Course 2BA1: Michaelmas Term 2006
Section 2: Trigonometric and Exponential Functions

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2 Trigonometric and Exponential Functions

2.1 Basic Trigonometric Identities

An anticlockwise rotation about the origin through an angle of θ radians sends a point (x, y) of the plane to the point (x', y'), where

$$\left\{ \begin{array}{rcl} x' & = & x\cos\theta - y\sin\theta \\ y' & = & x\sin\theta + y\cos\theta \end{array} \right. \qquad (1)$$

(This follows easily from the fact that such a rotation takes the point $(1, 0)$ to the point $(\cos\theta, \sin\theta)$ and takes the point $(0, 1)$ to the point $(-\sin\theta, \cos\theta)$.)

An anticlockwise rotation about the origin through an angle of ϕ radians then sends the point (x', y') of the plane to the point (x'', y''), where

$$\left\{ \begin{array}{rcl} x'' & = & x' \cos\phi - y' \sin\phi \\ y'' & = & x' \sin\phi + y' \cos\phi \end{array} \right. \tag{2}$$

Now an anticlockwise rotation about the origin through an angle of $\theta + \phi$ radians sends the point (x, y), of the plane to the point (x'', y''), and thus

$$\left\{ \begin{array}{rcl} x'' & = & x\cos(\theta + \phi) - y\sin(\theta + \phi) \\ y'' & = & x\sin(\theta + \phi) + y\cos(\theta + \phi) \end{array} \right. \tag{3}$$

But if we substitute the expressions for x' and y' in terms of x, y and θ provided by equation (1) into equation (2), we find that

$$\left\{ \begin{array}{rcl} x'' & = & x(\cos\theta\,\cos\phi - \sin\theta\,\sin\phi) - y(\sin\theta\,\cos\phi + \cos\theta\,\sin\phi) \\ y'' & = & x(\sin\theta\,\cos\phi + \cos\theta\,\sin\phi) + y(\cos\theta\,\cos\phi - \sin\theta\,\sin\phi) \end{array} \right. \tag{4}$$

On comparing equations (3) and (4) we see that

$$\cos(\theta + \phi) = \cos\theta\,\cos\phi - \sin\theta\,\sin\phi, \tag{5}$$

and

$$\sin(\theta + \phi) = \sin\theta\,\cos\phi + \cos\theta\,\sin\phi. \tag{6}$$

On replacing ϕ by $-\phi$, and noting that $\cos(-\phi) = \cos\phi$ and $\sin(-\phi) = -\sin\phi$, we find that

$$\cos(\theta - \phi) = \cos\theta\,\cos\phi + \sin\theta\,\sin\phi, \tag{7}$$

and

$$\sin(\theta - \phi) = \sin\theta\,\cos\phi - \cos\theta\,\sin\phi. \tag{8}$$

If we add equations (5) and (7) we find that

$$\cos\theta\,\cos\phi = \tfrac{1}{2}(\cos(\theta + \phi) + \cos(\theta - \phi)). \tag{9}$$

If we subtract equation (5) from equation (7) we find that

$$\sin\theta\,\sin\phi = \tfrac{1}{2}(\cos(\theta - \phi) - \cos(\theta + \phi)). \tag{10}$$

And if we add equations (6) and (8) we find that

$$\sin\theta\,\cos\phi = \tfrac{1}{2}(\sin(\theta + \phi) + \sin(\theta - \phi)). \tag{11}$$

If we substitute $\phi = \theta$ in equations (5) and (6), and use the identity $\cos^2\theta + \sin^2\theta = 1$, we find that

$$\sin 2\theta = 2\sin\theta\,\cos\theta \tag{12}$$

and

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta. \tag{13}$$

It then follows from equation (13) that

$$\sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta) \tag{14}$$

$$\cos^2\theta = \tfrac{1}{2}(1 + \cos 2\theta). \tag{15}$$

Remark Equations (1) and (2) may be written in matrix form as follows:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$
$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

Also equation (3) may be written

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

It follows from basic properties of matrix multiplication that

$$\begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

and therefore

$$\begin{aligned} \cos(\theta+\phi) &= \cos\theta\,\cos\phi - \sin\theta\,\sin\phi \\ \sin(\theta+\phi) &= \sin\theta\,\cos\phi + \cos\theta\,\sin\phi. \end{aligned}$$

This provides an alternative derivation of equations (5) and (6).

