Course 2BA1: Michaelmas Term 2006
Section 3: Differential Equations

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Contents

3 Differential Equations

3.1 Examples of Differential Equations

A *differential equation* is an equation that relates a function y of a variable x to its derivatives. Such a differential equation can usually be written in the form

$$F\left(\frac{d^p y}{dx^p}, \frac{d^{p-1}y}{dx^{p-1}}, \ldots \frac{dy}{dx}, y, x\right) = 0,$$

where p is a positive integer and F is a real-valued (or complex-valued) function with $p+2$ arguments. If the differential equation can be expressed in the above form for some positive integer p, but cannot be expressed in this form with p replaced by any smaller integer, then the differential equation is said to be of *order p*.

The following are typical examples of differential equations:

$$\frac{dy}{dx} + 2y = 0; \tag{1}$$

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0; \tag{2}$$

$$\frac{dy}{dx} - 2xy = 0; \tag{3}$$

$$\left(\frac{dy}{dx}\right)^2 + y^2 - 1 = 0. \tag{4}$$

Equation (2) is a 2nd order differential equation. The other three equations are first order differential equations.

The function $y = e^{-2x}$ is the solution to the differential equation (1), since

$$\frac{d}{dx}e^{-2x} + 2e^{-2x} = -2e^{-2x} + 2e^{-2x} = 0.$$

It follow easily from this that the function $y = Ae^{-2x}$ solves this differential equation for any constant A.

The function $y = e^{2x}$ solves the differential equation (2), since

$$\frac{d^2}{d^2x}e^{2x} - 4\frac{d}{dx}e^{2x} + 4e^{2x} = 4e^{2x} - 8e^{2x} + 4e^{2x} = 0.$$

The function $y = xe^{2x}$ also solves this differential equation, since

$$\begin{aligned}
&\frac{d^2}{d^2x}(xe^{2x}) - 4\frac{d}{dx}(xe^{2x}) + 4xe^{2x} \\
&\qquad = \frac{d}{dx}((2x+1)e^{2x}) - 4(2x+1)e^{2x} + 4xe^{2x} \\
&\qquad = (4x+4)e^{2x} - 4(2x+1)e^{2x} + 4xe^{2x} = 0
\end{aligned}$$

Now if $y = (Ax + B)e^{2x}$ then $y = Au + Bv$, where $u = xe^{2x}$ and $v = e^{2x}$, and therefore

$$\frac{d^2y}{d^2x} - 4\frac{dy}{dx} + 4y = A\left(\frac{d^2u}{d^2x} - 4\frac{du}{dx} + 4u\right) + B\left(\frac{d^2v}{d^2x} - 4\frac{dv}{dx} + 4v\right) = 0.$$

We conclude that, for any given values of the constants A and B, the function $(Ax + B)e^{2x}$ solves the differential equation (2).

The function $y = e^{x^2}$ is a solution of the differential equation (3). And the functions $y = \sin x$ and $y = \cos x$ are solutions of the differential equation (4).

3.2 The Differential Equation $\dfrac{dy}{dx} + ay = 0$

Let a be a non-zero real number, and let us seek solutions to the differential equation

$$\frac{dy}{dx} + ay = 0. \tag{5}$$

We suppose that our function y can be represented as a power series in x, of the form

$$y = \sum_{n=0}^{\infty} \frac{y_n}{n!} x^n,$$

where $y_0, y_1, y_2, y_3, \ldots$ are constants to be determined. Now

$$y = y_0 + \sum_{n=0}^{\infty} \frac{y_{n+1}}{(n+1)!} x^{n+1},$$

and

$$\frac{d}{dx}\left(\frac{y_{n+1}}{(n+1)!} x^{n+1}\right) = \frac{(n+1)y_{n+1}}{(n+1)!} x^n = \frac{y_{n+1}}{n!} x^n.$$

