Course 2BA1, 2008–09
Section 3: Graph Theory

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Contents

1 Graph Theory

1.1 Undirected Graphs

An *undirected graph* can be thought of as consisting of a finite set V of points, referred to as the *vertices* of the graph, together with a finite set E of *edges*, where each edge joins two distinct vertices of the graph.

We now proceed to formulate the definition of an undirected graph in somewhat more formal language.

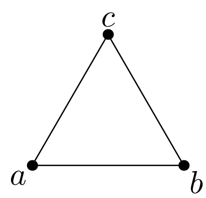
Let V be a set. We denote by V_2 the set consisting of all subsets of V with exactly two elements. Thus, for any set V,

$$V_2 = \{A \in \mathcal{P}V : |A| = 2\},$$

where $\mathcal{P}V$ denotes the power set of V (i.e., the set consisting of all subsets of V), and $|A|$ denotes the number of elements in a subset A of V.

Definition An *undirected graph* (V, E) consists of a finite set V together with a subset E of V_2 (where V_2 is the set consisting of all subsets of V with exactly two elements). The elements of V are the *vertices* of the graph; the elements of E are the *edges* of the graph.

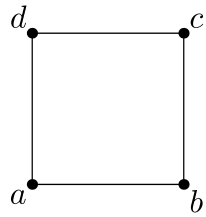
Example Let a, b and c label the three vertices of a triangle in the plane. Then there is an undirected graph (V, E) which consists of the vertices and edges of this triangle.



Here

$$\begin{aligned} V &= \{a, b, c\}; \\ E &= \{\{a, b\}, \{b, c\}, \{c, a\}\}. \end{aligned}$$

Example Let a, b, c and d label the four vertices of a square in the plane (labelled in cyclic order around the square). Then there is an undirected graph (V, E) which consists of the vertices and edges of this square.



Here

$$V = \{a, b, c, d\};$$
$$E = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}\}.$$

Note that, in this example, not every subset of V_2 with exactly two elements is an edge of the graph. (Indeed the diagonals $\{a, c\}$ and $\{b, d\}$ are not edges of this graph.)

Let (V, E) be an undirected graph. In order to simplify notation, we shall often denote by $a\,b$ an edge $\{a, b\}$ of the graph whose endpoints are the vertices a and b.

Definition A graph is said to be *trivial* if it consists of a single vertex.

We may denote a graph by a single letter such as G. Writing $G = (V, E)$ indicates that V is the set of vertices and E is the set of edges of some graph G.

1.2 Incidence and Adjacency

Definition If v is a vertex of some graph, if e is an edge of the graph, and if $e = v\,v'$ for some vertex v' of the graph, then the vertex v is said to be *incident* to the edge e, and the edge e is said to be *incident* to the vertex v.

(We see therefore that an edge of a graph is *incident* to a vertex of the graph, and the vertex is *incident* to the edge, if and only if the vertex is one of the endpoints of the edge.)

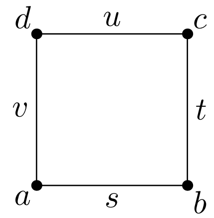
Definition Two distinct vertices v and v' of a graph (V, E) are said to be *adjacent* if and only if $v\,v' \in E$.

(We see therefore that two distinct vertices of a graph are *adjacent* if and only if they are the endpoints of an edge of the graph.)

1.3 Incidence and Adjacency Tables and Matrices

The following example illustrates how we may associate incidence and adjacency tables or matrices with graphs.

Example Let a, b, c and d represent the four vertices of a square in the plane, and consider the graph consisting of the vertices and edges of this square. Let s, t, u and v denote the four edges of the square, where $s = a\,b$, $t = b\,c$, $u = c\,d$ and $v = d\,a$.



The incidence relations between the vertices a, b, c and d and the edges s, t, u and v can be expressed by the following table:

	s	t	u	v
a	1	0	0	1
b	1	1	0	0
c	0	1	1	0
d	0	0	1	1

Such a table is known as the *incidence table* for the graph.

If a vertex is incident to an edge then the corresponding entry in the table has the value 1; otherwise that entry has the value 0.

If the vertices are ordered (as first vertex, second vertex, etc.) and if the edges are also ordered, then this information may be encoded in a matrix, known as an *incidence matrix.* In this example, if the vertices are ordered as a, b, c, d (so that a is the first vertex, b is the second vertex, c is the third vertex, and d is the fourth vertex), and if the edges are ordered as s, t, u, v, then the corresponding incidence matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Definition Let (V, E) be a graph with m vertices and n edges. Let the vertices be ordered as $v_1, v_2, \ldots, v_m$, and let the edges be ordered as $e_1, e_2, \ldots, e_n$. The *incidence matrix* for such a graph then takes the form

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix},$$

where the entry a_{ij} in the ith row and jth column has the value 1 if the ith vertex is incident to the jth edge, but has the value 0 otherwise.

One may introduce in a similar fashion the *adjacency table* and the *adjacency matrix* of a graph.

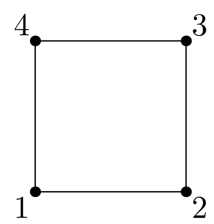
Definition Let (V, E) be a graph with m vertices, and let the vertices be ordered as $v_1, v_2, \ldots, v_m$. The *adjacency matrix* for such a graph then takes the form

$$\begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1m} \\ b_{21} & b_{22} & \ldots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & a_{m2} & \ldots & b_{mm} \end{pmatrix},$$

where the entry b_{ij} in the ith row and jth column has the value 1 if the ith vertex is adjacent to the jth vertex but has the value 0 otherwise.

Note that the adjacency matrix of any (undirected) graph is symmetric: $b_{ij} = b_{ji}$ for all indices i and j, where b_{ij} denotes the entry in the ith row and jth column of the adjacency matrix.

Example Consider the graph consisting of the vertices and edges of a square in the plane. Suppose that the vertices are ordered in anticlockwise order around the square, starting from some chosen vertex of the square.



Then the adjacency matrix for this graph is the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

1.4 Complete Graphs

Definition A graph (V, E) is said to be *complete* if and only if, $\{v, v'\} \in E$ for all $v \in V$ and $v' \in V$ satisfying $v \neq v'$.