2.2 Basic Trigonometric Integrals

On differentiating the sine and cosine function, we find that

$$\frac{d}{dx}\sin kx = k\cos kx \tag{16}$$

$$\frac{d}{dx}\cos kx = -k\sin kx. \tag{17}$$

for all real numbers k.

It follows that

$$\int \sin kx = -\frac{1}{k}\cos kx + C \qquad (18)$$

$$\int \cos kx = \frac{1}{k}\sin kx + C, \qquad (19)$$

for all non-zero real numbers k, where C is a constant of integration.

Theorem 2.1 *Let m and n be positive integers. Then*

$$\int_{-\pi}^{\pi} \cos nx\, dx = 0, \qquad (20)$$

$$\int_{-\pi}^{\pi} \sin nx\, dx = 0, \qquad (21)$$

$$\int_{-\pi}^{\pi} \cos mx\, \cos nx\, dx = \begin{cases} \pi & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases} \qquad (22)$$

$$\int_{-\pi}^{\pi} \sin mx\, \sin nx\, dx = \begin{cases} \pi & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases} \qquad (23)$$

$$\int_{-\pi}^{\pi} \sin mx\, \cos nx\, dx = 0. \qquad (24)$$

Proof First we note that

$$\int_{-\pi}^{\pi} \cos nx\, dx = \left[\frac{1}{n}\sin nx\right]_{-\pi}^{\pi} = \frac{1}{n}(\sin n\pi - \sin(-n\pi)) = 0$$

and

$$\int_{-\pi}^{\pi} \sin nx\, dx = \left[-\frac{1}{n}\cos nx\right]_{-\pi}^{\pi} = -\frac{1}{n}(\cos n\pi - \cos(-n\pi)) = 0$$

for all non-zero integers n, since $\cos n\pi = \cos(-n\pi) = (-1)^n$ and $\sin n\pi = \sin(-n\pi) = 0$ for all integers n.

Let m and n be positive integers. It follows from equations (9) and (10) that

$$\int_{-\pi}^{\pi} \cos mx\, \cos nx\, dx = \tfrac{1}{2}\int_{-\pi}^{\pi} (\cos((m-n)x) + \cos((m+n)x))\, dx.$$

and

$$\int_{-\pi}^{\pi} \sin mx\, \sin nx\, dx = \tfrac{1}{2}\int_{-\pi}^{\pi} (\cos((m-n)x) - \cos((m+n)x))\, dx$$

But

$$\int_{-\pi}^{\pi} \cos((m+n)x)\,dx = 0$$

(since $m+n$ is a positive integer, and is thus non-zero). Also

$$\int_{-\pi}^{\pi} \cos((m-n)x)\,dx = 0 \text{ if } m \neq n,$$

and

$$\int_{-\pi}^{\pi} \cos((m-n)x)\,dx = 2\pi \text{ if } m = n$$

(since $\cos((m-n)x) = 1$ when $m = n$). It follows that

$$\begin{aligned}\int_{-\pi}^{\pi} \cos mx\, \cos nx\,dx &= \int_{-\pi}^{\pi} \sin mx\, \sin nx\,dx = \tfrac{1}{2}\int_{-\pi}^{\pi} \cos((m-n)x)\,dx \\ &= \begin{cases} \pi & \text{if } m = n; \\ 0 & \text{if } m \neq n. \end{cases}\end{aligned}$$

Using equation (11), we see also that

$$\int_{-\pi}^{\pi} \sin mx\, \cos nx\,dx = \tfrac{1}{2}\int_{-\pi}^{\pi} (\sin((m+n)x) + \sin((m-n)x))\,dx = 0$$

for all positive integers m and n. (Note that $\sin((m-n)x) = 0$ in the case when $m = n$). ∎

2.3 The Exponential Function

The exponential function $x \mapsto e^x$ is characterized by the properties that $e^0 = 1$ and

$$\frac{d}{dx} e^x = e^x.$$

This last identity is an example of a *differential equation*, and it follows from the general theory of differential equations that the conditions described above uniquely characterize the exponential function amongst differentiable functions, so that any differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ with the properties that $f(0) = 1$ and

$$\frac{d}{dx} f(x) = f(x)$$

must satisfy the equation $f(x) = e^x$ for all real numbers x. We can apply this result in order to prove that $e^{a+b} = e^a e^b$ for all real numbers a and b.

