Course 223, 1987–88, Assignment 1 (SF Michaelmas Term)

Assignment due 16th November 1987

One of the purposes of this assignment is to give you practice in constructing simple 'epsilon-delta' arguments and to encourage you to come to grips with the rigourous definition of continuity, so that you will (hopefully) improve your understanding of this concept. For your convenience, the formal definition of continuity is reproduced here.

A real-valued function $f\colon D \to \mathbb{R}$ defined on a subset D of $\mathbb{R}^n$ is said to be *continuous* at a point $\mathbf{a}$ of D if and only if, for all $\varepsilon > 0$ there exists some $\delta > 0$ (which may depend on $\mathbf{a}$) such that

$$|f(\mathbf{x}) - f(\mathbf{a})| < \varepsilon$$

whenever $\mathbf{x}$ belongs to D and

$$|\mathbf{x} - \mathbf{a}| < \delta.$$

Observe that, as a consequence of this definition, a real-valued function f defined on a subset D of $\mathbb{R}^n$ fails to be continuous at a point $\mathbf{a}$ of D if and only if one can find some strictly positive real number $\varepsilon_0 > 0$ with the property that, for every $\delta > 0$ (no matter how small), there exists some point $\mathbf{x}$ of D (which will depend on δ) with the properties that $|\mathbf{x} - \mathbf{a}| < \delta$ and $|f(\mathbf{x}) - f(\mathbf{a})| \geq \varepsilon_0$. If you have difficulty in seeing this, study carefully the definition of continuity given above and convince yourself that the statement that I have just made is correct.

We come now to the questions set for this assignment.

1. (a) Let D be a subset of $\mathbb{R}^n$. Let f, g and h be real-valued functions defined on D which have the property that

$$f(\mathbf{x}) \leq g(\mathbf{x}) \leq h(\mathbf{x})$$

for all $\mathbf{x} \in D$. Let $\mathbf{a}$ be a point of D, let c be a real number, and let us suppose that

$$f(\mathbf{a}) = g(\mathbf{a}) = h(\mathbf{a}) = c$$

and that both f and h are continuous at $\mathbf{a}$. The objective of this question is to show that g is also continuous at $\mathbf{a}$.

Let $\varepsilon > 0$ be given.

(i) Using the fact that the function h is continuous at $\mathbf{a}$, show that there exists some $\delta_1 > 0$ such that $g(\mathbf{x}) < c + \varepsilon$ for all points $\mathbf{x}$ of D that satisfy $|\mathbf{x} - \mathbf{a}| < \delta_1$.

(ii) Show also that there exists some $\delta_2 > 0$ such that $g(\mathbf{x}) > c - \varepsilon$ for all points $\mathbf{x}$ of D that satisfy $|\mathbf{x} - \mathbf{a}| < \delta_2$.

(ii) Using these results, explain why the function g is continuous at $\mathbf{a}$.

2. (a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be the real valued function on $\mathbb{R}$ defined by

$$f(x) = \begin{cases} 3x \cos \dfrac{\pi}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

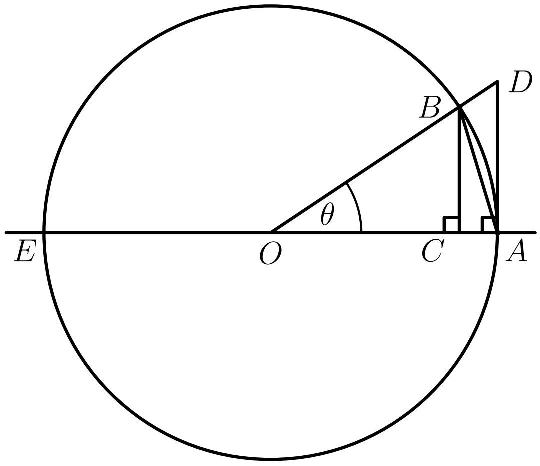
Let $\varepsilon > 0$ be given. Find a strictly positive real number δ (depending on ϵ) such that $|f(x)| < \varepsilon$ whenever $|x| < \delta$. Is the function f continuous at $x = 0$?

(b) Let $g\colon \mathbb{R} \to \mathbb{R}$ be the real valued function on $\mathbb{R}$ defined by

$$g(x) = \begin{cases} 4 \cos \dfrac{\pi}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Show that if $\delta > 0$ is a strictly positive real number, no matter how small then there will always exist some real number x with the properties that $0 < |x| < \delta$ and $g(x) \geq 1$. Thus decide whether or not the function g is continuous at $x = 0$.