It follows that

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} \frac{y_{n+1}}{n!} x^n.$$

(Here we have differentiated the power series for the function y term by term. It can be proved that we are justified in doing so, but we do not attempt such a proof here.) Therefore

$$0 = \frac{dy}{dx} + ay = \sum_{n=0}^{\infty} \frac{y_{n+1} + ay_n}{n!} x^n.$$

Now if the right hand side is to be the zero function, then the coefficient of x^n must be zero for all non-negative integers n, and therefore $y_{n+1} + ay_n = 0$ for all non-negative integers n. Thus $y_n = C(-a)^n$ for all non-negative integers n, where $C = y_0$. But then

$$y = \sum_{n=0}^{\infty} \frac{C(-a)^n x^n}{n!} = C \sum_{n=0}^{\infty} \frac{(-ax)^n}{n!} = Ce^{-ax}.$$

We conclude, therefore, that any solution to the differential equation 5 that can be represented as a power series must be a function y of the variable x that is given by an equation of the form $y = Ce^{-ax}$ for some constant C. (There are no other solutions to this differential equation.)

3.3 The Differential Equation $\dfrac{d^2y}{dx^2} - k^2y = 0$

We now use the method of power series to find solutions to the equation

$$\frac{d^2y}{dx^2} - k^2y = 0, \tag{6}$$

where k is a real number satisfing $k \neq 0$. Let

$$y = \sum_{n=0}^{\infty} \frac{y_n}{n!} x^n.$$

Then

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} \frac{y_{n+1}}{n!} x^n,$$

and hence

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} \frac{y_{n+2}}{n!} x^n,$$

It follows that the function y satisfies the differential equation 6 if and only if

$$\sum_{n=0}^{\infty} \frac{y_{n+2} - k^2y_n}{n!} x^n = 0,$$

and thus if and only if

$$y_{n+2} - k^2y_n = 0$$

for all non-negative integers n. It is then easy to see that the values of $y_2, y_3, y_4, y_5, \ldots$ are determined by the values of y_0 and y_1. Now we can find constants A and B such that $y_0 = A+B$ and $y_1 = Ak-Bk$. (These constants are given by the formulae $A = (ky_0+y_1)/(2k)$ and $B = (ky_0-y_1)/(2k)$.) One then readily verify that $y_n = Ak^n + B(-k)^n$ for all non-negative integers n. Therefore

$$y = A\sum_{n=0}^{\infty} \frac{(kx)^n}{n!} + B\sum_{n=0}^{\infty} \frac{(-kx)^n}{n!} = Ae^{kx} + Be^{-kx}.$$

One can readily verify that any function of this form satisfies the differential equation. There are no other solutions.

3.4 The Differential Equation $\dfrac{d^2y}{dx^2} + k^2 y = 0$

Let y be a solution to the differential equation

$$\frac{d^2y}{dx^2} + k^2 y = 0, \tag{7}$$

where k is a real number satisfing $k \neq 0$, and let

$$y = \sum_{n=0}^{\infty} \frac{y_n}{n!} x^n.$$

Then

$$y_{n+2} + k^2 y_n = 0$$

for all non-negative integers n. It is then easy to see that the values of $y_2, y_3, y_4, y_5, \ldots$ are determined by the values of y_0 and y_1. Let $A = y_0$ and $B = y_1/k$. Then $y_{2m} = (-1)^m A k^{2m}$ and $y_{2m+1} = (-1)^m B k^{2m+1}$ for all non-negative integers m. On referring to the Taylor series for the sine and cosine functions, we find easily that

$$y = A \sum_{m=0}^{\infty} \frac{(-)^m (kx)^{2m}}{(2m)!} + B \sum_{m=0}^{\infty} \frac{(-1)^m (kx)^{2m+1}}{(2m+1)!} = A \cos kx + B \sin kx.$$

It is then easy to verify that the function $A \cos kx + B \sin kx$ does indeed satisfy the differential equation for any values of the constants A and B. There are no other solutions.

