Course 2BA1: Michaelmas Term 2006
Section 1: The Principle of Mathematical Induction

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Contents

1 The Principle of Mathematical Induction

1.1 Integers and Natural Numbers

An *integer* is a whole number. Such numbers are of three types, *positive*, *negative* and *zero*. The *positive integers* (or positive whole numbers) are $1, 2, 3, 4, \ldots$. Similarly the *negative integers* or negative whole numbers) are $-1, -2, -3, -4, \ldots$. There is of course exactly one integer that is zero, namely 0 itself.

The *non-negative integers* are therefore $0, 1, 2, 3, \ldots$. Similarly the *non-positive integers* are $0, -1, -2, -3, \ldots$.

It is customary in mathematics to denote the set (or collection) of integers by $\mathbb{Z}$. (The word for 'number' in German is 'Zahl'.)

The *natural numbers* are the positive integers $1, 2, 3, 4, \ldots$. It is customary to denote the set of natural numbers by $\mathbb{N}$.

(Note therefore that terms 'natural number' and 'positive integer' are synonyms, i.e., they refer to the same objects.)

1.2 Introduction to the Principle of Mathematical Induction

For each natural number n, let S_n denote the sum of the first n (positive) odd numbers. Calculating S_1, S_2, S_3, S_4, S_5, we find

$$\begin{array}{lllr} S_1 & = 1 & = & 1, \\ S_2 & = 1+3 & = & 4, \\ S_3 & = 1+3+5 & = & 9, \\ S_4 & = 1+3+5+7 & = & 16, \\ S_5 & = 1+3+5+7+9 & = & 25. \end{array}$$

You may notice a pattern beginning to emerge. Does this pattern continue? Suppose that we see whether or not the pattern continues to S_6. Adding up, we find

$$S_6 = 1 + 3 + 5 + 7 + 9 + 11 = 36.$$

We are thus led to conjecture that

$$S_n = n^2$$

for all natural numbers n?

Can we prove it? If so, how?

Merely testing the proposition for a few values of n, no matter how many, cannot in itself suffice to *prove* that the proposition holds for *all* natural numbers n. Moreover propositions may turn out to be true in a very large number of cases, and yet fail for others. Such a proposition is the following:

$$\text{“}n < 1,000,000,000\text{”}.$$

This proposition holds for a large number of natural numbers n (indeed for $999,999,999$ of them, to be precise), yet it obviously fails to hold for all natural numbers n.

One might ask what strategies are available for proving that some conjectured result does indeed hold for all natural numbers n. One such is the *Principle of Mathematical Induction*.

Suppose that, for each natural number n, $P(n)$ denotes some proposition, such as "$S_n = n^2$". For each value of n, the proposition $P(n)$ would be either true or false. Our task is to prove that it is true for all values of n. The Principle of Mathematical Induction states that this is true provided that (i) $P(1)$ is true, and (ii) if $P(m)$ is true for any natural number m then $P(m+1)$ is also true.

We can express this more informally as follows. Suppose that we are required to prove that some statement is true for all values of a natural number n. To do this, it suffices to prove (i) that the statement is true when $n = 1$, and (ii) that if the statement is true when $n = m$ for some natural number m, then it is also true when $n = m + 1$ (no matter what the value of m).

To understand the justification for the Principle of Mathematical Induction, consider the following. For each natural number n, let $P(n)$ denote (as above) a proposition (that is either true or false). We suppose that we have proved that $P(1)$ is true, and that if $P(m)$ is true then $P(m + 1)$ is true. Now

> $P(1)$ is true.
>
> If $P(1)$ is true then $P(2)$ is true. Moreover $P(1)$ is true. Therefore $P(2)$ is true.
>
> If $P(2)$ is true then $P(3)$ is true. Moreover $P(2)$ is true. Therefore $P(3)$ is true.
>
> If $P(3)$ is true then $P(4)$ is true. Moreover $P(3)$ is true. Therefore $P(4)$ is true.
>
> $\vdots$
>
> If $P(n - 2)$ is true then $P(n - 1)$ is true. Moreover $P(n - 2)$ is true. Therefore $P(n - 1)$ is true.
>
> If $P(n - 1)$ is true then $P(n)$ is true. Moreover $P(n - 1)$ is true. Therefore $P(n)$ is true.

The pattern exhibited in these statements should convince you that $P(n)$ is true for any natural number n, no matter how large.