(Thus a graph is *complete* if and only if any two distinct vertices of the graph are the endpoints of an edge of the graph.)

A complete graph with n vertices is denoted by K_n.

1.5 Bipartite Graphs

Definition A graph (V, E) is said to be *bipartite* if there exist subsets V_1 and V_2, such that

(i) $V_1 \cup V_2 = V$;

(ii) $V_1 \cap V_2 = \emptyset$;

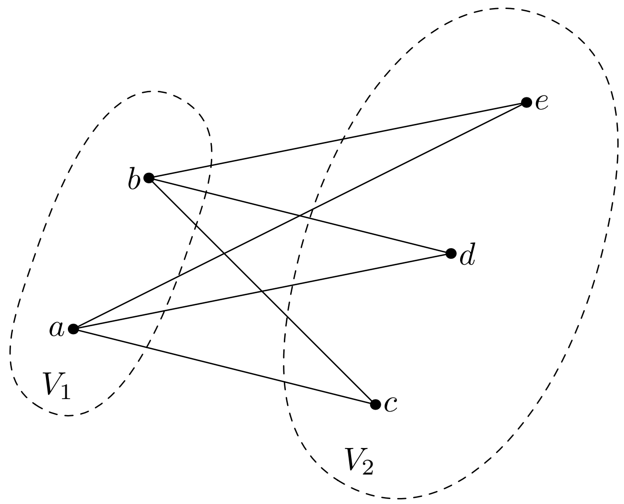
(iii) each edge in E is of the form $\{v, w\}$ with $v \in V_1$ and $w \in V_2$.

If in addition $\{v, w\}$ is an edge of the graph for every $v \in V_1$ and $w \in V_2$ then the graph (V, E) is said to be a *complete bipartite graph*. In the case when V_1 has p elements and V_2 has q elements, such a complete bipartite graph is denoted by $K_{p,q}$.

Example Let (V, E) be a graph with

$$\begin{aligned} V &= \{a, b, c, d, e\}, \\ E &= \{a\,c, a\,d, a\,e, b\,c, b\,d, b\,e\}. \end{aligned}$$

Let $V_1 = \{a, b\}$ and $V_2 = \{c, d, e\}$. Then the conditions in the above definition are satisfied by the graph (V, E) and the subsets V_1 and V_2, and therefore the graph is bipartite. Moreover it is a complete bipartite graph.



1.6 Isomorphism of Graphs

Definition An *isomorphism* between two graphs (V, E) and (V', E') is a bijective function $\varphi: V \to V'$ with the following property: for any two distinct vertices a and b belonging to V, $\{a, b\} \in E$ if and only if $\{\varphi(a), \varphi(b)\} \in E'$. If there exists such an isomorphism $\varphi: V \to V'$ between two graphs (V, E) and (V', E') then these graphs are said to be *isomorphic*.

We recall that a function $\varphi: V \to V'$ is *bijective* if and only if it has a well-defined inverse $\varphi^{-1}: V' \to V$. Thus a bijection $\varphi: V \to V'$ sets up a one-to-one correspondence between the vertices of V and V': to every vertex of V there corresponds a vertex of V', and vice versa. Such a one-to-one correspondence between the vertices belonging to V and V' is an isomorphism between the graphs (V, E) and (V', E') when it has the following additional property: a pair of distinct vertices belonging to V are the endpoints of an edge of (V, E) if and only if the corresponding vertices belonging to V' are the endpoints of an edge of (V', E'). There is then a one-to-one correspondence between the edges of the two graphs, induced by the one-to-one correspondence between their vertices.

1.7 Subgraphs

Definition Let (V, E) and (V', E') be graphs. The graph (V', E') is said to be a *subgraph* of (V, E) if and only if $V' \subset V$ and $E' \subset E$ (i.e., if and only if the vertices and edges of (V', E') are all vertices and edges of (V, E)).

Let (V, E) be a graph, and let V' be a subset of V. Let

$$E' = \{\{a, b\} \in E : a \in V' \text{ and } b \in V'\},$$

(so that E' be the set of all edges $\{a, b\}$ belonging to E whose endpoints a and b belong to V'). Then (V', E') is a subgraph of (V, E). It is referred to as the *restriction* of the graph (V, E) to V', or as the graph *induced* on V' by the graph (V, E). If the graph (V, E) is denoted by G, then its restriction (V', E') to V' may be denoted by $G|_{V'}$.

1.8 Vertex Degrees

Definition Let (V, E) be a graph. The *degree* $\deg v$ of a vertex v of this graph is defined to be the number of edges of the graph that are incident to v (i.e., the number of edges of the graph which have v as one of their endpoints).

Definition A vertex of a graph of degree 0 is said to be an *isolated* vertex.

Definition A vertex of a graph of degree 1 is said to be an *pendant* vertex.

Theorem 1.1 *Let (V, E) be a graph. Then*

$$\sum_{v \in V} \deg v = 2|E|,$$

where $\sum_{v \in V} \deg v$ denotes the sum of the degrees of all the vertices of the graph, and $|E|$ denotes the number of edges of the graph.

Proof Clearly $\sum_{v \in V} \deg v$ counts the number of times an edge of a graph is incident on a vertex of the graph. But this quantity must be twice the number of edges of the graph, since each edge is incident on exactly two vertices. ∎

Corollary 1.2 $\sum_{v \in V} \deg v$ *is an even integer.*

Corollary 1.3 *In any graph, the number of vertices of odd degree must be even.*

Definition A graph is said to be *k-regular*, for some non-negative integer k, if every vertex of the graph has degree equal to k. A *regular graph* is a graph that is k-regular for some non-negative integer k.

Corollary 1.4 *Let (V, E) be a k-regular graph. Then $k|V| = 2|E|$, where $|V|$ denotes the number of vertices and $|E|$ denotes the number of edges of the graph.*

Proof If the graph is k-regular then the sum of the degrees of the vertices of the graph is equal to $k|V|$. The result then follows immediately from Theorem 1.1. ∎

Example The graph consisting of the vertices and edges of a square is 2-regular, since every vertex (i.e., every corner of the square) is incident to exactly two edges.

Example A complete graph with n vertices is $(n-1)$-regular, since each vertex is adjacent to all the remaining $n-1$ vertices.

Example A complete bipartite graph $K_{p,q}$ is regular if and only if $p = q$.