3.5 The Differential Equation $\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0$

Let y be a solution to the differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0, \tag{8}$$

and let $u = e^{\frac{bx}{2}} y$. Then $y = e^{-\frac{bx}{2}} u$, and therefore

$$\begin{aligned}
\frac{dy}{dx} &= e^{-\frac{bx}{2}} \frac{du}{dx} - \tfrac{1}{2} b e^{-\frac{bx}{2}} u, \\
\frac{d^2y}{dx^2} &= e^{-\frac{bx}{2}} \frac{d^2u}{dx^2} - b e^{-\frac{bx}{2}} \frac{du}{dx} + \tfrac{1}{4} b^2 e^{-\frac{bx}{2}} u.
\end{aligned}$$

On substituting these values into the differential equation, we find that

$$e^{-\frac{bx}{2}}\left(\frac{d^2u}{dx^2} - \tfrac{1}{4}b^2u + cu\right) = 0.$$

Thus the function u is a solution to the differential equation

$$\frac{d^2u}{dx^2} - \tfrac{1}{4}(b^2 - 4c)u = 0.$$

If $b^2 - 4c > 0$, then our previous results show that $u = Ae^{kx} + Be^{-kx}$, where $k = \sqrt{b^2 - 4c}$. It follows that

$$y = Ae^{px} + Be^{qx}$$

where

$$p = \frac{1}{2}(-b + \sqrt{b^2 - 4c}), \quad q = \frac{1}{2}(-b - \sqrt{b^2 - 4c}).$$

Note that p and q are roots of the quadratic polynomial $s^2 + bs + c$.

If $b^2 - 4c = 0$, then the second derivative of the function u vanishes, and therefore $u = Ax + B$. But then

$$y = (Ax + B)e^{-\frac{bx}{2}}.$$

In this case $-\frac{1}{2}b$ is a repeated root of the quadratic polynomial $s^2 + bs + c$.

If $b^2 - 4c < 0$, then $u = A\cos kx + B\sin kx$, where $k = \sqrt{4c - b^2}$. It follows that

$$y = e^{-\frac{bx}{2}}(A\cos kx + B\sin kx) \qquad (k = \sqrt{4c - b^2})$$

In this case $-\frac{1}{2}b \pm ik$ are the roots of the quadratic polynomial $s^2 + bs + c$.

From these observations, we see that the solutions of the differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

can be found from the roots of the associated *auxiliary polynomial* s^2+bs+c, as described in the following theorem.

Theorem 3.1 *Let b and c be real numbers. Then the solutions of the differential equation*

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0,$$

are determined by the roots of the auxiliary polynomial

$$s^2 + bs + c$$

as follows:—

(i) *if* $b^2 > 4c$ *then the auxiliary polynomial* s^2+bs+c *has two real roots* r_1 *and* r_2*, and the general solution of the differential equation is given by*

$$y = Ae^{r_1 x} + Be^{r_2 x},$$

where A *and* B *are constants;*

(ii) *if* $b^2 = 4c$ *then the auxiliary polynomial* s^2+bs+c *has a repeated root* r*, and the general solution of the differential equation is given by*

$$y = (Ax + B)e^{rx},$$

where A *and* B *are constants;*

(iii) *if* $b^2 < 4c$ *then the auxiliary polynomial* $s^2 + bs + c$ *has two non-real roots* $p+iq$ *and* $p-iq$ *(where* p *and* q *are real numbers), and the general solution of the differential equation is given by*

$$y = e^{px} \left(A \sin qx + B \cos qx \right),$$

where A *and* B *are constants.*

Example Consider the differential equation

$$\frac{d^2y}{dx^2} - 11\frac{dy}{dx} + 24y = 0.$$

The auxiliary polynomial associated to this equation is the quadratic polynomial $s^2 - 11s + 24$. This polynomial has two real roots with values 3 and 8. The general solution of this differential equation is therefore of the form

$$y = Ae^{3x} + Be^{8x},$$

where A and B are arbitrary real constants.