3. Consider the situation described by the following diagram:—



Here the circle passing through A, B and E is a circle of unit radius with centre O. The lines BC and AD are perpendicular to the line OA. The angle between OA and OB measured in radians is θ, where $0 < \theta < \pi/2$.

(a) Write down the area of the triangle OBC, the area of the triangle ODA and the area of the sector OAB (where the *sector* OAB is defined to be the region bounded by the line segments OA and OB and the circular arc AB of length θ).

(b) Show that if $0 < \theta < \pi/2$ then

$$\sin\theta\,\cos\theta < \theta < \frac{\sin\theta}{\cos\theta}..$$

Deduce that

$$\cos\theta < \frac{\sin\theta}{\theta} < \frac{1}{\cos\theta}$$

when $0 < \theta < \pi/2$. Explain why these inequalities also hold when $-\pi/2 < \theta < 0$.

(c) Let h be the real-valued function on $\mathbb{R}$ defined by

$$h(\theta) = \begin{cases} \dfrac{\sin\theta}{\theta} & \text{if } \theta \neq 0; \\ 1 & \text{if } \theta = 0. \end{cases}$$

Using the inequalities proved in part (b), or otherwise, show that the function h is continuous at $\theta = 0$.

Course 223, 1987–88, Assignment 2 (SF Michaelmas Term)

Assignment due 10th December 1987

The first question of this assignment concerns sequences and continuous functions, as dealt with in the first two sections of the course. The second and third questions relate to the Riemann integral. You should try to set out your work so that the logical structure of your proofs are correct (each line following on from the next). In particular, do not mix up your 'for all' and 'there exists' e.t.c., particularly in question 1.

1. Let $(\mathbf{x}_i : i \in \mathbb{N})$ and $(\mathbf{y}_i : i \in \mathbb{N})$ be sequences in $\mathbb{R}^n$. Let $\mathbf{c}$ be a point of $\mathbb{R}^n$. Suppose that

$$\begin{aligned} \lim_{i \to +\infty} \mathbf{x}_i &= \mathbf{c}, \\ \lim_{i \to +\infty} |\mathbf{x}_i - \mathbf{y}_i| &= 0. \end{aligned}$$

   (a) Using the formal definition of limits, show that, given any $\delta > 0$, there exists some positive integer N such that the inequalities

$$\begin{aligned} |\mathbf{x}_i - \mathbf{c}| &< \frac{\delta}{2}, \\ |\mathbf{x}_i - \mathbf{y}_i| &< \frac{\delta}{2}, \end{aligned}$$

   both hold for all integers i satisfying $i \geq N$. Hence prove that

$$\lim_{i \to +\infty} \mathbf{y}_i = \mathbf{c}.$$

   (b) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous function on $\mathbb{R}^n$. Using (a), show that

$$\lim_{i \to +\infty} |f(\mathbf{x}_i) - f(\mathbf{y}_i)| = 0.$$

   Explain why, given any $\varepsilon > 0$, there exists some positive integer N' such that

$$|f(\mathbf{x}_i) - f(\mathbf{y}_i)| < \varepsilon$$

   for all $i \geq N'$.

   (c) Explain why these results are relevent to the proof of the theorem that states that continuous functions are uniformly continuous on closed bounded subsets of $\mathbb{R}^n$).

2. The purpose of this question is to show from first principles that the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is Riemann-integrable on the interval $[0, a]$ (where a is any positive real number) and to evaluate the Riemann integral of f on this interval.

   (a) Prove by induction on n that

$$\begin{aligned}\sum_{i=1}^{n} i &= \frac{1}{2} n(n+1), \\ \sum_{i=1}^{n} i^2 &= \frac{1}{6} n(n+1)(2n+1).\end{aligned}$$

   (b) Let n be a positive integer and let P_n $(= \{t_0, t_1, \ldots, t_n\})$ be the partition of $[0, a]$ defined by

$$t_i = \frac{ia}{n} \qquad (i = 0, 1, \ldots, n)$$

   (i.e., P_n represents a partition of $[0, a]$ into n subintervals of length a/n. Show that

$$L(P_n, f) = \frac{a^3}{3}\left(1 - \frac{3}{2n} + \frac{1}{2n^2}\right)$$

   and calculate $U(P_n, f)$ for all positive integers n, where $f(x) = x^2$ and $a > 0$. [The quantities $L(P_n, f)$ and $U(P_n, f)$ are defined as in lectures.]

