Course 2BA1: Trinity 2006
Section 10: Introduction to Fourier Methods

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Contents

10 Introduction to Fourier Methods

10.1 Representation of Doubly-Periodic Sequences

Definition A *doubly-infinite* sequence $(z_n : n \in \mathbb{Z})$ of complex numbers associates to every integer n a corresponding complex number z_n.

Definition We say that doubly-infinite sequence $(z_n : n \in \mathbb{Z})$ of complex numbers is *m-periodic* if $z_{n+m} = z_n$ for all integers n.

Lemma 10.1 *Let m be a positive integer, and let $\omega_m = e^{2\pi i/m}$. Then the value of $\sum_{k=0}^{m-1} \omega_m^{kn}$ is determined, for any integer n, as follows:*

$$\sum_{k=0}^{m-1} \omega_m^{kn} = \begin{cases} m & \text{if } n \text{ is divisible by } m; \\ 0 & \text{if } n \text{ is not divisible by } m. \end{cases}$$

Proof The complex number ω_m has the property that $\omega_m^m = 1$. Also

$$(1 - z)(1 + z + z^2 + \cdots + z^{m-1}) = 1 - z^m$$

for any complex number z. It follows that

$$(1-\omega_m^n)\sum_{k=0}^{m-1}\omega_m^{kn} = 1-\omega_m^{mn} = 0$$

for all integers n, and therefore

$$\sum_{k=0}^{m-1}\omega_m^{kn} = 0 \quad \text{provided that} \quad \omega_m^n \neq 1.$$

Now $\omega_m^n = 1$ if and only if the integer n is divisible by m. We can therefore conclude that

$$\sum_{k=0}^{m-1}\omega_m^{kn} = \begin{cases} m & \text{if } n \text{ is divisible by } m, \\ 0 & \text{if } n \text{ is not divisible by } m, \end{cases}$$

as required. ∎

Theorem 10.2 *Let $(z_n : n \in \mathbb{Z})$ be a doubly-infinite sequence of complex numbers which is m-periodic . Then*

$$z_n = \sum_{k=0}^{m-1} c_k \omega_m^{kn},$$

for all integers n, where $\omega_m = e^{2\pi i/m}$ and

$$c_k = \frac{1}{m}\sum_{j=0}^{m-1} z_j \omega_m^{-kj}.$$

Proof It follows from the definition of the numbers c_k that

$$\sum_{k=0}^{m-1} c_k\omega_m^{kn} = \frac{1}{m}\sum_{k=0}^{m-1}\sum_{j=0}^{m-1} z_j\omega_m^{-kj}\omega_m^{kn} = \frac{1}{m}\sum_{j=0}^{m-1}\left(z_j\sum_{k=0}^{m-1}\omega_m^{(n-j)k}\right),$$

for all integers n. Now it follows from Lemma 10.1 that

$$\sum_{k=0}^{m-1}\omega_m^{(n-j)k} = 0$$

unless $n-j$ is divisible by m, in which case

$$\sum_{k=0}^{m-1}\omega_m^{(n-j)k} = m.$$

Moreover, given any integer n, there is a unique integer r between 0 and $m-1$ for which $n-r$ is divisible by m. It follows that

$$\sum_{k=0}^{m-1} c_k \omega_m^{kn} = z_r \quad \text{where } 0 \le r < m \text{ and } r \equiv n \pmod{m}.$$

Moreover $z_r = z_n$, because the sequence $(z_n : n \in \mathbb{Z})$ is m-periodic. Thus

$$\sum_{k=0}^{m-1} c_k \omega_m^{kn} = z_n$$

for all integers n, as required. ∎

Example Let $(z_n : n \in \mathbb{Z})$ be an 3-periodic sequence with $z_0 = 2$, $z_1 = 4$, $z_2 = 5$. Let $\omega = \omega_3 = e^{2\pi i/3}$. It follows from Theorem 10.2 that

$$z_n = c_0 + c_1 \omega^n + c_2 \omega^{2n}$$

for all integers n, where $\omega_m = e^{2\pi i/m}$ and

$$c_k = \frac{1}{3}\left(z_0 + z_1 \omega^{-k} + z_2 \omega^{-2k}\right).$$

for $k = 0, 1, 2$. Now $\omega^{-1} = \omega^2$ and $\omega^{-2} = \omega$, because $\omega^3 = 1$. Therefore

$$c_k = \frac{1}{3}\left(z_0 + z_1 \omega^{2k} + z_2 \omega^{k}\right),$$

and thus

$$\begin{aligned} c_0 &= \tfrac{1}{3}(2+4+5) = \tfrac{11}{3}, \\ c_1 &= \tfrac{1}{3}(2+4\omega^2+5\omega), \\ c_2 &= \tfrac{1}{3}(2+4\omega+5\omega^2). \end{aligned}$$