1.9 Walks, Trails and Paths

Definition Let (V, E) be a graph. A *walk* $v_0 v_1 v_2 \ldots v_n$ of length n in the graph from a vertex a to a vertex b is determined by a finite sequence $v_0, v_1, v_2, \ldots, v_n$ of vertices of the graph such that $v_0 = a$, $v_n = b$ and $v_{i-1} v_i$ is an edge of the graph for $i = 1, 2, \ldots, n$.

A walk $v_0 v_1 v_2 \ldots v_n$ in a graph is said to *traverse* the edges $v_{i-1}v_i$ for $i = 1, 2, \ldots, n$ and to *pass through* the vertices $v_0, v_1, \ldots, v_n$.

Each vertex v in a graph determines a walk of length of length zero in the graph, consisting of the single vertex v; such a walk is said to be *trivial.*

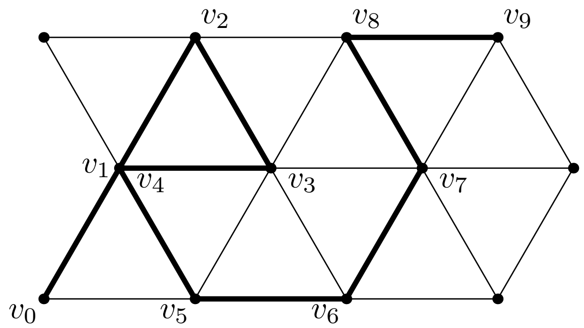
Definition Let (V, E) be a graph. A *trail* $v_0 v_1 v_2 \ldots v_n$ of length n in the graph from a vertex a to a vertex b is a walk of length n from a to b with the property that the edges $v_{i-1}v_i$ are distinct for $i = 1, 2, \ldots, n$.

A trail in a graph is thus a walk in the graph which traverses edges of the graph at most once.

Definition Let (V, E) be a graph. A *path* $v_0 v_1 v_2 \ldots v_n$ of length n in the graph from a vertex a to a vertex b is a walk of length n from a to b with the property that the vertices $v_0, v_1, \ldots, v_n$ are distinct.

A path in a graph is thus a walk in the graph which passes through vertices of the graph at most once.

Definition A walk, trail or path in a graph is said to be *trivial* if it is a walk v of length zero determined by a single vertex v of v; otherwise it is said to be non-trivial.



A trail $v_0\, v_1\, \ldots\, v_9$ in a graph

1.10 Connected Graphs

Definition An undirected graph is said to be *connected* if, given any two vertices u and v of the graph, there exists a path in the graph from u to v.

Theorem 1.5 *Let u and v be vertices of a graph. Then there exists a path in the graph from u to v if and only if there exists a walk in the graph from u to v.*

Proof Any path in a graph from one vertex to another is a walk. It therefore only remains to show that if there exists a walk in the graph from a vertex u to a vertex v, then there must also exist a path in the graph from u to v.

Now if there exists at least one walk from u to v, then there must exist a walk from u to v whose length is less than or equal to that of every other walk from u to v. Let this walk be $a_0\, a_1\, \ldots\, a_n$, where $a_0 = u$ and $a_n = v$. We claim that this walk is in fact a path from u to v. Indeed were it the case that $a_j = a_k$ for some integers j and k satisfying $0 \leq j < k \leq n$ then the walk $a_0\, \ldots\, a_j\, a_{k+1}\, \ldots\, a_n$ from u to v obtained on omitting the edges $a_j\, a_{j+1}, \ldots, a_{k-1}\, a_k$ would be a walk from u to v whose length was strictly less than that of the given walk (which is the *shortest* walk from u to v). But this is clearly impossible. Hence $a_0, a_1, \ldots, a_n$ must be distinct, and thus the the walk $a_0\, a_1\, \ldots\, a_n$ is a path from u to v. ∎

Corollary 1.6 *An undirected graph is connected if and only if, given any two vertices u and v of the graph, there exists a walk in the graph from u to v.*

1.11 The Components of a Graph

Let (V, E) be an undirected graph. We can define a relation $\sim$ on the set V of vertices of the graph, where two vertices a and b of the graph satisfy $a \sim b$

if and only if there exists a walk in the graph from a to b.

Lemma 1.7 *Let (V, E) be an undirected graph. Then the relation $\sim$ on the set V of vertices of the graph is an equivalence relation, where two vertices u and v of the graph satisfy $u \sim v$ if and only if there exists a walk in the graph from u to v.*

Proof We must prove that the relation $\sim$ on V is reflexive, symmetric and transitive.

Clearly $v \sim v$ for any vertex v of the graph, since the trivial walk v is walk from v to itself. Thus the relation $\sim$ is reflexive.

Let u and v be vertices of the graph satisfying $u \sim v$. Then there exists a walk $u\, a_1\, a_2\, \ldots\, a_{n-1}\, v$ from u to v. This walk may be reversed to obtain a walk $v\, a_{n-1}\, a_{n-2}\, \ldots\, a_1\, u$ from v to u. We conclude that if $u \sim v$ then $v \sim u$. Thus the relation $\sim$ is symmetric.

Finally let u, v and w be vertices of the graph for which $u \sim v$ and $v \sim w$. Then there exists a walk $u\, a_1\, a_2\, \ldots\, a_{n-1}\, v$ from u to v, and a walk $v\, b_1\, b_2\, \ldots\, b_{r-1}\, w$ from v to b. These two walks may be concatenated to yield a walk

$$u\, a_1\, a_2\, \ldots\, a_{n-1}\, v\, b_1\, b_2\, \ldots\, b_{r-1}\, w$$

from u to w, showing that $u \sim w$. Thus the relation $\sim$ is transitive. We have shown that this relation is reflexive, symmetric and transitive. It is therefore an equivalence relation. ∎

The equivalence relation $\sim$ on the set V of vertices of the graph (V, E) gives rise to a partition of V as the disjoint union of subsets $V_1, V_2, \ldots, V_m$, where

(i) $V_1 \cup V_2 \cup \cdots \cup V_k = V$;

(ii) $V_i \cap V_j = \emptyset$ if $i \neq j$;

(iii) two vertices u and v belong to a single subset V_i if and only if there exists a walk in (V, E) from u to v (i.e., if and only if $u \sim v$).