Example Consider the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0.$$

The auxiliary polynomial associated to this equation is the quadratic polynomial $s^2 + 4s + 4$. This polynomial has a repeated real root with value -2. The general solution of this differential equation is therefore of the form

$$y = (Ax + B)e^{-2x},$$

where A and B are arbitrary real constants.

Example Consider the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0.$$

The auxiliary polynomial associated to this equation is the quadratic polynomial $s^2 - 4s + 5$. This polynomial has a a pair of non-real roots with values $2 + i$ and $2 - i$. The general solution of this differential equation is therefore of the form

$$y = Ae^{2x} \sin x + Be^{2x} \cos x,$$

where A and B are arbitrary real constants.

3.6 Inhomogeneous Linear Differential Equations of the Second Order with Constant Coefficients

We now discuss the general solution of an *inhomogenous linear differential equation of the second order with constant coefficients*. Such a differential equation is of the form

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x),$$

where b and c are real numbers.

Suppose that y_P is some function of the variable x which satisfies this differential equation. Let y be any twice-differentiable function of the variable x, and let $y_C = y - y_P$. Then

$$\begin{aligned} a\frac{d^2y_C}{dx^2} + b\frac{dy_C}{dx} + cy_C &= a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy - a\frac{d^2y_P}{dx^2} - b\frac{dy_P}{dx} - cy_P \\ &= a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy - f(x). \end{aligned}$$

It follows that the function y satisfies the inhomogeneous differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x),$$

if and only if y_C satisfies the corresponding homogeneous differential equation

$$\frac{d^2y_C}{dx^2} + b\frac{dy_C}{dx} + cy_C = 0,$$

We see therefore that, once a particular solution y_P of the inhomogeneous differential equation has been found, any other solution of the inhomogeneous differential equation may be obtained by adding to y_P a solution y_C of the corresponding homogeneous differential equation. The function y_P is referred to as a *particular integral* of the inhomogeneous differential equation, and the function y_C is referred to as the *complementary function.* Any solution y of the given inhomogeneous differential equation

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x),$$

is the sum of the particular integral y_P, which satisfies the same differential equation, and a complementary function y_C, which satisfies the corresponding homogeneous linear differential equation

$$\frac{d^2y_C}{dx^2} + b\frac{dy_C}{dx} + cy_C = 0.$$

Example Let us find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 10y = x^2.$$

We first find a particular integral of this equation. Examination of this equation shows that it might be sensible to look for a particular integral which is a quadratic polynomial in x of the form $px^2 + qx + r$, where the coefficients p, q and r are chosen appropriately. Now if $y = px^2 + qx + r$ then

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 10y = 10px^2 + (10q + 14p)x + 10r + 7q + 2p.$$

If the right hand side of this equation is to equal x^2, then p, q and r must be chosen so as to satisfy the equations

$$10p = 1, \quad 10q + 14p = 0, \quad 10r + 7q + 2p = 0.$$

The solution of these equations is given by

$$p = \frac{1}{10}, \quad q = -\frac{7}{50}, \quad r = -\frac{39}{500}.$$

We conclude that a particular integral y_P of the differential equation is given by

$$y_P = \frac{1}{10}x^2 - \frac{7}{50}x - \frac{39}{500}.$$

The complementary function y_C must satisfy the differential equation

$$\frac{d^2y_C}{dx^2} + 7\frac{dy_C}{dx} + 10y_C = 0.$$

The roots of auxiliary polynomial $s^2 + 7s + 10$ associated to this differential equation are -2 and -5. The complementary function y_C is then of the form

$$y_C = Ae^{-2x} + Be^{-5x}.$$

where A and B are arbitrary real constants. The general solution of the differential equation

