Course 223, 1988–89, Annual Examination (SF Trinity Term)

1. (a) State and prove *Rolle's Theorem.*

   (b) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a 3 times differentiable function on $\mathbb{R}$, and let a and b be real numbers satisying $a < b$. Suppose that

   $$f(a) = 0, \qquad f'(a) = 0, \qquad f(b) = 0, \qquad f'(b) = 0.$$

   Prove that there exists some ξ satisfying $a < \xi < b$ for which $f'''(\xi) = 0$.

   (c) State the *Mean Value Theorem*, and prove this theorem (either by using *Rolle's Theorem*, or otherwise).

2. (a) Let $f\colon [a, b] \to \mathbb{R}$ be a a real-valued function defined over a closed bounded interval $[a, b]$. Suppose that the function f is bounded above and below on the interval $[a, b]$. Let P be a partition of the interval $[a, b]$, given by

   $$P = \{t_0, t_1, \ldots, t_n\},$$

   where $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$. Define the *upper sum* $U(P, f)$ and the *lower sum* $L(P, f)$ of f for the partition P of $[a, b]$.

   (b) Let $f\colon [a, b] \to \mathbb{R}$ be a function on the interval $[a, b]$ which is bounded above and below on $[a, b]$. Define what is meant by saying that the function f is *Riemann-integrable* on $[a, b]$, and define the *Riemann integral* $\int_a^b f(t)\,dt$ of a Riemann-integrable function f on the interval $[a, b]$.

   (c) Let $f\colon [a, b] \to \mathbb{R}$ and $g\colon [a, b] \to \mathbb{R}$ be real-valued functions on the interval $[a, b]$ which are bounded above and below. Suppose that f and g are both Riemann-integrable on $[a, b]$. Prove that their sum $f + g$ is also Riemann-integrable on $[a, b]$ and that

   $$\int_a^b (f(t) + g(t))\,dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt.$$

3. (a) Let $f\colon D \to \mathbb{R}$ be a real-valued function defined over some open interval D in $\mathbb{R}$ which contains a and $a + h$. Suppose that f is

k times differentiable on D and that the kth derivative $f^{(k)}$ of f is continuous on D. Prove that

$$f(a+h) = f(a) + \sum_{j=1}^{k-1} \frac{h^j}{j!} f^{(j)}(a) + r_k(a;h),$$

where

$$r_k(a;h) = \frac{h^k}{(k-1)!} \int_0^1 (1-t)^{k-1} f^{(k)}(a+th)\, dt.$$

(b) Let $f\colon (-1,1) \to \mathbb{R}$ be defined by $f(x) = \log(1+x)$. Show that if $|h| < 1$ then

$$(1-t)^{k-1} |f^{(k)}(th)| < \frac{(k-1)!}{(1+th)}$$

for all $t \in [0,1]$. Hence show that

$$\log(1+h) = \sum_{j=0}^{+\infty} \frac{(-1)^{j-1} h^j}{j}$$

whenever $|h| < 1$.

4. (a) Let $f_1, f_2, f_3, f_4, \ldots$ be a sequence of real-valued functions defined over some subset D of $\mathbb{R}$. Let f be a real-valued function on D. Define what is meant by saying that the sequence $f_1, f_2, f_3, f_4, \ldots$ of functions *converges uniformly* on D to the function f.

(b) Give an example of a *uniformly convergent* sequence $f_1, f_2, f_3, f_4, \ldots$ of continuous real-valued functions defined over the whole of $\mathbb{R}$ with the property that

$$\lim_{j\to+\infty} \int_{-\infty}^{+\infty} f_j(t)\, dt \neq \int_{-\infty}^{+\infty} \left(\lim_{j\to+\infty} f_j(t) \right) dt.$$

[N.B., you should verify that your example does indeed possess the required properties.]

[The Examination contained two further questions supplied by another examiner.]