If u and v are the endpoints of some edge $u\,v$ of the graph (V, E), then $u \sim v$ (since an edge can be considered as a walk of length one), and thus u and v belong to the same set V_i. Thus, if we define

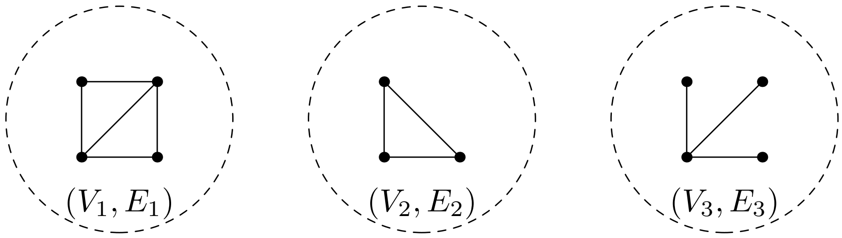
$$E_i = \{u\,v \in E : u \in V_i \text{ and } v \in V_i\},$$

then (V_1, E_1), $(V_2, E_2), \ldots, (V_k, E_k)$ are subgraphs of (V, E), and

$$V = V_1 \cup V_2 \cup \cdots \cup V_k, \quad E = E_1 \cup E_2 \cup \cdots \cup E_k.$$

These subgraphs are *disjoint* since $V_i \cap V_j = \emptyset$ and $E_i \cap E_j = \emptyset$ if $i \neq j$. Moreover the graph (V_i, E_i) is the restriction of the graph (V, E) to V_i (also describable as the graph induced on V_i by (V, E)) for $i = 1, 2, \ldots, k$.

The subgraphs (V_i, E_i) of (V, E) are referred to as the *components* (or *connected components*) of the graph (V, E).



A graph with three components

Lemma 1.8 *The vertices and edges of any walk in an undirected graph are all contained in a single component of that graph.*

Proof Let $v_0 \, v_1 \ldots v_n$ be a walk in a graph (V, E). Then $v_0 \, v_1 \ldots v_r$ is a walk in (V, E) from v_0 to v_r for each integer r between 1 and m. It follows that each vertex v_r through which the walk passes must belong to the same component of the graph as v_0. Therefore all the vertices and edges of this walk belong to a single component of the graph, namely that component which contains the vertex v_0. ∎

Lemma 1.9 *Each component of an undirected graph is connected.*

Proof Let (V, E) be a graph, and let u and v be vertices belonging to V_i, where (V_i, E_i) is one of the components of this graph. Then there exists a walk in (V, E) from u to v. But the vertices and edges of this walk are contained in a single component of the graph (V, E), by Lemma 1.8, and that component must obviously be the component (V_i, E_i) that contains the vertices u and v. Thus there exists a walk in (V_i, E_i) from u to v. We conclude that the graph (V_i, E_i) is connected. ∎

Remark The importance of the concept of the components of a graph is that it enables us to reduce the study of undirected graphs in general to the study of connected graphs. Indeed any undirected graph can be represented as a disjoint union of connected subgraphs: these subgraphs are the components of the given graph. These connected components may then be studied individually. Moreover properties of any one component do not affect those of any other, since no edge of the graph passes from any one component of the graph to any other.

1.12 Circuits

Definition Let (V, E) be a graph. A *walk* $v_0\, v_1\, v_2\, \ldots\, v_n$ in the graph is said to be *closed* if $v_0 = v_n$.

Thus a walk in a graph is closed if and only if it starts and ends at the same vertex.

Definition Let (V, E) be a graph. A *circuit* in the graph is a non-trivial closed trail in the graph.

We see therefore that a circuit in a graph is a closed walk with no repeated edges, and passing though at least two vertices.

Definition A circuit $v_0\, v_1\, v_2\, \ldots\, v_{n-1}\, v_0$ in a graph is said to be *simple* if the vertices $v_0, v_1, v_2, \ldots\, v_{n-1}$ are distinct.

Remark Some authors use the term *cycle* to denote a simple circuit in a graph. Others use the term *cycle* to refer to a circuit in the graph, irrespective of whether or not it is simple.

We now prove two theorems that provide sufficient conditions for a graph to contain simple circuits.

Theorem 1.10 *If a graph has no isolated or pendant vertices then it contains at least one simple circuit.*

Proof Let (V, E) be a graph with no isolated or pendant vertices. The length of any path in this graph cannot exceed $|V| - 1$, where $|V|$ denotes the number of vertices of the graph, since a path of length m passes through $m + 1$ distinct vertices. Therefore there exists a path $v_0\, v_1\, v_2\, \ldots\, v_m$ in the graph whose length m is greater than or equal to the length of every other path in the graph. Now the final vertex v_m of the graph is adjacent to at least two vertices of the graph, since the graph contains no isolated or pendant vertices. One of these vertices is v_{m-1}. If none of the vertices $v_0, v_1, \ldots, v_{m-2}$ were incident to v_m then there would exist a vertex w adjacent to v_m that was distinct from $v_0, v_1, \ldots, v_m$, and then $v_0, v_1, \ldots, v_m\, w$ would be a path in the graph with length exceeding m, which is impossible. It follows that at least one of the vertices $v_0, v_1, \ldots, v_{m-2}$ is incident to v_m; let that vertex be v_k, where $0 \leq k \leq m-2$. Then $v_k\, v_{k+1}\, \ldots\, v_m\, v_k$ is a simple circuit in the graph. Thus a graph with no isolated or pendant vertices always contains a simple circuit. ∎

Theorem 1.11 *Let u and v be vertices of a graph, where $u \neq v$. Suppose that there exist at least two distinct paths in the graph from u to v. Then the graph contains at least one simple circuit.*

Proof Let $a_0\, a_1\, a_2\, \ldots\, a_m$ and $b_0\, b_1\, b_2\, \ldots\, b_n$ be two distinct paths in the graph with $a_0 = b_0 = u$ and $a_m = b_n = v$. We may suppose that $m \leq n$. Now the fact that paths are distinct ensures that there exists at least integer i satisfying $0 < i \leq m$ for which $a_i \neq b_i$. Let the smallest such integer i be $r+1$, where r is an integer in the range $0 \leq r < m$. Then $a_r = b_r$ and $a_{r+1} \neq b_{r+1}$. Now the condition $a_i \in \{b_j : r < j \leq n\}$ is satisfied when $i = m$, since $a_m = b_n$. Let s be the smallest integer satisfying $r < s \leq m$ for which $a_s \in \{b_j : r < j \leq n\}$. Then $a_s = b_t$ for some integer t satisfying $r < t \leq n$. Moreover none of the vertices a_i with $r < i < s$ belong to the set $\{b_j : r < j < t\}$. It follows that

$$a_r\, a_{r+1}\, \ldots\, a_s\, b_{t-1}\, \ldots\, b_{r+1} a_r$$

is a simple circuit in the graph. Thus the graph has at least one simple circuit, as required. ∎

1.13 Eulerian Trails and Circuits

Definition An *Eulerian trail* in a graph is a trail that traverses every edge of the graph.