We now consider how to apply the Principle of Mathematical Induction to prove that $S_n = n^2$ for all natural numbers n, where S_n denotes the sum of the first n odd numbers. Obviously $S_1 = 1$, so that the conjectured result holds when $n = 1$. Suppose that $S_m = m^2$ for some natural number m. Then

$$S_{m+1} = S_m + (2m + 1) = m^2 + 2m + 1 = (m + 1)^2$$

Thus if the identity $S_n = n^2$ holds when $n = m$ then it also holds when $n = m + 1$. We conclude from the Principle of Mathematical Induction that $S_n = n^2$ for all natural numbers n.

We can write out the argument rather more formally as follows. For each natural number n, let $P(n)$ denote the proposition "$S_n = n^2$". Clearly, for any given natural number n, such a proposition $P(n)$ is either true or false. We want to show that $P(n)$ is true for all natural numbers n. This however follows on applying the Principle of Mathematical Induction, given that we have noted that $P(1)$ is true, and have demonstrated that if $P(m)$ is true for any natural number m then $P(m + 1)$ is also true.

1.3 Some examples of proofs using the Principle of Mathematical Induction

Example We claim that

$$\sum_{i=1}^{n} i = \tfrac{1}{2}n(n+1)$$

for all natural numbers n, where

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n.$$

We prove this result using the Principle of Mathematical Induction.

For any natural number n let $P(n)$ denote the proposition

$$\text{“}\sum_{i=1}^{n} i = \tfrac{1}{2}n(n+1)\text{”}.$$

One can easily see that the proposition $P(1)$ is true, since both sides of the above identity reduce to the value 1 in this case.

Suppose that $P(m)$ is true for some natural number m. Then

$$\sum_{i=1}^{m} i = \tfrac{1}{2}m(m+1).$$

But then

$$\sum_{i=1}^{m+1} i = \sum_{i=1}^{m} i + (m+1) = \tfrac{1}{2}m(m+1) + (m+1) = \tfrac{1}{2}(m+1)(m+2),$$

and therefore the proposition $P(m+1)$ is also true. We can therefore conclude from the Principle of Mathematical Induction that $P(n)$ is true for all natural numbers, which is the result we set out to prove.

Example We prove by induction on n that

$$\sum_{i=1}^{n} i^2 = \tfrac{1}{6}n(n+1)(2n+1)$$

for all natural numbers n, where

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2.$$

To achieve this, we have to verify that the formula holds when $n = 1$, and that if the formula holds when $n = m$ for some natural number m, then the formula holds when $n = m + 1$.

The formula does indeed hold when $n = 1$, since $1 = \frac{1}{6} \times 1 \times 2 \times 3$.

Suppose that the formula holds when $n = m$. Then

$$\sum_{i=1}^{m} i^2 = \tfrac{1}{6}m(m+1)(2m+1).$$

But then

$$\begin{aligned}
\sum_{i=1}^{m+1} i^2 &= \sum_{i=1}^{m} i^2 + (m+1)^2 \\
&= \tfrac{1}{6}m(m+1)(2m+1) + (m+1)^2 \\
&= \tfrac{1}{6}(m+1)\left(m(2m+1) + 6(m+1)\right) = \tfrac{1}{6}(m+1)\left(2m^2 + 7m + 6\right) \\
&= \tfrac{1}{6}(m+1)(m+2)(2m+3),
\end{aligned}$$

and therefore the formula holds when $n = m+1$. The required result therefore follows using the Principle of Mathematical Induction.

Example We prove by induction on n that

$$1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 + \cdots + n(n+3) = \tfrac{1}{3}n(n+1)(n+5).$$

for all natural numbers n. The left hand side of the above identity may be written as $\displaystyle\sum_{i=1}^{n} i(i+3)$.

The required identity

$$\sum_{i=1}^{n} i(i+3) = \tfrac{1}{3}n(n+1)(n+5)$$

holds when $n = 1$, since both sides are then equal to 4. Suppose that this identity holds when n is equal to some natural number m, so that

$$\sum_{i=1}^{m} i(i+3) = \tfrac{1}{3}m(m+1)(m+5).$$

Then

$$\sum_{i=1}^{m+1} i(i+3) \quad = \quad \sum_{i=1}^{m} i(i+3) + (m+1)(m+4)$$

$$\begin{aligned}
&= \tfrac{1}{3}m(m+1)(m+5) + (m+1)(m+4) \\
&= \tfrac{1}{3}(m+1)\Big(m(m+5) + 3(m+4)\Big) \\
&= \tfrac{1}{3}(m+1)(m^2+8m+12) \\
&= \tfrac{1}{3}(m+1)(m+2)(m+6),
\end{aligned}$$

and therefore the required identity $\sum_{i=1}^{n} i(i+3) = \frac{1}{3}n(n+1)(n+5)$ holds when $n = m+1$. It now follows from the Principle of Mathematical Induction that this identity holds for all natural numbers m.