Let a be some fixed real number. A simple application of the Chain Rule for differentiation shows that

$$\frac{d}{dx}e^{x-a} = e^{x-a}.$$

It follows that if we define $f(x) = e^a e^{x-a}$ for all real numbers x then $f(0) = 1$ and

$$\frac{d}{dx}f(x) = f(x),$$

and therefore $f(x) = e^x$ for all real numbers x. Thus $e^x = e^a e^{x-a}$ for all real numbers x and a. On setting $x = a + b$, we find that $e^{a+b} = e^a e^b$ for all real numbers a and b. This standard property of the exponential function is therefore a consequence of the differential equation $(d/dx)\, e^x = e^x$ and the initial condition $e^0 = 1$ satisfied by the exponential function.

It follows from the theory of Taylor series that this exponential function may be expanded as an infinite series as follows:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

The infinite series on the right hand side of this formula converges for all values of the real number x. The following result will enable us to analyse the convergence of this series.

Proposition 2.2 *Let x be a real number, and let $a_n = x^n/n!$ for all positive integers n. Let t be a real number satisfying $0 < t < 1$, and let N be a positive integer chosen large enough to ensure that $tN > |x|$. Then $|a_{N+k}(x)| < t^k |a_N(x)|$ and*

$$|a_{N+1}(x)| + |a_{N+2}(x)| + \cdots + |a_{N+k}(x)| < \frac{t}{1-t}|a_N(x)|$$

for all positive integers k.

Proof The definition of $a_n(x)$ ensures that $a_{n+1}(x) = x a_n(x)/(n+1)$ for all non-negative integers n. It follows that

$$|a_{n+1}(x)| = \frac{|x|\,|a_n(x)|}{|n+1|} < t|a_n(x)| \quad \text{whenever} \quad n \geq N.$$

Using the Principle of Mathematical Induction, it follows that $|a_{N+k}(x)| < t^k |a_N(x)|$ for all non-negative integers k. Therefore

$$|a_{N+1}(x)| + |a_{N+2}(x)| + \cdots + |a_{N+k}(x)| \leq (t + t^2 + \cdots + t^k)|a_N(x)|$$

for all positive integers k. But

$$(1-t)(t+t^2+\cdots+t^k) = (t-t^2)+(t^2-t^3)+\cdots(t^k-t^{k+1}) = t-t^{k+1}$$

and therefore

$$t+t^2+\cdots+t^k = \frac{t-t^{k+1}}{1-t}.$$

It follows from this that

$$t+t^2+\cdots+t^k < \frac{t}{1-t} \quad \text{provided that} \quad 0<t<1.$$

We find therefore that

$$|a_{N+1}(x)|+|a_{N+2}(x)|+\cdots+|a_{N+k}(x)| < \frac{t}{1-t}|a_N(x)|$$

as required. ∎

Corollary 2.3 *Let x be a real number, and let $a_n = x^n/n!$ for all positive integers n. Let t be a real number satisfying $0<t<1$, and let N be a positive integer chosen large enough to ensure that $N > 2|x|$. Then $|a_{N+k}(x)| < 2^{-k}|a_N(x)|$ and*

$$-|a_N(x)| < a_{N+1}(x)+a_{N+2}(x)+\cdots+a_{N+k}(x) < |a_N(x)|$$

for all positive integers k.

Proof If u and v are real numbers then $-|u| \leq u \leq |u|$, and $-|v| \leq v \leq |v|$, and therefore $-(|u|+|v|) \leq u+v \leq |u|+|v|$. It follows that $|u+v| \leq |u|+|v|$ for all real numbers u and v. A straightforward proof by induction on k now shows that

$$\left|\sum_{j=1}^{k} a_{N+j}(x)\right| \leq \sum_{j=1}^{k} |a_{N+j}(x)|$$

for all positive integers k. On applying the results of Proposition 2.2 with $t=\frac{1}{2}$, we find that the right hand side of this inequality is bounded above by $|a_N(x)|$. Therefore

$$-|a_N(x)| < a_{N+1}(x)+a_{N+2}(x)+\cdots+a_{N+k}(x) < |a_N(x)|,$$

for all positive integers k. Also $|a_{N+k}(x)| \leq 2^{-k}|a_n(x)|$, as required. ∎

Suppose now that we wish to calculate the value of e^x, for some value of x, to within an error of at most ε, where $\varepsilon > 0$. Let us define

$$p_m(x) = \sum_{n=0}^{m} \frac{x^n}{n!} = 1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^m}{m!}$$

for all positive integers m. Now we can pick a value of N (depending on the choice of x) which is large enough to ensure that $|x|^N/N! < \varepsilon$. Indeed suppose we first pick some natural number M satisfying $M > 2|x|$. Then $|a_{M+k}(x)| < \dfrac{1}{2^k}|a_M(x)|$ for all positive integers k, where $a_n(x) = x^n/n!$. We can therefore make $a_{M+k}(x)$ as close to zero as we wish by choosing a sufficiently large value of k.