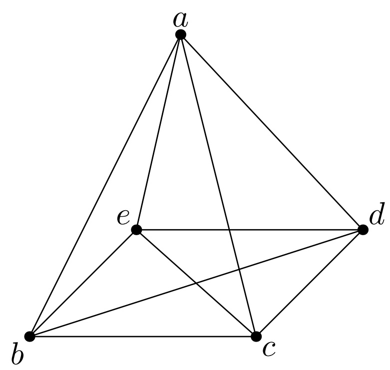
Note that an Eulerian trail in a graph must traverse every edge of the graph exactly once, since a trail traverses an edge of the graph at most once.

Definition An *Eulerian circuit* in a graph is a circuit that traverses every edge of the graph.

It follows from these definitions that any closed Eulerian trail is an Eulerian circuit.

Example Let (V, E) be the complete graph K_5 on five vertices a, b, c, d and e, where

$$\begin{aligned} V &= \{a,\, b,\, c,\, d,\, e\}, \\ E &= \{a\,b,\, a\,c,\, a\,d,\, a\,e,\, b\,c,\, b\,d,\, b\,e,\, c\,d,\, c\,e,\, d\,e\}. \end{aligned}$$



This graph has Eulerian circuits. One of them is the following:

$$a\,b\,c\,a\,d\,e\,c\,d\,b\,e\,a.$$

Remark Eulerian trails and circuits are named after the Swiss mathematician Leonhard Euler (1707–1783), who first studied the problem of the existence of such circuits in connection with the problem of the Seven Bridges of Königsberg. The citizens of this city used to amuse themselves by attempting to devise a walk around the city that would cross each of the seven bridges exactly once. They always failed in this attempt, for reasons explained by Euler.

We shall derive necessary and sufficient conditions for the existence of Eulerian trails and circuits in a connected graph. The following theorem will give rise to a necessary condition for the existence of an Eulerian trail or circuit.

Theorem 1.12 *Let $v_0\, v_1\, \ldots\, v_m$ be a trail in a graph, and let v be a vertex of that graph. Then the number of edges of the trail incident to the vertex v is even, except in the case when the trail is not closed and the trail starts or finishes at v, in which case the number of edges of the trail incident to the vertex v is odd.*

Proof First suppose that $v \neq v_0$ and $v \neq v_m$. The edges of the trail that are incident to v are then those of the form $v_{i-1}\, v_i$ and $v_i\, v_{i+1}$ with $0 < i < m$ and $v_i = v$. It follows that the number of edges of the trail incident to v is then equal to twice the number of integers i satisfying $0 < i < m$ for which $v = v_i$, and is thus even.

If $v = v_0$, and if the trail is not closed (i.e., if $v_m \neq v_0$), then the edges of the trail incident to v are the edge $v_0\,v_1$ together with the edges $v_{i-1}\,v_i$ and $v_i\,v_{i+1}$ for those integers i satisfying $1 < i < m$ for which $v = v_i$. Therefore the number of edges of the trail incident to v is then equal to one plus twice the number of integers i satisfying $1 < i < m$ for which $v = v_i$, and is thus odd. Similarly the number of edges of the trail incident to v is odd in the case when $v = v_m$ and the trail is not closed. Finally, in the case when the trail is closed and $v = v_0 = v_m$, the edges incident to v are $v_0\,v_1$ and $v_{m-1}\,v_m$, together with the edges $v_{i-1}\,v_i$ and $v_i\,v_{i+1}$ for those integers i satisfying $1 < i < m$ for which $v = v_i$. The total number of edges of the trail incident to v is therefore even. ∎

Corollary 1.13 *Let v be a vertex of a graph. Then, given any circuit in the graph, the number of edges incident to v that are traversed by that circuit is even.*

Corollary 1.14 *If a graph admits an Eulerian circuit then the degree of every vertex of the graph must be even.*

Proof Let v be a vertex of the graph. It follows from Corollary 1.13 that the number of edges of any Eulerian circuit incident to v is even. But every edge incident to v is an edge of an Eulerian circuit, since an Eulerian circuit by definition traverses every edge of the graph. It follows that the degree of the vertex v is even, as required. ∎

Example Any attempt to find an Eulerian circuit in the complete graph K_4 on four vertices is guaranteed to fail, since such a graph is 3-regular (i.e., the degree of each of the four vertices of the graph is equal to 3).

Corollary 1.15 *If a graph admits an Eulerian trail that is not a circuit then the degrees of exactly two vertices of the graph must be odd, and the degrees of the remaining vertices must be even. The two vertices with odd degrees will then be the initial and final vertices of the Eulerian trail.*

Proof As in the proof of Corollary 1.15 we see from Theorem 1.12 that the degree of a vertex of the graph must be even unless that vertex is one of the two endpoints of the trail, in which case the degree must be odd. ∎

We shall now work towards a proof of the fact that a non-trivial connected graph has an Eulerian circuit if the degree of each of its vertices is even. For this we use the results of the following lemmas.