   (c) Show that

$$\begin{aligned}\lim_{n \to +\infty} L(P_n, f) &= \frac{a^3}{3}, \\ \lim_{n \to +\infty} U(P_n, f) &= \frac{a^3}{3}.\end{aligned}$$

   Hence show that f is Riemann-integrable on $[a, b]$ and that

$$\int_0^a f(x)\, dx = \frac{a^3}{3}.$$

3. Let $g: [0, 1] \to \mathbb{R}$ be the function on $[0, 1]$ defined by

$$g(x) = \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

   Let P be a partition of $[0, 1]$. Show that $U(P, g) = 1$. Calculate $L(P, g)$. Is g Riemann-integrable on $[a, b]$?

Course 223, 1987–88, Assignment 3 (SF Hilary Term)

Assignment due 25th January 1988

1. The purpose of these questions is to work through the proof of Rolle's Theorem and the Mean Value Theorem. We now state these theorems.

   Rolle's Theorem. Let a and b be real numbers satisfying $a < b$ and let $f\colon [a,b] \to \mathbb{R}$ be a continuous function on the closed interval $[a,b]$ that is differentiable on (a,b). Suppose that $f(a) = f(b)$. Then there exists some $\xi \in (a,b)$ such that $f'(\xi) = 0$.

   The Mean Value Theorem. Let a and b be real numbers satisfying $a < b$ and let $f\colon [a,b] \to \mathbb{R}$ be a continuous function on the closed interval $[a,b]$ that is differentiable on (a,b). Then there exists some $\xi \in (a,b)$ such that

   $$f(b) - f(a) = f'(\xi)(b-a).$$

   (a) Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function defined on a closed bounded interval $[a,b]$. It follows from results proved in this course (and also in course 121) that there exist $u, v \in [a,b]$ such that

   $$f(u) \le f(t) \le f(v)$$

   for all $t \in [a,b]$. Write down a proof of this result (referring to your lecture notes for this course or for Course 121, if necessary).

   (b) Let $f\colon [a,b]$ be a continuous function on a closed bounded interval $[a,b]$ and let v be a real number satisfying $a < v < b$ which has the property that $f(t) \le f(v)$ for all $t \in [a,b]$. Suppose that f is differentiable at v. Prove that

   $$\lim_{h \to 0^+} \frac{f(v+h) - f(v)}{h} \le 0$$

   and that

   $$\lim_{h \to 0^-} \frac{f(v+h) - f(v)}{h} \ge 0$$

   Hence show that $f'(v) = 0$. Similarly, show that if u is a real number satisfying $a < u < b$ which has the property that $f(t) \ge f(u)$ for all $t \in [a,b]$ and if f is differentiable at u then $f'(u) = 0$.

(c) Let a and b be real numbers satisfying $a < b$ and let $f\colon [a,b] \to \mathbb{R}$ be a continuous function on the closed interval $[a,b]$ that is differentiable on (a,b). Suppose that $f(a) = f(b)$. Let $u, v \in [a,b]$ be real numbers with the property that

$$f(u) \leq f(t) \leq f(v).$$

(Such real numbers exist by (a).) By considering separately the cases when $f(v) > f(a)$, when $f(u) < f(a)$ and when $f(u) = f(v) = f(a) = f(b)$, prove Rolle's Theorem (i.e., show that there exists some real number ξ such that $a < \xi < b$ and $f'(\xi) = 0$.

(d) By applying Rolle's Theorem to the function $g\colon [a,b] \to \mathbb{R}$ defined by

$$g(t) = \frac{(b-t)f(a) - (t-a)f(b)}{b-a},$$

show that if $f\colon [a,b] \to \mathbb{R}$ is a continuous function on the closed interval $[a,b]$ that is differentiable on (a,b), then there exists some ξ satisfying $a < \xi < b$ such that $f(b) - f(a) = (b-a)f'(\xi)$ (i.e., prove the Mean Value Theorem).

2. The purpose of this question is to prove Taylor's Theorem (with remainder). Taylor's Theorem states that if f is an $(n+1)$-times differentiable function defined on an open interval containing the real numbers a and $a+h$ then there exists some θ satisfying $0 < \theta < 1$ such that

$$f(a+h) = \sum_{j=0}^{n} \frac{h^j}{j!} f^{(j)}(a) + \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(a + \theta h).$$

Thus let $f\colon D \to \mathbb{R}$ be an $(n+1)$-times differentiable function defined on an open interval D in $\mathbb{R}$ and let x and $x+h$ be points of D, where $h \neq 0$. (Note that D contains all real numbers between x and $x+h$, since D is an interval.) Consider the function $g\colon D \to \mathbb{R}$ defined by

$$g(t) = f(t) - \sum_{j=0}^{n} \frac{(t-a)^j}{j!} f^{(j)}(a) - \frac{M(t-a)^{n+1}}{(n+1)!},$$

where M is the constant defined by

$$M = \frac{(n+1)!}{h^{n+1}} \left(f(a+h) - \sum_{j=0}^{n} \frac{h^j}{j!} f^{(j)}(a) \right).$$