Now

$$\begin{aligned} \omega &= \cos\tfrac{2\pi}{3} + i\sin\tfrac{2\pi}{3} = \tfrac{1}{2}(-1+\sqrt{3}\,i), \\ \omega^2 &= \cos\tfrac{4\pi}{3} + i\sin\tfrac{4\pi}{3} = \tfrac{1}{2}(-1-\sqrt{3}\,i). \end{aligned}$$

It follows that

$$c_1 = \tfrac{1}{6}(-5+\sqrt{3}\,i), \quad c_2 = \tfrac{1}{6}(-5-\sqrt{3}\,i).$$

Example Let $(z_n : n \in \mathbb{Z})$ be an 4-periodic sequence with $z_0 = 2$, $z_1 = 4$, $z_2 = 5$, $z_3 = 1$. Now if ω_4 is defined as in the statement of Theorem 10.2 then $\omega_4 = e^{2\pi i/4} = i$. It follows from Theorem 10.2 that

$$z_n = c_0 + c_1 i^n + c_2(-1)^n + c_3(-i)^n$$

for all integers n, where $\omega_m = e^{2\pi i/m}$ and

$$\begin{aligned} c_k &= \frac{1}{4}\left(z_0 + z_1 i^{-k} + z_2 i^{-2k} + z_3 i^{-3k}\right) \\ &= \frac{1}{4}\left(2 + 4 \times (-i)^k + 5 \times (-1)^k + i^k\right). \end{aligned}$$

Thus

$$c_0 = 3, \quad c_1 = -\tfrac{3}{4} - \tfrac{3}{4}i, \quad c_2 = \tfrac{1}{2}, \quad c_3 = -\tfrac{3}{4} + \tfrac{3}{4}i.$$

10.2 Periodic Sequences of Real Numbers

Theorem 10.3 *Let $(x_n : n \in \mathbb{Z})$ be a doubly-infinite sequence of real numbers which is m-periodic . Then*

$$x_n = \sum_{k=0}^{m-1} \left(a_k \cos\frac{2\pi kn}{m} + b_k \sin\frac{2\pi kn}{m}\right),$$

for all integers n, where $\omega_m = e^{2\pi i/m}$ and

$$a_k = \frac{1}{m}\sum_{j=0}^{m-1} x_j \cos\frac{2\pi kj}{m}, \quad b_k = \frac{1}{m}\sum_{j=0}^{m-1} x_j \sin\frac{2\pi kj}{m}.$$

Proof It follows from Theorem 10.2 that

$$x_n = \sum_{k=0}^{m-1} c_k \omega_m^{kn},$$

for all integers n, where $\omega_m = e^{2\pi i/m}$ and

$$c_k = \frac{1}{m}\sum_{j=0}^{m-1} x_j \omega_m^{-kj}.$$

Now

$$\begin{aligned} \omega_m^n &= \cos\frac{2n\pi}{m} + i \sin\frac{2n\pi}{m} \\ \omega_m^{-n} &= \cos\frac{2n\pi}{m} - i \sin\frac{2n\pi}{m} \end{aligned}$$

for all integers n. Now $c_k = a_k - b_k i$ for $k = 0, 1, \ldots, m-1$, where

$$a_k = \frac{1}{m} \sum_{j=0}^{m-1} x_j \cos \frac{2\pi kj}{m}, \quad b_k = \frac{1}{m} \sum_{j=0}^{m-1} x_j \sin \frac{2\pi kj}{m}.$$