Lemma 1.16 *Let $v\,w$ be an edge of a graph in which the degree of every vertex is even. Then there exists a circuit of the graph which traverses the edge $v\,w$.*

Proof Let $v_0 = v$ and $v_1 = w$. Suppose that, for some positive integer k a trail $v_0\,v_1\,\ldots\,v_k$ has been constructed in the graph starting at the vertex v and traversing the edge $v\,w$. Suppose also that $v_k \neq v$. It follows from Theorem 1.12 that the number of edges of the trail incident to v_k must be odd. But the degree of v_k is even. It follows that the number of edges of the trail incident to v_k must be strictly less than the degree of v_k, and therefore there must exist at least one edge of the graph incident to v_k which is not traversed by the trail $v_0\,v_1\,\ldots\,v_k$. Let that edge be $v_k\,v_{k+1}$, where v_{k+1} is a vertex adjacent to v_k. Then $v_0\,v_1\,\ldots\,v_k\,v_{k+1}$ is a trail of length $k+1$ in the graph which starts at v and traverses the edge $v\,w$.

Now the length of any trail in a graph cannot exceed the number of edges of the graph, since each edge of the graph is traversed at most once by any trail. It follows that successive extensions of the trail $v\,w$ will ultimately result in a trail that cannot be extended to a longer trail. This must then be closed (since we have just shown that if the trail is not closed then it can always be extended). This closed trail is then the required circuit. ∎

Lemma 1.17 *Suppose that a graph contains a circuit of length m and a circuit of length n. Suppose also that no edge of the graph is traversed by both circuits, and that at least one vertex of the graph is common to both circuits. Then the graph contains a circuit of length $m+n$.*

Proof Let u be a vertex of the graph which is common to both circuits. We may clearly suppose that both circuits start from and finish at this vertex u. Let the first circuit be $u\,v_1\,\ldots\,v_{m-1}\,u$ and let the second circuit be $u\,w_1\,\ldots\,w_{n-1}\,u$. We can then concatenate these two circuits together to obtain a third circuit

$$u\,v_1\ldots\,v_{m-1}\,u\,w_1\,\ldots\,w_{n-1}u$$

of length $m+n$. ∎

Lemma 1.18 *Let (V,E) be a connected graph, and let some trail in this graph be given. Suppose that no vertex of the graph has the property that some but not all of the edges of the graph incident to that vertex are traversed by the trail. Then the given trail is an Eulerian trail.*

Proof Let V_1 be the set of vertices through which the trail passes, and let V_2 denote the set consisting of any remaining vertices of the graph. Then $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$. We will prove that $V_2 = \emptyset$.

Now any vertex belonging to V_1 is incident to at least one edge traversed by the trail. But then all edges incident to a vertex belonging to V_1 must be traversed by the trail. But then any vertex of V adjacent to a vertex in V_1 must itself belong to V_1, and thus no edge can join a vertex in V_1 to a vertex in V_2. If the set V_2 were non-empty then there could not exist any path joining a vertex in V_2 to a vertex in V_1, and thus the graph would not be connected. Therefore V_2 must be empty, and $V_1 = V$. But then every edge of (V, E) must be traversed by the trail, and thus the trail is an Eulerian trail. ∎

Lemma 1.19 *Let (V, E) be a connected graph with the property that the degree of every vertex of the graph is even, and let some circuit in this graph be given. Suppose that there is some vertex v of the graph with the property that some but not all of the edges of the graph incident to that vertex are traversed by the given circuit. Then there exists a second circuit in the graph (V, E) which passes through the vertex v and which does not traverse any edge which is traversed by the given circuit.*

Proof let E' denote the subset of E consisting of those edges of the graph that are not traversed by the given circuit. Then (V, E') is a subgraph of the given graph (V, E). Given any vertex w, the number of edges of the given circuit that are incident to w is equal to $d(w) - d'(w)$, where $d(w)$ is the number of edges in E incident to w, and $d'(w)$ is the number of edges in E' incident to w. It follows from Lemma 1.13 that $d(w) - d'(w)$ is an even integer. But the degree $d(w)$ of each vertex w of the graph (V, E) is even, by assumption, and therefore $d'(w)$ is also even. Thus the degree of every vertex in the subgraph (V, E') is even.

Now the vertex v of the graph has the property that some but not all of edges incident to this vertex are traversed by the given trail. Therefore some at least of the edges of the graph (V, E) incident to v are edges also of the subgraph (V, E'). It then follows from Lemma 1.16 that the subgraph (V, E') contains a circuit which passes through the vertex v. This circuit is of course a circuit in the graph (V, E), it passes through the vertex v, and it does not traverse any edge of the graph (V, E) that is traversed by the given circuit. ∎

Theorem 1.20 *A non-trivial connected graph contains an Eulerian circuit if the degree of every vertex of the graph is even.*

Proof Let (V, E) be a non-trivial connected graph with the property that the degree of every vertex is even. An easy application of Lemma 1.16 shows that such a graph contains at least one circuit. It therefore contains a circuit which is at least as long as every other circuit in the graph. We shall show that this circuit of maximal length is an Eulerian circuit.

Now if the graph were to contain some vertex v with the property that some but not all of the edges of the graph incident to that vertex are traversed by this circuit of maximal length, then it would follow from Lemma 1.19 that there would exist a second circuit in the graph (V, E) which would also pass through the vertex v, and which would not traverse any edge traversed by the circuit of maximal length. But it would then follow immediately from Lemma 1.17 that the graph would contain a circuit which was longer than the circuit of maximal length, which is clearly impossible.

We conclude therefore that the graph cannot contain any vertex v with the property that some but not all of the edges of the graph incident to that vertex are traversed by the circuit of maximal length. It now follows from Lemma 1.18 that such a circuit of maximal length must be an Eulerian circuit. ∎

Remark A careful examination of the proofs of Lemma 1.16, Corollary 1.17 and Lemma 1.19 shows that they provide an algorithm for constructing an Eulerian circuit in a non-trivial connected graph whose vertices all have even degree. Indeed the proof of Lemma 1.16 shows how circuits can be constructed in such a graph, and the proofs of Corollary 1.17 and Lemma 1.19 show how to replace a circuit that is not an Eulerian circuit by a strictly longer circuit. A finite number of such replacements must ultimately result in an Eulerian circuit.

On combining the results of Corollary 1.14 and Theorem 1.20 we conclude that a non-trivial connected graph has an Eulerian circuit if and only if the degree of each of its vertices is even.

We now prove the result corresponding to Theorem 1.20 for non-trivial connected graphs with exactly two vertices whose degree is odd.