Example Suppose that $x = 5$. We can apply the above results with $M = 11$. Now $a_{11}(x) = 5^{11}/11! = 1.2232474798$ to 10 decimal places. Using the fact that $2^{10} = 1024 > 1000$, we find that $a_{21}(x) < 0.00123$, $a_{31}(x) < 0.00000123$, $a_{41}(x) < 0.00000000123$, etc. Indeed $a_{11+10q}(x) < 1.23 \times 10^{-3q}$ for all positive integers q.

Suppose then that we choose N large enough to ensure that $N > 2|x|$ and $|x|^N/N! < \varepsilon$. It then follows from Corollary 2.3 that $p_N(x) - \varepsilon < p_{N+k}(x) < p_N(x) + \varepsilon$ for all positive integers k. This fact is sufficient to guarantee the convergence of the Taylor series for the exponential function. The sum of this Taylor series is the limit $\lim\limits_{k\to\infty} p_{N+k}(x)$ of the partial sums $p_{N+k}(x)$ is $k \to \infty$. Moreover $p_N(x) - \varepsilon < e^x < p_N(x) + \varepsilon$ provided that N is chosen large enough to ensure that $N > 2|x|$ and $|x|^N/N! < \varepsilon$. Thus suppose we wish, for example, to find the value of e^x, for some real number x, to an accuracy of r decimal places. We can choose a natural number N that is large enough to ensure that $N > 2|x|$ and $|x|^N/N! < 10^{-r}$. Then the sum $p_N(x)$ of the first $N + 1$ terms of the Taylor series for e^x will approximate to the value of e^x to within an error less than 10^{-r}. We conclude from this that, no matter how large the real number x that we choose, the values of the partial sums $p_n(x)$ of the Taylor series for e^x will always converge to some real number $\lim\limits_{n\to\infty} p_n(x)$, and this limit is the value of the exponential e^x of x.

Example Suppose we need to calculate the value of e^3 to 6 decimal places. Let $a_n(3) = 3^n/n!$ for all non-negative integers n. Then $a_7(3) = 0.4339285714$ to 10 decimal places. Now $2^{20} > 10^6$. The results of Corollary 2.3 guarantee that $a_{27}(3) < 10^{-6}$, and that e^3 will agree with the sum of the first 28 terms of the Taylor series for the exponential function to within an accuracy of 10^{-6}. And indeed

$$3^{27} = 7625597484987, \quad 27! = 10888869450418352160768000000,$$

and therefore $a_{27}(3) \approx 7.00311223283 \times 10^{-16}$. It follows that in fact the sum of these first 28 terms is guaranteed to agree with the value of e^3 to within an accuracy of 10^{-15}.

2.4 Basic Properties of Complex Numbers

We shall extend the definition of the exponential function so as to define a value of e^z for any complex number z. First we note some basic properties of complex numbers.

A *complex number* is a number that may be represented in the form $x+iy$, where x and y are real numbers, and where $i^2 = -1$. The real numbers x and y are referred to as the *real* and *imaginary* parts of the complex number $x + iy$, and the symbol i is often denoted by $\sqrt{-1}$. One adds or subtracts complex numbers by adding or subtracting their real parts, and adding or subtracting their imaginary parts. Thus

$$(x+iy)+(u+iv) = (x+u)+i(y+v). \qquad (x+iy)-(u+iv) = (x-u)+i(y-v).$$

Multiplication of complex numbers is defined such that

$$(x + iy) \times (u + iv) = (xu - yv) + i(xv + uy).$$

The reciprocal $(x + yi)^{-1}$ of a non-zero complex number $x + iy$ is given by the formula

$$(x + iy)^{-1} = \frac{x}{x^2 + y^2} - i\,\frac{y}{x^2 + y^2}$$

One can readily verify that the complex numbers constitute a commutative group with respect to the operation of addition, and that the non-zero complex numbers constitute a commutative group with respect to the operation of multiplication.