(Note that a_k and b_k are real numbers for all k. It follows that

$$x_n = \operatorname{Re}\left(\sum_{k=0}^{m-1} c_k \omega_m^{kn} \right) = \sum_{k=0}^{m-1} \left(a_k \cos \frac{2\pi kn}{m} + b_k \sin \frac{2\pi kn}{m} \right),$$

where $\operatorname{Re}\left(\sum_{k=0}^{m-1} c_k \omega_m^{kn} \right)$ denotes the real part of $\sum_{k=0}^{m-1} c_k \omega_m^{kn}$. ∎

10.3 Fourier Series of Periodic Function

Proposition 10.4 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function of a real variable with the property that $f(t+1) = f(t)$ for all real numbers t, let m be a positive integer, and let $t_n = n/m$ for all integers n. Then*

$$f(t_n) = \sum_{k=0}^{m-1} (a_{k,m} \cos 2\pi k t_n + b_{k,m} \sin 2\pi k t_n),$$

for all integers n, where $\omega_m = e^{2\pi i/m}$ and

$$\begin{aligned} a_{k,m} &= \frac{1}{m} \sum_{j=0}^{m-1} f(t_j) \cos 2\pi k t_j, \\ b_{k,m} &= \frac{1}{m} \sum_{j=0}^{m-1} f(t_j) \sin 2\pi i t_j. \end{aligned}$$

Moreover

$$\begin{aligned} a_{m-k,m} \cos 2\pi(m-k) t_n &= a_{k,m} \cos 2\pi k t_n, \\ b_{m-k,m} \sin 2\pi(m-k) t_n &= b_{k,m} \sin 2\pi k t_n \end{aligned}$$

for all integers k and n with $0 < k < m$. Thus if m is odd then

$$f(t_n) = a_{0,m} + \sum_{k=1}^{r} (2a_{k,m} \cos 2\pi k t_n + 2b_{k,m} \sin 2\pi k t_n),$$

where $r = \frac{1}{2}(m-1)$*, and if* m *is even then*

$$\begin{aligned} f(t_n) &= a_{0,m} + \sum_{k=1}^{r} (2a_{k,m} \cos 2\pi k t_n + 2b_{k,m} \sin 2\pi k t_n) \\ &\quad + a_{r+1,m} \cos 2\pi (r+1) t_n + b_{r+1,m} \sin 2\pi (r+1) t_n, \end{aligned}$$

where $r = \frac{1}{2}(m-2)$*.*

Proof The sequence $(f(t_n) : n \in \mathbb{Z})$ is an m-periodic sequence of real numbers. The identity

$$f(t_n) = \sum_{k=1}^{m-1} (a_{k,m} \cos 2\pi k t_n + b_{k,m} \sin 2\pi k t_n),$$

therefore follows directly on applying Theorem 10.3 to this sequence.

Also mt_n is an integer for all integers n, and therefore

$$\begin{aligned} \cos 2\pi(m-k)t_n &= \cos(-2\pi k t_n) = \cos 2\pi k t_n, \\ \sin 2\pi(m-k)t_n &= \sin(-2\pi k t_n) = -\sin 2\pi k t_n. \end{aligned}$$

The expressions for $f(t_n)$ in the cases when m is odd and when m is even therefore following on grouping the terms involving $a_{k,m}$ and $b_{k,m}$ with those involving $a_{m-k,m}$ and $b_{m-k,m}$ for values of k satisfying $0 < k < m/2$. ∎

Now it can be shown that, if the function f is sufficiently well-behaved, if $k < m/2$, and if k and m are both sufficiently large, then the values of the coefficients $a_{k,m}$ and $b_{k,m}$ approximate to zero, and can be made as close to zero as desired on taking both k and m sufficiently large. It follows from Proposition 10.4 that if we choose a large value of N, and then choose m very much larger than N, we can obtain an approximation for $f(t)$, for values of t that are integer multiples of $1/m$, that takes the form

$$f(t) \approx a_{0,m} + \sum_{k=1}^{N} (2a_{k,m} \cos 2\pi k t + 2b_{k,m} \sin 2\pi k t).$$