Corollary 1.21 *Suppose that a connected graph has exactly two vertices whose degrees are odd. Then there exists an Euler trail in the graph joining the two vertices with odd degrees.*

Proof Let (V, E) be the graph, and let v and w be the two vertices of this graph whose degree is odd. We may embed the graph (V, E) as a subgraph of a larger graph (V', E') whose vertices all have even degree. We choose the graph (V', E') such that $V' = V \cup \{u\}$ and $E' = E \cup \{v\,u,\, u\,w\}$, where u is a vertex of V' that does not belong to V, and is the only such vertex of

V'. The graph (V', E') is then non-trivial and connected, and every vertex of (V', E') has even degree. (Indeed the degree of the vertex u in the graph (V', E') is equal to 2, and the degrees of the vertices v and w in the graph (V', E') exceed by one their degrees in the graph (V, E).) It follows from Theorem 1.20 that the graph (V', E') has an Eulerian circuit. We may order the vertices of this circuit so that the final two edges of the circuit are $w\,u$ and $u\,v$. Deletion of these two edges from the circuit yields the required Eulerian trail in the graph (V, W) from v to w. ∎

1.14 Hamiltonian Paths and Circuits

Definition A *Hamiltonian path* in a graph is a path that passes (exactly once) through every vertex of the graph.

Thus a path $v_0\, v_1\, v_2\, \ldots\, v_n$ in a graph (V, E) is a Hamiltonian path if and only if $V = \{v_0, v_1, \ldots, v_n\}$. A Hamiltonian path passes can have no repeated vertices (since it is a path) and therefore passes through each vertex of the graph exactly once.

Definition A *Hamiltonian circuit* in a graph is a simple circuit that passes through every vertex of the graph.

Thus a circuit $v_0\, v_1\, v_2\, \ldots\, v_{n-1}\, v_0$ in a graph (V, E) is a Hamiltonian circuit if and only if every vertex of the graph occurs exactly once in the list $v_0, v_1, \ldots, v_{n-1}$.

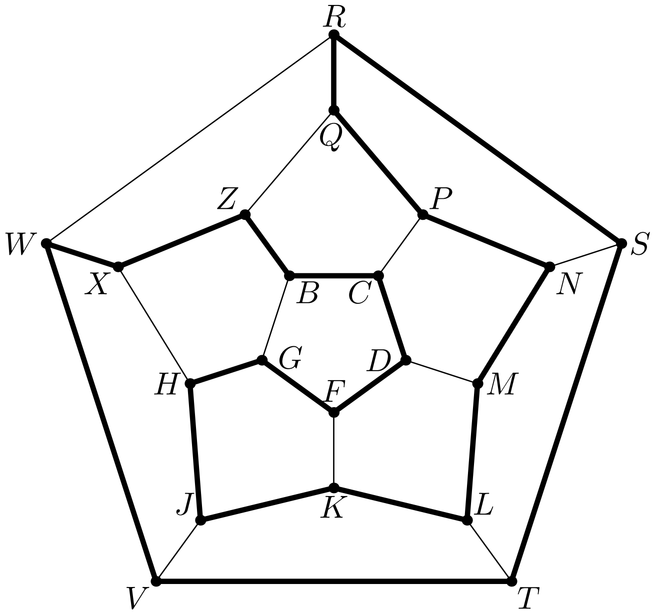
Remark Hamiltonian circuits are are named after William Rowan Hamilton (1805–1865), who showed in 1856 that such circuits could be found in the graph consisting of the vertices and edges of a dodecahedron. Hamilton developed an 'icosian calculus' for the study such circuits in the dodecahedron, and formulated a game, the *icosian game*, in which people were challenged to complete any path of length two in this graph to a Hamilton circuit in the graph.

1.15 Forests and Trees

Definition A graph is said to be *acyclic* if it contains no circuits.

Definition A *forest* is an acyclic graph.

Definition A *tree* is a connected forest.



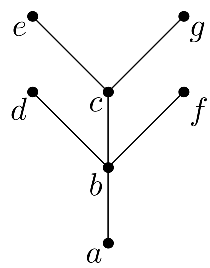
Hamilton's circuit round the edges of a dodecahedron

Note that the components of any forest are trees.

Example The graph (V, E), where

$$\begin{aligned} V &= \{a,\, b,\, c,\, d,\, e,\, f,\, g\}, \\ E &= \{a\,b,\, b\,c,\, b\,d,\, c\,e,\, b\,f,\, c\,g\}, \end{aligned}$$

is a tree.



The vertices a, d, e, f and g are pendant vertices (i.e., each of these vertices is incident to exactly one edge of the graph, and is therefore of degree one.) The tree has 7 vertices and 6 edges.

Theorem 1.22 *Every forest contains at least one isolated or pendant vertex.*

Proof If a graph has no isolated or pendant vertices, then it contains a circuit (Theorem 1.10). But a forest contains no circuits. Therefore must have at least one isolated or pendant vertex. ∎

Theorem 1.23 *A non-trivial tree contains at least one pendant vertex.*

Proof A non-trivial graph has more than one vertex. If a non-trivial graph has an isolated vertex then there does not exist any path or walk from that vertex to any other vertex of the graph, and therefore the graph is not connected. But a tree is by definition connected. Therefore a non-trivial tree cannot have any isolated vertex. However a tree is a forest, and therefore contains at least one vertex that is either an isolated vertex or a pendant vertex (Theorem 1.22). Such a vertex must then be a pendant vertex. ∎

Theorem 1.24 *Let (V, E) be a tree. Then $|E| = |V| - 1$, where $|V|$ and $|E|$ denote respectively the number of vertices and the number of edges of the tree.*

Proof We can prove the result by induction on the number $|V|$ of vertices of the tree. The result is clearly true when the tree is trivial, since it then consists of one vertex and no edges.

Suppose that every tree with m vertices has $m - 1$ edges. Let (V, E) be a tree with $m + 1$ vertices. At least of these vertices is a pendant vertex (Theorem 1.23). Let v be a pendent vertex, let w be the vertex that is adjacent to v, let $V' = V \setminus \{v\}$, and let $E' = E \setminus \{v\,w\}$. Then (V', E') is a subgraph of (V, E), and this subgraph has m vertices. (This subgraph is obtained from the original graph by deleting the vertex v and the edge $v\,w$ from that graph.) We claim that this subgraph (V', E') is in fact a tree.