Complex numbers may be represented by points of the plane (through the Argand diagram). A complex number $x + iy$ represents, and is represented by, the point of the plane whose Cartesian coordinates are (x, y). One often therefore refers to the set of all complex numbers as the *complex plane*. This complex plane is pictured as a flat plane, containing lines, circles etc., and distances and angles are defined in accordance with the usual principles of plane geometry and trigonometry.

The *modulus* of a complex number $x + iy$ is defined to be the quantity $\sqrt{x^2 + y^2}$: it represents the distance of the corresponding point (x, y) of the complex plane from the origin $(0, 0)$. The modulus of a complex number z is denoted by $|z|$.

Let z and w be complex numbers. Then z lies on a circle of radius $|z|$ centred at 0, and the point $z + w$ lies on a circle of radius $|w|$ centred at z. But this circle of radius $|w|$ centred at z is contained within the disk bounded by a circle of radius $|z| + |w|$ centred at the origin, and therefore $|z + w| \leq |z| + |w|$. This basic inequality is essentially a restatement of the basic geometric result that the length of any side of a triangle is less than or equal to the sum of the lengths of the other two sides. Indeed the complex numbers 0, z and $z + w$ represent the vertices of a triangle in the complex plane whose sides are of length $|z|$, $|w|$ and $|z+w|$. The inequality is therefore often referred to as the *Triangle Inequality.*

Lemma 2.4 *Let $z_1, z_2, \ldots, z_r$ be complex numbers. Then*

$$|z_1 + z_2 + \cdots + z_r| \leq |z_1| + |z_2| + \cdots + |z_r|.$$

Proof The result follows easily by induction on the number r of complex numbers involved. The result is clearly true when $r = 1$. Suppose that the result is true when $r = m$, where m is some natural number. Let $z_1, z_2, \ldots, z_{m+1}$ be a collection of $m + 1$ complex numbers. Then

$$|z_1 + z_2 + \cdots + z_m| \leq |z_1| + |z_2| + \cdots + |z_m|,$$

and therefore

$$\begin{aligned} |z_1 + z_2 + \cdots + z_m + z_{m+1}| &\leq |z_1 + z_2 + \cdots + z_m| + |z_{m+1}| \\ &\leq |z_1| + |z_2| + \cdots + |z_m| + |z_{m+1}|. \end{aligned}$$

The result is therefore true for all finite collections of complex numbers, by the Principle of Mathematical Induction. ∎

Let z and w be complex numbers, and let $z = x + iy$, $w = u + iv$. Then $zw = (xu - yv) + i(xv + yu)$ and therefore

$$\begin{aligned} |zw|^2 &= (xu - yv)^2 + (xv + yu)^2 \\ &= (x^2u^2 + y^2v^2 - 2xyuv) + (x^2v^2 + y^2u^2 + 2xyuv) \\ &= (x^2 + y^2)(u^2 + v^2) = |z|^2|w|^2. \end{aligned}$$

It follows that $|zw| = |z|\,|w|$ for all complex numbers z and w. A straightforward proof by induction on n then shows that $|z^n| = |z|^n$ for all complex numbers z and non-negative integers n.

2.5 Complex Numbers and Trigonometrical Identities

Let θ and φ be real numbers, and let

$$z = \cos\theta + i\sin\theta, \quad w = \cos\varphi + i\sin\varphi,$$

where $i = \sqrt{-1}$. Then

$$\begin{aligned} zw &= (\cos\theta\,\cos\varphi - \sin\theta\,\sin\varphi) + i(\sin\theta\,\cos\varphi + \cos\theta\,\sin\varphi) \\ &= \cos(\theta+\varphi) + i\sin(\theta+\varphi). \end{aligned}$$

2.6 The Exponential of a Complex Number

Let z be a complex number. We shall define $\exp(z)$ to be the sum of the infinite series

$$\exp z = \sum_{n=0}^{\infty} z^n/n!.$$

This infinite series defining $\exp(z)$ converges for all values of the complex number z, as we shall see.

Proposition 2.5 *For each complex number z, and for all positive integers m and n, let us define $a_n(z) = z^n/n!$ and*

$$p_m(z) = \sum_{n=0}^{m} a_n(z) = \sum_{m=0}^{m+1} \frac{z^n}{n!}.$$

(Thus $p_m(z)$ denotes the sum of the first $m+1$ terms of the infinite series defining $\exp(z)$.) Let R be a real number satisfying $R \geq 0$. Then $|a_n(z)| \leq a_n(R)$ and

$$|p_{m+k}(z) - p_m(z)| \leq p_{m+k}(R) - p_m(R)$$

for all non-negative integers m, n and k and for all complex numbers z satisfying $|z| \leq R$.