Show that $g(a) = g(a+h) = 0$ and that $g^j(a) = 0$ for all j satisfying $j \leq n$. Use Rolle's Theorem, applied to the function g and its derivatives, to show that there exists some θ satisfying $0 < \theta < 1$ with the property that $g^{(n+1)}(a + \theta h) = 0$. Hence show that

$$f(a+h) = \sum_{j=0}^{n} \frac{h^j}{j!} f^{(j)}(a) + \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(a + \theta h)$$

for some θ satisfying $0 < \theta < 1$. This result is known as *Taylor's Theorem.*

3. Using the fact that

$$\frac{d}{dx}(\sin x) = \cos x, \qquad \frac{d}{dx}(\cos x) = -\sin x,$$

and using Taylor's Theorem (with remainder) in the form proved in Question 2, prove that

$$\begin{aligned} \sin x &= \lim_{m \to +\infty} \sum_{j=0}^{m} (-1)^j \frac{x^{2j+1}}{(2j+1)!}, \\ \cos x &= \lim_{m \to +\infty} \sum_{j=0}^{m} (-1)^j \frac{x^{2j}}{(2j)!}. \end{aligned}$$

(You must consider the behaviour of the remainder term occuring in the statement of Taylor's Theorem as $m \to +\infty$.)

4. Let f be an $(n+1)$-times differentiable function on an open interval containing a and $a+h$. Suppose also that $f^{(n+1)}$ is continuous. The purpose of this question is to show that

$$f(a+h) = \sum_{j=0}^{n} \frac{h^j}{j!} f^{(j)}(a) + r_n(h)$$

where

$$r_n(h) = \frac{h^{(n+1)}}{n!} \int_0^1 (1-t)^n f^{(n+1)}(a+th)\, dt.$$

(a) Using the fact that

$$r_n(h) = \frac{1}{n!} \int_0^1 (1-t)^n \frac{d^{n+1}}{dt^{n+1}} \left(f(a+th) \right) dt$$

show that

$$r_{n-1}(h) = \frac{h^n}{n!} f^{(n)}(a) + r_n(h).$$

(b) Show by induction on n that if f is $(n+1)$ times differentiable on an open interval containing a and $a+h$ and if $f^{(n+1)}$ is continuous then

$$f(a+h) = \sum_{j=0}^{n} \frac{h^j}{j!} f^{(j)}(a) + r_n(h),$$

where $r_n(h)$ is defined as above.

Course 223, 1987–88, Assignment 4 (SF Hilary Term)

Assignment due 8th February 1988

1. Throughout this question, let $f: \mathbb{R} \to \mathbb{R}$ and $p: \mathbb{R} \to \mathbb{R}$ be the functions defined by $f(x) = 2x^3 + 5x^2$ and $p(x) = 6x^2 + 10x$. Also $e(a,h)$ is defined by $e(a,h) = f(a+h) - f(a) - p(a)h$.

   (a) Evaluate $e(a,h)$ as a function of a and h. Show that

   $$|e(a,h)| < (6|a| + 5 + 2|h|)|h|^2.$$

   Hence prove that

   $$\lim_{h \to 0} \frac{e(a,h)}{|h|} = 0.$$

   (b) Using standard properties of limits, or otherwise, prove that

   $$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = p(a).$$

2. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = 3x^2 - 7xy + 5y^2$. Prove that

   $$\lim_{(x,y) \to (0,0)} \frac{f(x,y)}{\sqrt{x^2 + y^2}} = 0.$$

3. Let $g: \mathbb{R} \to \mathbb{R}$ be the function defined by

   $$g(x) = \begin{cases} x^2 \cos\left(\dfrac{1}{x}\right) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

   Show that the function g is differentiable at 0 (using an epsilon-delta argument or otherwise). Evaluate the derivative $g'(x)$ of g at all $x \in \mathbb{R}$. Show that the derivative g' of g is *not* continuous at 0. (You may assume without proof that the function $s: \mathbb{R} \to \mathbb{R}$ defined by

   $$s(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   is not continuous at 0.)

Course 223, 1987–88, Assignment 5 (SF Hilary Term)

Assignment due 22nd February 1988

1. (a) Let $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the function defined by

$$f(u, v, w) = (e^u \sin v \cos w, e^u \sin v \sin w, e^u \cos v).$$

Calculate the first order partial derivatives of the components of f, and explain why they are continuous on $\mathbb{R}^3$. (You may assume without proof that standard functions such as sin, cos, exp etc. are continuous and have derivatives of all orders.) Explain why the function f is differentiable at every point of $\mathbb{R}^3$ (by appealing to an appropriate theorem proved in lectures), and write down the matrix representing the derivative of f at any point in $\mathbb{R}^3$.

