Course 223, 1987–88, Supplemental Examination (SF)

1. Let $\mathbb{R}^n$ and $\mathbb{R}^m$ be Euclidean spaces of dimensions n and m respectively, and let $\varphi\colon \mathbb{R}^n \to \mathbb{R}^m$ be a function mapping $\mathbb{R}^n$ into $\mathbb{R}^m$.

   (a) State precisely what it means to say that the function φ is *continuous*.

   (b) State precisely what is meant by saying that a subset V of $\mathbb{R}^m$ is *open*.

   (c) Prove that the function $\varphi\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if $\varphi^{-1}(V)$ is an open set in $\mathbb{R}^n$ for every open set V in $\mathbb{R}^m$

2. Let K be a subset of $\mathbb{R}^n$ and let $f\colon K \to \mathbb{R}$ be a continuous real-valued function on K.

   (a) State precisely what is meant by saying that the function f is *uniformly continuous* on K.

   (b) Use the formal definition of uniform continuity to show that if a real-valued function f is not uniformly continuous on a subset K of $\mathbb{R}^n$ then there exists a strictly positive real number ε_0 and sequences $(\mathbf{x}_i : i \in \mathbb{N})$ and $(\mathbf{y}_i : i \in \mathbb{N})$ of points of K such that such that $|\mathbf{x}_i - \mathbf{y}_i| < 1/i$ and $|f(\mathbf{x}_i) - f(\mathbf{y}_i)| \geq \varepsilon_0$.

   (c) Every sequence of points in a closed bounded subset K of $\mathbb{R}^n$ possesses a subsequence which converges to a point of K. By using this fact, or otherwise, prove that if $f\colon K \to \mathbb{R}$ is a continuous function on a closed bounded set K then f is uniformly continuous on K.

3. (a) Let a and b be real numbers satisfying $a < b$, and let $f\colon [a,b] \to \mathbb{R}$ be a bounded function on the closed bounded interval $[a,b]$. Define the *upper* and *lower Riemann integrals* of f on $[a,b]$, and define what is meant by saying that the function f is *Riemann integrable* on $[a,b]$. [If you use the quantities $L(P,f)$ and $U(P,f)$ employed in lectures then you should state the precise definition of these quantities.]

   (b) Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function on the closed bounded interval $[a,b]$. Prove that f is Riemann-integrable on $[a,b]$. [You may assume without proof the theorem that states that continuous functions are uniformly continuous on closed bounded sets.]

4. (a) State the Fundamental Theorem of Calculus.

(b) Let $F: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$F(x) = \begin{cases} \displaystyle\int_0^x t \cos \frac{1}{t}\, dt & \text{if } x > 0; \\ 0 & \text{if } x \leq 0. \end{cases}$$

Is the function F differentiable over the whole of $\mathbb{R}$. [Justify your answer.]

(c) Find the derivative of the function $G: \mathbb{R} \to \mathbb{R}$ defined by

$$G(x) = \int_{-(x-1)^2}^{(x+1)^2} e^{-t^2}\, dt.$$

[State clearly all theorems that you use.]

5. Let $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ be a function mapping $\mathbb{R}^n$ into $\mathbb{R}^m$, and let $\mathbf{a}$ be a point of $\mathbb{R}^n$. Let the components of the map φ be denoted by $\varphi_1, \varphi_2, \ldots, \varphi_m$.

(a) State precisely what it means to say that the function φ is *differentiable* at the point $\mathbf{a}$.

(b) Show that if the function φ is differentiable at $\mathbf{a}$ then the derivative of φ at the point $\mathbf{a}$ is represented by the Jacobian matrix

$$\begin{pmatrix} \dfrac{\partial \varphi_1}{\partial x_1} & \dfrac{\partial \varphi_1}{\partial x_2} & \cdots & \dfrac{\partial \varphi_1}{\partial x_n} \\ \dfrac{\partial \varphi_2}{\partial x_1} & \dfrac{\partial \varphi_2}{\partial x_2} & \cdots & \dfrac{\partial \varphi_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial \varphi_m}{\partial x_1} & \dfrac{\partial \varphi_m}{\partial x_2} & \cdots & \dfrac{\partial \varphi_m}{\partial x_n} \end{pmatrix},$$

where the partial derivatives occuring in this matrix are evaluated at the point $\mathbf{a}$.

(c) Write down an example of a function $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ (where n and m are suitably chosen positive integers) which has the property that all the partial derivatives occuring in the above Jacobian matrix exist at some point $\mathbf{a}$ of $\mathbb{R}^n$, yet the function φ itself is not differentiable at the point $\mathbf{a}$.

6. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous function defined on $\mathbb{R}^2$. Suppose that the functions

$$\frac{\partial f(x,y)}{\partial x}, \qquad \frac{\partial f(x,y)}{\partial y}, \qquad \frac{\partial^2 f(x,y)}{\partial x \partial y}, \qquad \frac{\partial^2 f(x,y)}{\partial y \partial x}$$

exist and are continuous at each point (x,y) of $\mathbb{R}^2$. Prove that

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\partial^2 f(x,y)}{\partial y \partial x}.$$

7. Let (x,y,z) denote the standard Cartesian coordinates on $\mathbb{R}^3$.

   (a) Prove that $d(g\omega) = g\,d\omega + dg \wedge \omega$ for all smooth 1-forms ω and for all smooth real-valued functions g on $\mathbb{R}^3$.

   (b) Let ω be the smooth 1-form on $\mathbb{R}^3$ defined by

   $$\omega = e^{y^2} \cos x \cos z\, dx + 2y e^{y^2} \sin x \cos z\, dy - e^{y^2} \sin x \sin z\, dz.$$

   Show by direct calculation that $d\omega = 0$. Write down a smooth function f on $\mathbb{R}^3$ with the property that $\omega = df$.

   (c) Let η be the smooth 2-form on $\mathbb{R}^3$ defined by

   $$\eta = \sin x \cos y \sin z\, dx \wedge dy - \cos x \sin y \cos z\, dx \wedge dz.$$

   Calculate $\omega \wedge \eta$ and $d\eta$ (where ω is the 1-form defined in (a)). Does there exist a smooth 1-form ξ on $\mathbb{R}^3$ with the property that $d\xi = \eta$? [Justify your answer.]

   (d) Let β be a smooth 3-form on $\mathbb{R}^3$. Does there always exist a smooth 2-form α on $\mathbb{R}^3$ with the property that $d\alpha = \beta$? [Justify your answer.]

8. Let (x,y,z) denote the standard Cartesian coordinates on $\mathbb{R}^3$. Let D and E be open sets in $\mathbb{R}^3$ and let $\varphi\colon D \to E$ be a smooth map from D into E. Let φ_1, φ_2 and φ_3 denote the Cartesian components of the map φ

   (a) Let P, Q and R be smooth real-valued functions on E, and let η be the smooth 1-form on E defined by

   $$\eta = P(x,y,z)\, dx + Q(x,y,z)\, dy + R(x,y,z)\, dz.$$

   Derive an expression for the pullback $\varphi^*\eta$ of the 1-form η in terms of the functions P, Q, R, the components of the map φ and their partial derivatives.

(b) Prove that $\varphi^*(df) = d(f \circ \varphi)$ for all smooth real-valued functions f on E.

(c) Prove that $d(\varphi^* \eta) = \varphi^* d\eta$ for all smooth 1-forms η on E.

(d) Let $\varphi \colon \mathbb{R}^3 \to \mathbb{R}^3$ be the smooth map defined by

$$\varphi(x, y, z) = (x \sin y \cos z, x \sin y \sin z, x \cos z).$$

Let ω be the 2-form on $\mathbb{R}^3$ defined by $\omega = dx \wedge dy$. Calculate $\varphi^* \omega$.

9. Let (x, y, z, t) denote the standard Cartesian coordinates on $\mathbb{R}^4$.

(a) Let $\gamma \colon [0, 2\pi] \to \mathbb{R}^4$ be the closed curve in $\mathbb{R}^4$ defined by

$$\gamma(\theta) = (\cos\theta, \sin\theta, \cos 2\theta, \sin 2\theta).$$

Show that $\int_\gamma \eta = 0$, where η is the smooth 1-form on $\mathbb{R}^3$ defined by

$$\eta = x\,dz + z\,dx - y\,dt - t\,dy.$$

(b) Let M be the 3-dimensional oriented submanifold of $\mathbb{R}^4$ (with boundary) defined by

$$\begin{aligned} M \quad = \quad & \{(x, y, z, t) \in \mathbb{R}^4 : \\ & \quad t > 0, \quad x^2 + y^2 + z^2 \leq 1, \quad x^2 + y^2 + z^2 - t^2 = 1\}, \end{aligned}$$

where the orientation on M is chosen such that the restriction to M of the Cartesian coordinates (x, y, z) defines a positively-oriented coordinate system on M. Let ω be the 3-form on $\mathbb{R}^4$ defined by

$$\omega = 7t^2\,dx \wedge dy \wedge dz - zt\,dx \wedge dy \wedge dt + yt\,dx \wedge dz \wedge dt - xt\,dy \wedge dz \wedge dt.$$

Calculate $\int_M \omega$ (where M is oriented as described above).