Proof Let z be a complex number satisfying $|z| \leq R$, where R is some real number satisfying $R \geq 0$, and let n, m and k be non-negative integers. Then

$$|a_n(z)| = |z^n/n!| = |z|^n/n! = a_n(|z|) \leq a_n(R).$$

Moreover a straightforward application of Lemma 2.4 shows that

$$\begin{aligned} |p_{m+k}(z) - p_m(z)| &= \left|\sum_{n=m+1}^{m+k} a_n(z)\right| \leq \sum_{n=m+1}^{m+k} |a_n(z)| \leq \sum_{n=m+1}^{m+k} a_n(R) \\ &\leq p_{m+k}(R) - p_m(R), \end{aligned}$$

as required. ∎

Corollary 2.6 *Let R and ε be real numbers satisfying $R \geq 0$ and $\varepsilon > 0$. If N is any natural number N chosen large enough to ensure that $N > 2R$ and $a_N(R) < \varepsilon$ (which is always possible), then $|p_{N+k}(z) - p_N(z)| < \varepsilon$ for all complex numbers z satisfying $|z| \leq R$.*

Proof It follows from Corollary 2.3 that $p_{N+k}(R) - p_N(R) < \varepsilon$ for all non-negative integers k, provided that $N > 2R$ and $a_N(R) < \varepsilon$. The required result therefore follows directly from Proposition 2.5. ∎

The result stated in Corollary 2.6 is sufficient to ensure that, given any complex number z, the sequence $p_1(z), p_2(z), p_3(z), \ldots$ converges to a well-defined complex number $\exp z$. For, in order that a complex number be well-defined, we need to have a definite procedure that would enable one to calculate the real and imaginary parts of this complex number to any desired degree of accuracy. Suppose that the maximum allowable margin of error is represented by ε, where $\varepsilon > 0$. Then we could choose a non-negative real number R satisfying $|z| \leq R$, and determine a positive integer N that is large enough to satisfy the conditions stated in Corollary 2.6. Then the distance in the complex plane between $p_N(z)$ and any subsequent member $p_{N+k}(z)$ of the above sequence is always less than ε, and therefore the real and imaginary parts of the complex number $p_N(z)$ will determine those of the limit $\exp z$ to an error of at most ε. Thus the value of $\exp z$ may be determined to whatever degree of accuracy is required, simply by choosing a sufficiently large value of N.

The following result now follows immediately from Corollary 2.6.

Corollary 2.7 *Let R and ε be real numbers satisfying $R \geq 0$ and $\varepsilon > 0$. If N is any natural number N chosen large enough to ensure that $N > 2R$ and $a_N(R) < \varepsilon$ (which is always possible), then*

$$\left| \exp z - \sum_{n=0}^{N} \frac{z^n}{n!} \right| \leq \varepsilon,$$

for all complex numbers z satisfying $|z| \leq R$.

Corollary 2.7 ensures that if real numbers R and ε are given, where $R \leq 0$ and $\varepsilon > 0$, and if a natural number N is determined that satisfies the conditions stated in that corollary, then the sum of the first $N + 1$ terms of the infinite series defining $\exp z$ will agree with the value of $\exp z$ to within an error of at most ε throughout the disk in the complex plane that is bounded by a circle of radius R centred at 0.

Example Here are the first fifteen values of the infinite series that defines $\exp(1 + \frac{1}{2}i)$:

$$\begin{aligned}
p_0(1+0.5i) &= 1,\\
p_1(1+0.5i) &= 2+0.5i,\\
p_2(1+0.5i) &= 2.375+i,\\
p_3(1+0.5i) &= 2.41666666667+1.22916666667i,\\
p_4(1+0.5i) &= 2.3984375+1.29166666667i,\\
p_5(1+0.5i) &= 2.38854166667+1.30234375i,\\
p_6(1+0.5i) &= 2.38600260417+1.30329861111i,\\
p_7(1+0.5i) &= 2.38557167659+1.30325365823i,\\
p_8(1+0.5i) &= 2.38552062019+1.30322110615i,\\
p_9(1+0.5i) &= 2.38551675571+1.30321465279i,\\
p_{10}(1+0.5i) &= 2.38551669193+1.30321381423i,\\
p_{11}(1+0.5i) &= 2.38551672425+1.30321373509i,\\
p_{12}(1+0.5i) &= 2.38551673024+1.30321372985i,\\
p_{13}(1+0.5i) &= 2.3855167309+1.30321372967i,\\
p_{14}(1+0.5i) &= 2.38551673096+1.30321372968i,
\end{aligned}$$

Now $|1 + 0.5i| < 2$. Now $a_{10}(2) = 0.0002821869488536155$. It follows that the real and imaginary parts of $p_m(1 + 0.5i)$ should agree with those of $\exp(1+0.5i)$ to at least three decimal places, provided that $m \geq 10$, and this is borne out on examining the above table of values of $p_m(1 + 0.5i)$.