But it follows from the expressions for $a_{k,n}$ and $b_{k,n}$ given in Proposition 10.4 that

$$\lim_{m\to\infty} a_{k,m} = \int_0^1 f(t) \cos 2\pi k t \, dt, \quad \lim_{m\to\infty} b_{k,m} = \int_0^1 f(t) \sin 2\pi k t \, dt,$$

for any continuous function f. (This follows from the fact that

$$\int_0^1 g(t)\,dt = \lim_{m\to\infty} \frac{1}{m} \sum_{j=0}^{m-1} g\left(\frac{j}{m}\right)$$

for any continuous function g.) It follows that if the function f is well-behaved, and if we choose a sufficiently large value of the positive integer N, then the periodic function f may be approximated as closely as desired by a sum of the form

$$f(t) \approx a_0 + \sum_{k=1}^{N} \left(2a_k \cos 2\pi k t + 2b_k \sin 2\pi k t\right),$$

where

$$a_k = \int_0^1 f(t) \cos 2\pi k t\,dt, \quad b_k = \int_0^1 f(t) \sin 2\pi k t\,dt.$$

Thus any well-behaved function f that satisfies the identity $f(t+1) = f(t)$ for all real numbers t may be represented in the form

$$f(t) = a_0 + \sum_{k=1}^{\infty} \left(2a_k \cos 2\pi k t + 2b_k \sin 2\pi k t\right).$$

For this identity to be valid, it suffices if the function f is continuous and has at most finitely many local maxima and minima in the interval $[0, 1]$. The result also applies to periodic functions with at most finitely many points of discontinuity, and finitely many local maxima and minima, in this interval, provided that the value of the function is appropriately defined at those points of discontinuity.

This expression on the right-hand side of the above identity is referred to as the *Fourier series* for the function f.

We can easily extend this result other periodic functions f. Let p be a positive real number, and let $f\colon \mathbb{R} \to \mathbb{R}$ be a well-behaved periodic function with the property that $f(t+p) = f(t)$ for all real numbers x. If $g(t) = f(pt)$ for all real numbers t then $g\colon \mathbb{R} \to \mathbb{R}$ is a periodic function with the property that $g(t+1) = g(t)$ for all real numbers t. It follows that the function f has a Fourier series of the form

$$f(t) = g\left(\frac{t}{p}\right) = a_0 + \sum_{k=1}^{\infty} \left(2a_k \cos \frac{2\pi k t}{p} + 2b_k \sin \frac{2\pi k t}{p}\right),$$

where

$$\begin{aligned}
a_k &= \int_0^1 f(pt)\cos 2\pi kt\,dt = \frac{1}{p}\int_0^p f(t)\cos\frac{2\pi kt}{p}\,dt,\\
b_k &= \int_0^l f(pt)\sin 2\pi kt\,dt = \frac{1}{p}\int_0^p f(t)\sin\frac{2\pi kt}{p}\,dt.
\end{aligned}$$

Moreover the periodicity of the function f ensures that the integrals may be taken over any interval of length p, and the value of the integral will be independent of the interval chosen. Thus we may write

$$a_k = \frac{1}{p}\int_c^{c+p} f(t)\cos\frac{2\pi kt}{p}\,dt, \quad b_k = \frac{1}{p}\int_c^{c+p} f(t)\sin\frac{2\pi kt}{p}\,dt.$$

where the value of c may be chosen at will. In particular, we may choose $c = -\frac{1}{2}p$, obtaining the expressions

$$a_k = \frac{1}{p}\int_{-p/2}^{p/2} f(t)\cos\frac{2\pi kt}{p}\,dt, \quad b_k = \frac{1}{p}\int_{-p/2}^{p/2} f(t)\sin\frac{2\pi kt}{p}\,dt.$$

for a_k and b_k.

A function f is said to be *even* if $f(-t) = f(t)$ for all real numbers t. Let $f\colon \mathbb{R} \to IR$ be a well-behaved even function with period p. On examination of the expressions for a_k and b_k above, we see that

$$a_k = \frac{2}{p}\int_0^{p/2} f(t)\cos\frac{2\pi kt}{p}\,dt \quad \text{and} \quad b_k = 0 \quad \text{if } f \text{ is even.}$$

A function f is said to be *odd* if $f(-t) = -f(t)$ for all real numbers t. Let $f\colon \mathbb{R} \to IR$ be a well-behaved odd function with period p. On examination of the expressions for a_k and b_k above, we see that

$$b_k = \frac{2}{p}\int_0^{p/2} f(t)\sin\frac{2\pi kt}{p}\,dt \quad \text{and} \quad a_k = 0 \quad \text{if } f \text{ is odd.}$$