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 10y = x^2$$

is then

$$y = \frac{1}{10}x^2 - \frac{7}{50}x - \frac{39}{500} + Ae^{-2x} + Be^{-5x}.$$

Remark Suppose that one is seeking a particular integral of an inhomogeneous differential equation of the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x),$$

where $f(x)$ is a polynomial in x, and $c \neq 0$. There will exist a particular integral y_P of the form $y_P = g(x)$, where $g(x)$ is a polynomial in x of the same degree as $f(x)$. Let

$$f(x) = p_0 + p_1x + p_2x^2 + \cdots + p_nx^n, \quad g(x) = q_0 + q_1x + q_2x^2 + \cdots + q_nx^n,$$

If we equate coefficients of powers of x on both sides of the differential equation

$$a\frac{d^2}{dx^2}g(x) + b\frac{d}{dx}g(x) + cg(x) = f(x),$$

we obtain a system of simultaneous linear equations which determine the coefficients $q_0, q_1, \ldots, q_n$ of the polynomial $g(x)$ in terms of the coefficients $p_0, p_1, \ldots, p_n$ of the polynomial $f(x)$. This enables us to find a particular integral of the differential equation.

Example Let us find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \sin x.$$

First we seek a particular integral of this equation. Now

$$\text{if } y = \sin x \text{ then } y'' - 6y' + 9y = 8\sin x - 6\cos x,$$

$$\text{if } y = \cos x \text{ then } y'' - 6y' + 9y = 8\cos x + 6\sin x.$$

Thus if

$$y_P = \frac{1}{50}\,(4\sin x + 3\cos x)$$

then $y_P'' - 6y_P' + 9y_P = \sin x$, and thus y_P is a particular integral of the inhomogeneous differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \sin x.$$

The complementary function y_C is then a solution of the corresponding homogeneous differential equation $y_C'' - 6y_C' + 9y = 0$. The associated auxiliary polynomial $s^2 - 6s + 9$ has a repeated root, whose value is 3. The complementary function y_C is then given by $y_C = (Ax + B)e^{3x}$, where A and B are real constants. The general solution of the differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \sin x$$

is then given by

$$y = \frac{1}{50}\,(4\sin x + 3\cos x) + (Ax + B)e^{3x}.$$

Example Let us find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = xe^{3x}.$$

Examination of this differential equation suggests that it might be sensible to look for a particular integral of the form $y_P = (p + qx)e^{3x}$, where p and q are appropriately chosen real constants. Now if $y_P = (p + qx)e^{3x}$ then

$$y_P' = (3p + q + 3qx)e^{3x}, \quad y_P'' = (9p + 6q + 9qx)e^{3x},$$

and thus

$$y_P'' - 2y_P' + 5y_P = (8p + 4q + 8qx)e^{3x}.$$

Thus $y_P'' - 2y_P' + 5y_P = xe^{3x}$ if and only if $p = -\frac{1}{16}$ and $q = \frac{1}{8}$. A particular integral y_P of the differential equation is thus given by

$$y_P = \frac{1}{16}(2x - 1)e^{3x}.$$

The complementary function y_C satisfies the differential equation $y_C'' - 2y_C' + 5y_C = 0$. The roots of the associated auxiliary polynomial $s^2 - 2s + 5$ are $1 + 2i$ and $1 - 2i$. The complementary function y_C is therefore of the form

$$y_C = Ae^x \sin 2x + Be^x \cos 2x.$$

where A and B are arbitrary real constants. The general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = xe^{3x}$$

is thus given by

$$y = \frac{1}{16}(2x-1)e^{3x} + Ae^x \sin 2x + Be^x \cos 2x.$$

3.7 Homogeneous and Inhomogeneous Linear Differential Equations of the First Order

We shall describe a method for solving differential equations of the form

$$\frac{dy}{dx} + p(x)y = r(x).$$

Such an equation is a homogeneous linear first order differential equation if $r(x) = 0$ for all x. It is inhomogeneous if the function r is not everywhere zero.