The quantity $\exp z$ is customarily denoted by e^z for any complex number z.

2.7 Euler's Formula

Theorem 2.8 (Euler's Formula)

$$e^{i\theta} = \cos\theta + i\,\sin\theta$$

for all real numbers θ.

Proof Let us take the real and imaginary parts of the infinite series that defines $e^{i\theta}$. Now $i^2 = -1, i^3 = -i$ and $i^4 = 1$, and therefore

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{i^n\theta^n}{n!} = C(\theta) + iS(\theta),$$

where

$$
\begin{aligned}
C(\theta) &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \frac{\theta^{10}}{10!} + \frac{\theta^{12}}{12!} - \cdots \\
S(\theta) &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \frac{\theta^{11}}{11!} + \frac{\theta^{13}}{13!} - \cdots .
\end{aligned}
$$

However the infinite series that define these functions $C(\theta)$ and $S(\theta)$ are the Taylor series for the trigonometric functions $\cos\theta$ and $\sin\theta$. Thus $C(\theta) = \cos\theta$ and $S(\theta) = \sin\theta$ for all real numbers θ, and therefore $e^{i\theta} = \cos\theta + i\sin\theta$, as required. ∎

Note that if we set $\theta = \pi$ in Euler's formula we obtain the identity

$$e^{i\pi} + 1 = 0.$$

The following identities follow directly from Euler's formula.

Corollary 2.9

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right), \quad \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

for all real numbers θ.

It is customary to define the values $\cos z$ and $\sin z$ of the cosine and sine functions at any complex number z by the formulae

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right), \quad \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right).$$

Corollary 2.9 ensures that the cosine and sine functions defined for complex values of the argument in this fashion agree with the standard functions for real values of the argument defined through trigonometry.

2.8 Multiplication of Complex Exponentials

Let z and w be complex numbers. Then

$$e^z\, e^w = \left(\sum_{j=0}^{\infty} \frac{z^j}{j!}\right)\left(\sum_{k=0}^{\infty} \frac{w^k}{k!}\right) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{z^j w^k}{j!k!}.$$

Thus the value of the product $e^z\, e^w$ is equal to the value of the infinite double sum that is obtained on adding together the quantities $z^j w^k/(j!k!)$ for all ordered pairs (j, k) of non-negative integers. This double sum may be evaluated by adding together, for each non-negative integer n, the values of the quantities $z^j w^k/(j!k!)$ for all ordered pairs (j,k) of negative numbers with $j+k=n$, and then adding together the resultant quantities for all non-negative values of the integer n. Thus

$$e^z e^w = \sum_{n=0}^{\infty} \left(\sum_{\substack{(j,k) \\ j+k=n}} \frac{z^j w^k}{j!k!} \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{j=0}^{n} \frac{n!}{j!(n-j)!} z^j w^{n-j} \right).$$

(Here we have used the fact that if $j+k=n$ then $k=n-j$.) Now the quantity $\dfrac{n!}{j!(n-j)!}$ is the binomial coefficient $\dbinom{n}{j}$. It follows from the Binomial Theorem that

$$\sum_{j=0}^{n} \frac{n!}{j!(n-j)!} z^j w^{n-j} = (z+w)^n.$$

If we substitute this identity in the formula for the product $e^z e^w$, we find that

$$e^z e^w = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = e^{z+w}.$$

We have thus obtained the following result.

Theorem 2.10

$$e^z e^w = e^{z+w}$$

for all complex numbers z and w.