Example We can use the Principle of Mathematical Induction to prove that

$$\sum_{k=1}^{n} 5^k k = \frac{5}{16}\Big((4n-1)5^n + 1\Big).$$

for all natural numbers n. This equality holds when $n = 1$, since both sides are then equal to 5. Suppose that the equality holds when $n = m$ for some natural number m, so that

$$\sum_{k=1}^{m} 5^k k = \frac{5}{16}\Big((4m-1)5^m + 1\Big).$$

Then

$$\begin{aligned}
\sum_{k=1}^{m+1} 5^k k &= \sum_{k=1}^{m} 5^k k + 5^{m+1}(m+1) \\
&= \frac{5}{16}\Big((4m-1)5^m + 1\Big) + 5^{m+1}(m+1) \\
&= \frac{5}{16}\Big((4m-1)5^m + 1 + 16(m+1)5^m\Big) \\
&= \frac{5}{16}\Big((20m+15)5^m + 1\Big) = \frac{5}{16}\Big((4m+3)5^{m+1} + 1\Big) \\
&= \frac{5}{16}\Big((4(m+1)-1)5^{m+1} + 1\Big).
\end{aligned}$$

and thus the equality holds when $n = m+1$. It follows from the Principle of Mathematical Induction that the equality holds for all natural numbers n.

Example We now use Principle of Mathematical Induction to prove that $6^n - 1$ is divisible by 5 for all natural numbers n. The result is clearly true when $n = 1$. Suppose that the result is true when $n = m$ for some natural number m. Then $6^m - 1$ is divisible by 5. But then

$$6^{m+1} - 1 = 6^{m+1} - 6^m + (6^m - 1) = 5 \times 6^m + (6^m - 1),$$

and therefore $6^{m+1} - 1$ is also divisible by 5. It therefore follows that $6^n - 1$ is divisible by 5 for all natural numbers n.

Example Given any two positive integers n and k we define the *binomial coefficient* $\binom{n}{k}$ to be the number of ways of choosing k distinct objects from a collection consisting of n objects. We also define $\binom{n}{0} = 1$ for all natural numbers n, and we define

$$\binom{n}{k} = 0 \text{ whenever } k < 0.$$

Note that $\binom{n}{n} = 1$ (since the entire collection can be selected in exactly one way), and that $\binom{n}{k} = 0$ when $k > n$ (since it is clearly impossible to select more than n distinct objects from a collection consisting of n objects).

We wish to prove that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \text{ whenever } 0 \leq k \leq n$$

(where $0! = 1$ and where, for each natural number n, $n!$ (n factorial) denotes the product $1 \times 2 \times 3 \times \cdots \times n$ of all the natural numbers between 1 and n). We shall prove this result using the Principle of Mathematical Induction.

First, though, we derive a *recursion formula* for the binomial coefficients.

We are interested in the number $\binom{n}{k}$ of ways of choosing k objects from a collection consisting of n objects, in the case where $n > 1$. Let us suppose for the sake of argument that those n objects are coloured balls. Moreover let us suppose that exactly one of those balls is coloured black, and that the remaining balls are coloured red. There are then two distinct types of choices that we can make. We can make a choice consisting entirely of red balls: let us refer to such a choice as a *type I* choice. Alternatively we can make a choice consisting of the black ball together with $k - 1$ red balls: let us refer to such a choice as a *type II* choice. A *type I* choice requires us to choose

k red balls from a collection of $n-1$ red balls, and there are $\binom{n-1}{k}$ such choices. A *type II* choice requires us to choose $k-1$ red balls from a collection of $n-1$ red balls, and there are $\binom{n-1}{k-1}$ such choices. The total number of choices is obtained by adding together the number of *type I* choices and the number of *type II* choices. It follows that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

(Note that the definitions we have made ensure that this formula also holds when $k = 0$, and indeed when $k < 0$.)

We now proceed to prove the required formula for the binomial coefficients, using the Principle of Mathematical Induction. Let $P(n)$ denote the proposition

$$\text{“}\binom{n}{k} = \frac{n!}{k!(n-k)!} \text{ whenever } 0 \leq k \leq n\text{”}$$

The proposition $P(1)$ asserts that $\binom{1}{0} = \binom{1}{1} = 1$, which is certainly true.