Consider the function $q(x)$ where

$$q(x) = \exp\left(\int p(x)\, dx\right).$$

(Here $\exp u = e^u$ for all real numbers u, and $\int p(x)\, dx$ denotes some indefinite integral of the function p.) On applying the Chain Rule and the Fundamental Theorem of Calculus, we find that

$$\frac{d}{dx}q(x) = \exp\left(\int p(x)\, dx\right) \frac{d}{dx}\int p(x)\, dx = q(x)p(x).$$

Thus

$$p(x) = \frac{q'(x)}{q(x)},$$

where

$$q'(x) = \frac{dq(x)}{dx}.$$

It follows that a function y of x is a solution of the differential equation

$$y'(x) + p(x)y(x) = r(x).$$

if and only if

$$q(x)y'(x) + q'(x)y(x) = q(x)r(x).$$

But

$$q(x)y'(x) + q'(x)y(x) = \frac{d}{dx}\left(q(x)y(x)\right).$$

It follows that the function y satisfies the differential equation

$$y'(x) + p(x)y(x) = r(x)$$

if and only if

$$q(x)y(x) = \int q(x)r(x)\,dx + C,$$

where C is a constant of integration. The general solution of the differential equation. On dividing this equation by $q(x)$, we obtain the following result:

Theorem 3.2 *The general solution of the differential equation*

$$\frac{dy}{dx} + p(x)y = r(x).$$

is thus given by

$$y(x) = \frac{1}{q(x)}\int q(x)r(x)\,dx + \frac{C}{q(x)},$$

where

$$q(x) = \exp\left(\int p(x)\,dx\right),$$

and where C is some constant.

The function q is referred to as an *integrating factor* for the differential equation.

Example Consider the differential equation

$$\frac{dy}{dx} + cy = x.$$

The general solution then has the form

$$y(x) = \frac{1}{q(x)}\int q(x)r(x)\,dx + \frac{C}{q(x)},$$

where

$$q(x) = \exp\left(\int c\,dx\right) = e^{cx}$$

and $r(x) = x$. Using the method of Integration by Parts, we find that

$$\begin{aligned}\int_0^x q(s)r(s)\,ds &= \int_0^x se^{cs}\,ds = \left[\frac{1}{c}se^{cs}\right]_0^x - \frac{1}{c}\int_0^x e^{cs}\,ds \\ &= \frac{x}{c}e^{cx} - \frac{1}{c^2}(e^{cx} - 1).\end{aligned}$$

Using this function as an indefinite integral of $q(x)r(x)$, we find that the general solution of the differential equation is given by

$$\begin{aligned}y(x) &= \frac{1}{e^{cx}}\left(\frac{x}{c}e^{cx} - \frac{1}{c^2}(e^{cx} - 1)\right) + \frac{C}{e^{cx}} \\ &= \frac{x}{c} - \frac{1}{c^2}(1 - e^{-cx}) + Ce^{-cx}.\end{aligned}$$

where C is an arbitrary constant. We may write this general solution in the simpler form

$$y(x) = \frac{x}{c} - \frac{1}{c^2} + Ae^{-cx},$$

where A is an arbitrary constant. The constants A and C in these two forms of the general solution are related by the equation

$$A = C + \frac{1}{c^2}.$$

Remark The solution to the differential equation

$$\frac{dy}{dx} + cy = x.$$

is of the form $y_P + y_C$, where y_P is a particular integral given by

$$y_P(x) = \frac{x}{c} - \frac{1}{c^2},$$

and y_C is the complementary function, given by $y_C = Ae^{-cx}$.

Example Consider the differential equation

$$\frac{dy}{dx} + 2xy = 0.$$

The integrating factor $q(x)$ is given by

$$q(x) = \exp\left(\int 2x\,dx\right) = e^{x^2}.$$

The solution to the differential equation therefore takes the form

$$y(x) = \frac{C}{q(x)} = Ce^{-x^2}.$$