Remark The above theorem was derived by evaluating a double sum by grouping together terms in a certain fashion. One could question whether or not such processes can be justified when one is working with infinite series. In this particular case they can. But a more rigorous proof of Theorem 2.10 may be constructed by showing that, given any complex numbers z and w, we can make the quantity $|p_{2N}(z+w) - p_N(z)p_N(w)|$ as small as we please, provided that we choose a sufficiently large value of N. To see this, let

$$\begin{aligned} S_N &= \{(j,k) \in \mathbb{Z}^2 : 0 \leq j \leq N \text{ and } 0 \leq k \leq N\} \\ T_N &= \{(j,k) \in \mathbb{Z}^2 : j \geq 0,\ \ k \geq 0 \text{ and } j+k \leq N\} \end{aligned}$$

for all non-negative integers N. Then $S_N \subset T_{2N} \subset S_{2N}$ for all non-negative integers N. Moreover

$$p_N(z)p_N(w) = \sum_{(j,k)\in S_N} \frac{z^j w^k}{j!k!}, \quad p_{2N}(z+w) = \sum_{(j,k)\in T_{2N}} \frac{z^j w^k}{j!k!}$$

It follows that

$$p_{2N}(z+w) - p_N(z)p_N(w) = \sum_{(j,k)\in T_{2N}\setminus S_N} \frac{z^j w^k}{j!k!}$$

and therefore

$$\begin{aligned} |p_{2N}(z+w) - p_N(z)p_N(w)| &\leq \sum_{(j,k)\in T_{2N}\setminus S_N} \frac{|z|^j|w|^k}{j!k!} \leq \sum_{(j,k)\in S_{2N}\setminus S_N} \frac{|z|^j|w|^k}{j!k!} \\ &= \sum_{(j,k)\in S_{2N}} \frac{|z|^j|w|^k}{j!k!} - \sum_{(j,k)\in S_N} \frac{|z|^j|w|^k}{j!k!} \\ &= p_{2N}(|z|)p_{2N}(|w|) - p_N(|z|)p_N(|w|). \end{aligned}$$

Now the quantities $p_{2N}(|z|)p_{2N}(|w|)$ and $p_N(|z|)p_N(|w|)$ both converge to the value $e^{|z|}e^{|w|}$ as $N \to \infty$, and therefore approach one another ever more closely as N increases. It follows from this that $p_{2N}(|z|)p_{2N}(|w|) - p_N(|z|)p_N(|w|)$ may be made as small as we please by choosing a sufficiently large value of N. We can therefore make the quantity $|p_{2N}(z+w) - p_N(z)p_N(w)|$ as small as we please by choosing N sufficiently large. It then follows that $e^{z+w} = e^z e^w$, since $p_{2N}(z+w) \to e^{z+w}$ and $p_N(z)p_N(w) \to e^z e^w$ as $N \to \infty$.

On combining the results of Theorem 2.10 and Euler's Formula (Theorem 2.8), we obtain the following identity for the value of the exponential of a complex number.

Corollary 2.11

$$e^{x+iy} = e^x(\cos y + i \sin y)$$

for all complex numbers $x + iy$.

2.9 Complex Roots of Unity

Lemma 2.12 *Let ω be a complex number satisfying the equation $\omega^n = 1$ for some positive integer n. Then*

$$\omega = e^{\frac{2\pi m i}{n}} = \cos\frac{2\pi m}{n} + i\sin\frac{2\pi m}{n}$$

for some integer m.

Proof The modulus $|\omega|$ of ω is a positive real number satisfying the equation $|\omega|^n = |\omega^n| = 1$. It follows that $\omega = e^{i\theta} = \cos\theta + i\sin\theta$ for some real number θ. Now

$$(e^{i\theta})^2 = e^{i\theta}e^{i\theta} = e^{2i\theta}, \quad (e^{i\theta})^3 = e^{2i\theta}e^{i\theta} = e^{3i\theta}, \text{ etc.,}$$

and a straightforward proof by induction on r shows that

$$(e^{i\theta})^r = e^{ri\theta} = \cos r\theta + i\sin r\theta$$

for all positive integers r. Now $\omega^n = 1$. It follows that

$$1 = (e^{i\theta})^n = e^{ni\theta} = \cos n\theta + i\sin n\theta,$$

and thus $\cos n\theta = 1$ and $\sin n\theta = 0$. But these conditions are satisfied if and only if $n\theta = 2\pi m$ for some integer m, in which case $\omega = e^{2\pi mi/n}$, as required. ∎

We see that, for any positive integer n, there exist exactly n complex numbers ω satisfying $\omega^n = 1$. These are of the form $e^{2\pi mi/n}$ for $m = 0, 1, \ldots, n-1$. They lie on the unit circle in the complex plane (i.e., the circle of radius 1 centred on 0 in the complex plane) and are the vertices of a regular n-sided polygon in that plane.