}\colon \mathbb{R} \to \mathbb{R}$ be the function defined by

$$
\tilde{g}(x) = f(x - 2nl) \text{ if } 2nl \leq x \leq (2n+1)l \text{ for some integer } n
$$

and

$$
\tilde{g}(x) = f((2n+2)l - x) \text{ if } (2n+1)l \leq x \leq (2n+2)l \text{ for some integer } n.
$$

The function $\tilde{g}\colon \mathbb{R} \to \mathbb{R}$ is an even function with the property that $\tilde{g}(x+2l) = \tilde{g}(x)$ for all real numbers x. Indeed it is easily seen that $\tilde{g}\colon \mathbb{R} \to \mathbb{R}$ is the unique even function with this property which agrees with the function f on the interval $[0, l]$.

If the function f is sufficiently well-behaved then the function $\tilde{g}$ may be represented as a Fourier series. This Fourier series is of the form

$$
\tilde{g}(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right),
$$

where

$$
\begin{aligned}
a_n &= \frac{1}{l}\int_{-l}^{l} \tilde{g}(x) \cos\left(\frac{n\pi x}{l}\right)\, dx \\
&= \frac{2}{l}\int_{0}^{l} \tilde{g}(x) \cos\left(\frac{n\pi x}{l}\right)\, dx
\end{aligned}
$$

for all non-negative integers n. (This follows from equations (11), (12) and (13) on replacing l by $2l$, and then using the fact that $\tilde{g}(-x) = \tilde{g}(x)$ for all real numbers x.)

Therefore every sufficiently well-behaved function $f\colon [0, l] \to \mathbb{R}$ may be represented in the form

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \tag{17}$$

where

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)\,dx \tag{18}$$

for each positive integer n.

Example Consider the function $f\colon [0, 1] \to \mathbb{R}$ defined by

$$f(x) = x \quad \text{if } 0 \le x \le 1.$$

This function may be represented as a cosine series of the form

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x$$

where

$$a_0 = 2\int_0^1 f(x)\,dx = 2\int_0^1 x\,dx = 1,$$

and where

$$a_n = 2\int_0^1 f(x)\cos n\pi x\,dx$$

for all positive integers n. Using the method of integation by parts, and making use of the fact that $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$ for all integers n, we find that

$$\begin{aligned}
a_n &= 2\int_0^1 x\cos n\pi x\,dx = \frac{2}{n\pi}\int_0^1 x\frac{d}{dx}\left(\sin n\pi x\right)\,dx \\
&= \frac{2}{n\pi}\left[x\sin n\pi x\right]_0^1 - \frac{2}{n\pi}\int_0^1 \sin n\pi x\,dx = -\frac{2}{n\pi}\int_0^1 \sin n\pi x\,dx \\
&= \frac{2}{n^2\pi^2}\left[\cos n\pi x\right]_0^1 = -\frac{2}{n^2\pi^2}\left(1-(-1)^n\right) \\
&= \begin{cases} -\dfrac{4}{n^2\pi^2} & \text{if } n \text{ is odd;} \\ 0 & \text{if } n \text{ is even.} \end{cases}
\end{aligned}$$

Thus

$$x = \frac{1}{2} - \sum_{\substack{n \text{ odd} \\ n>0}} \frac{4}{n^2\pi^2} \cos n\pi x \quad \text{when } 0 \le x \le 1.$$

Remark The function $\tilde{g}$ defined by

$$\tilde{g}(x) = \frac{1}{2} - \sum_{\substack{n \text{ odd} \\ n>0}} \frac{4}{n^2\pi^2} \cos n\pi x$$

for all real numbers x is an even periodic function, with period equal to 2, which coincides with the function $f\colon [0,1] \to \mathbb{R}$ on the interval $[0,1]$, where $f(x) = x$ for all real numbers x satisfying $0 \le x \le 1$. It follows that

$$\tilde{g}(x) = |x - 2m| \quad \text{whenever } m \text{ is an integer and } 2m - 1 \le x \le 2m + 1.$$

Remark Setting $x = 1$ in the identity

$$x = \frac{1}{2} - \sum_{\substack{n \text{ odd} \\ n>0}} \frac{4}{n^2\pi^2} \cos n\pi x \quad \text{when } 0 \le x \le 1,$$

we find that

$$1 = \frac{1}{2} - \sum_{\substack{n \text{ odd} \\ n>0}} \frac{4}{n^2\pi^2} \cos n\pi = \frac{1}{2} + \sum_{\substack{n \text{ odd} \\ n>0}} \frac{4}{n^2\pi^2}$$

and thus

$$\sum_{\substack{n \text{ odd} \\ n>0}} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

But

$$\sum_{\substack{n \text{ odd} \\ n>0}} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \left(1 - \frac{1}{4}\right) \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$