First we show that (V', E') is connected. Now, given any two vertices in V', there exists a path in (V, E) from one vertex to the other. This path could not pass through the vertex v, since otherwise the path would have to pass through w twice (going out to v and then returning from v), which is impossible since a path by definition has no repeated vertices. Therefore this path is in fact a path in (V', E'). We conclude that (V', E') is connected.

Now the tree (V, E) does not contain any circuits. It follows immediately that the connected subgraph (V', E') does not contain any circuits, and is thus a tree. It has m vertices.

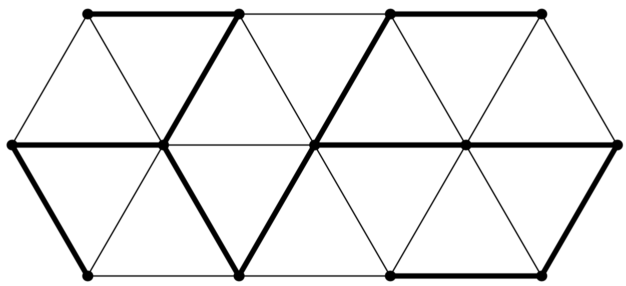
The induction hypothesis now ensures that the tree (V', E') has $m - 1$ edges, and therefore the tree (V, E) has m edges. The required result therefore follows by the Principle of Mathematical Induction. ∎

Theorem 1.25 *Given two distinct vertices of a tree, there exists a unique path in the tree from the first vertex to the the second.*

Proof Let u and v be distinct vertices of the tree. There must exist at least one path in the tree from u to v, since any tree is connected. Were there to exist more than one, then it would follow from Theorem 1.11 that there would exist at least one circuit in the tree, which is impossible, since that a tree cannot contain any circuits. Therefore there must exist exactly one path in the tree from u to v. ∎

1.16 Spanning Trees

Definition A *spanning tree* in a graph (V, E) is a subgraph of the graph (V, E) that is a tree which includes every vertex of the graph (V, E).



A spanning tree in a graph

Theorem 1.26 *Every connected graph contains a spanning tree*

Proof Let (V, E) be a connected graph. The collection consisting of all the connected subgraphs of (V, E) with the same vertices as (V, E) is non-empty, since it includes the graph (V, E) itself. Choose a subgraph (V, E') in this collection such that the number $|E'|$ of edges in this subgraph is less than or equal to the number of edges of any other subgraph in the collection. We claim that (V, E') is the required spanning tree. Clearly (V, E') is connected and has the same vertices as V. It only remains to show that (V, E') does not contain any circuits.

Suppose that (V, E') were to contain a circuit. Let $v\,w$ be an edge traversed by some circuit in (V, E'), and let $E'' = E \setminus \{v\,w\}$. There would then exist a walk from v to w whose edges belong to E''. (Such a walk could consist of the remaining edges of the circuit traversing the edge $v\,w$.) Moreover every vertex in V could be joined to v by a walk whose edges belong to E', and could therefore be joined either to v or to w by a walk whose edges belong to E''. It would then follow that every vertex of V could be joined to v by a walk whose edges belong to E'', and therefore the graph (V, E'') would be a connected subgraph of (V, E) with the same vertices as (V, E) and with fewer edges than (V, E'), which is impossible. We conclude therefore that the subgraph (V, E') of (V, E) cannot contain any circuits and is therefore the required spanning tree. ∎

Corollary 1.27 *Let (V, E) be a connected graph with $|V|$ vertices and $|E|$ edges. Suppose that $|E| = |V| - 1$. Then the graph (V, E) is a tree.*

Proof A connected graph (V, E) contains a spanning tree, by Theorem 1.26. This spanning tree must have $|V| - 1$ edges, by Theorem 1.24. But the spanning tree then has the same number of edges as the original graph (V, E), and must therefore be the same as this graph. It follows that the graph (V, E) must be a tree, since it is a spanning tree of itself. ∎

1.17 Directed Graphs

Definition An *directed graph* or *digraph* (V, E) consists of a finite set V together with a subset E of $V \times V$. The elements of V are the *vertices* of the digraph; the elements of E are the *edges* of the digraph.

An edge of a digraph (V, E) is an ordered pair (a, b) where a and b are vertices of the graph. These vertices need not be distinct: a digraph may contain *loops* of the form (a, a), where a is some vertex of the digraph. Also the vertices of an edge of a digraph are ordered: if a and b are distinct vertices of the graph then $(a, b) \neq (b, a)$, and moreover neither, one only, or both of (a, b) and (b, a) may be edges of the digraph.

Let (a, b) be an edge of a directed graph (V, E). We say that a is the *initial vertex* and b is the *terminal vertex* of the edge. Moreover we say that the vertex b is *adjacent from* the vertex a, and the vertex a is *adjacent to* the vertex b, and the edge (a, b) is *incident from* the vertex a and *incident to* the vertex b.

1.18 Adjacency Matrices of Directed Graphs

Definition Let (V, E) be a directed graph, and let the vertices of the graph be ordered as $v_1, v_2, \ldots, v_m$. The *adjacency matrix* of the directed graph is the $m \times m$ matrix (b_{ij}), or

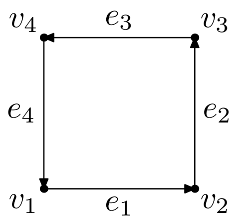
$$\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & a_{m2} & \dots & b_{mm} \end{pmatrix},$$

where

$$b_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E; \\ 0 & \text{otherwise.} \end{cases}$$

Example Let (V, E) be the directed graph whose vertices are ordered as v_1, v_2, v_3 and v_4, and whose edges are ordered as e_1, e_2, e_3 and e_4, where

$$e_1 = (v_1, v_2), \quad e_2 = (v_2, v_3), \quad e_3 = (v_3, v_4), \quad e_4 = (v_4, v_1).$$



The adjacency matrix of this digraph is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

1.19 Directed Graphs and Binary Relations

There is a correspondence between directed graphs and binary relations on finite sets.

Let V be a finite set. Corresponding to any relation R on V there is a directed graph (V, E), where

$$E = \{(a, b) \in V \times V : aRb\}.$$

Conversely any directed graph (V, E) gives rise to a relation R on the set V of vertices of the digraph, where vertices a and b of the graph satisfy aRy if and only if $(a, b) \in R$.