Now suppose that $P(n)$ is true for some natural number n. We show that $P(n+1)$ is true. If $P(n)$ is true and if the integer k satisfies $1 \leq k \leq n$ then

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{and} \quad \binom{n}{k-1} = \frac{n!}{(k-1)!(n+1-k)!}.$$

It then follows from the recursion formula derived above that

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n+1-k)!}.$$

But

$$\frac{1}{(n-k)!} = \frac{n+1-k}{(n+1-k)!} \quad \text{and} \quad \frac{1}{(k-1)!} = \frac{k}{k!}.$$

It follows that

$$\begin{aligned} \binom{n+1}{k} &= \frac{n!}{k!(n+1-k)!}\left((n+1-k)+k\right) \\ &= \frac{(n+1)!}{k!(n+1-k)!} \end{aligned}$$

The required identity $\binom{n+1}{k} = \dfrac{(n+1)!}{k!(n+1-k)!}$ holds also when $k = 0$ and $k = n+1$, since it is easily seen that both sides of the identity are equal to

1 in these cases. We conclude that if the proposition $P(n)$ is true for any natural number n then the proposition $P(n+1)$ is also true. We can therefore conclude from the Principle of Mathematical Induction that the proposition $P(n)$ is true for all natural numbers n, which is what we are required to prove.

Example We can use the Principle of Mathematical Induction to prove that $(2n)! < 4^n(n!)^2$ for all natural numbers n. This inequality holds when $n = 1$, since in that case $(2n)! = 2! = 2$ and $4^n(n!)^2 = 4$. Suppose that the inequality holds when $n = m$ for some natural number m. Then $(2m)! < 4^m(m!)^2$. Now

$$(2(m+1))! = (2m+2)! = (2m)!(2m+1)(2m+2).$$

Also

$$4^{m+1}((m+1)!)^2 = 4(4^m(m!)^2)(m+1)^2.$$

Moreover

$$(2m+1)(2m+2) < (2m+2)^2 = 4(m+1)^2.$$

On multiplying together the two inequalities

$$(2m)! < 4^m(m!)^2 \quad \text{and} \quad (2m+1)(2m+2) < 4(m+1)^2$$

(which we are allowed to do since the quantities on both sides of these inequalities are strictly positive), we find that

$$(2m)!(2m+1)(2m+2) < 4(4^m(m!)^2)(m+1)^2.$$

Thus if the inequality $(2n)! < 4^n(n!)^2$ holds when $n = m$ then it also holds when $n = m+1$. We conclude from the Principle of Mathematical Induction that it must hold for all natural numbers n.

Example We can use the Principle of Mathematical Induction to prove that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 > \tfrac{1}{4}(n^4 + 2n^3)$$

for all natural numbers n. This inequality holds when $n = 1$, since the left hand side is then equal to 1, and the right hand side is equal to $\frac{3}{4}$. Suppose that the inequality holds when $n = m$ for some natural number m, so that

$$\sum_{i=1}^{m} i^3 > \tfrac{1}{4}(m^4 + 2m^3).$$

Then

$$\begin{aligned}
\sum_{i=1}^{m+1} i^3 &= \sum_{i=1}^{m} i^3 + (m+1)^3 \\
&> \tfrac{1}{4}(m^4 + 2m^3) + (m+1)^3 \\
&= \tfrac{1}{4}\Big(m^4 + 2m^3 + 4(m+1)^3\Big) \\
&= \tfrac{1}{4}\Big(m^4 + 6m^3 + 12m^2 + 12m + 4\Big)
\end{aligned}$$

Now

$$\begin{aligned}
(m+1)^4 + 2(m+1)^3 &= (m^4 + 4m^3 + 6m^2 + 4m + 1) \\
&\quad + (2m^3 + 6m^2 + 6m + 2) \\
&= m^4 + 6m^3 + 12m^2 + 10m + 3
\end{aligned}$$

But $12m + 4 > 10m + 3$ (since $m > 0$), and therefore

$$m^4 + 6m^3 + 12m^2 + 12m + 4 > (m+1)^4 + 2(m+1)^3.$$

It follows that

$$\sum_{i=1}^{m+1} i^3 > \tfrac{1}{4}\Big(m^4 + 6m^3 + 12m^2 + 12m + 4\Big) > \tfrac{1}{4}((m+1)^4 + 2(m+1)^3).$$

Thus if the inequality

$$\sum_{i=1}^{n} i^3 > \tfrac{1}{4}(n^4 + 2n^3)$$

holds when $n = m$ for some natural number m, then it also holds when $n = m+1$. It follows from the Principle of Mathematical Induction that this identity holds for all natural numbers n.