   (b) Let $g\colon \mathbb{R}^3 \to \mathbb{R}$ be the function defined by $g(x, y, z) = x^2 + y^2 + z^2$. Calculate the matrix representing the derivative of g at each point of $\mathbb{R}^3$. Let $h\colon \mathbb{R}^3 \to \mathbb{R}$ be defined to be the composition $g \circ f$ (i.e., f followed by g), where f is the function defined in (a). Give an expression for h and evaluate the derivative of h at each point of $\mathbb{R}^3$. Verify the chain rule, which in this instance states that $h'(\mathbf{a}) = g'(f(\mathbf{a}))f'(\mathbf{a})$ for all points $\mathbf{a}$ of $\mathbb{R}^3$.

2. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \text{minimum}(|x|, |y|).$$

Do the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist at $(0, 0)$? Is f differentiable at $(0, 0)$?

Course 223, 1987–88, Assignment 6 (SF Hilary Term)

Assignment due 7th March 1988

1. Let (x, y, z) denote the standard coordinate system on $\mathbb{R}^3$. Let ω_1, ω_2, ω_3 and ω_4 denote the differential forms on $\mathbb{R}^3$ defined by

$$\begin{aligned}
\omega_1 &= x\,dy + y\,dz + z\,dx, \\
\omega_2 &= 4y^2z\,dx + 9x^3\,dy - 3xy^2\,dz, \\
\omega_3 &= 3z\,dx \wedge dy + 4x\,dy \wedge dz + 5y\,dz \wedge dx, \\
\omega_4 &= 2x\,dy \wedge dz + 4z^2\,dz \wedge dx + 7x\,dx \wedge dy.
\end{aligned}$$

Calculate the the following quantities:

(i)	$\omega_1 \wedge \omega_2$,	(ii)	$\omega_1 \wedge \omega_3$,	(iii)	$\omega_1 \wedge \omega_4$,	(iv)	$\omega_2 \wedge \omega_3$,
(v)	$\omega_2 \wedge \omega_4$,	(vi)	$d\omega_1$,	(vii)	$d\omega_2$,	(viii)	$d\omega_3$,
(ix)	$d\omega_4$,	(x)	$d(\omega_1 \wedge \omega_2)$,	(xi)	$d\omega_1 \wedge \omega_2$,	(xii)	$\omega_1 \wedge d\omega_2$.

Check that

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 - \omega_1 \wedge d\omega_2.$$

2. Let (x, y) denote the standard coordinate system on $\mathbb{R}^2$. Let ω be the 1-form on $\mathbb{R}^2$ defined by

$$\omega = \cos x\, \sin y\, dx + \sin x\, \cos y\, dy.$$

Show that $d\omega = 0$. Can you find a differentiable real-valued function f on $\mathbb{R}^2$ with the property that $df = \omega$?

Course 223, 1987–88, Assignment 7 (SF Trinity Term)

Assignment due 18th April 1988

1. Let D_1, D_2 and D_3 be open sets in $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^p$ respectively, and let $\varphi\colon D_1 \to D_2$ and $\psi\colon D_2 \to D_3$ be smooth maps. Prove that $\varphi^*(\psi^*\omega) = (\psi \circ \varphi)^*\omega$ for all differential forms ω on D_3.

2. Let (x, y, z, t) denote the standard Cartesian coordinates on $\mathbb{R}^4$ and let ω be the 1-form on $\mathbb{R}^4$ defined by

$$\omega = x\,dx + y\,dy + z\,dz - t\,dt.$$

Show that $\gamma^*\omega = 0$, where $\gamma\colon \mathbb{R} \to \mathbb{R}^4$ is defined by

$$\gamma(u) = (3u\cos u, 3u\sin u, 4u, 5u).$$

3. Let (u, v) denote the standard Cartesian coordinates on $\mathbb{R}^2$ and let (x, y, z) denote the standard Cartesian coordinates on $\mathbb{R}^3$. Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the smooth map defined by

$$\varphi(u, v) = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right).$$

Show that

$$\varphi^*(x\,dx + y\,dy + z\,dz) = 0$$

and calculate

$$\varphi * (x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy).$$

4. Use the Poincaré Lemma to show that if P and Q are smooth functions defined over the whole of $\mathbb{R}^2$ and if

$$\frac{\partial P(x, y)}{\partial y} = \frac{\partial Q(x, y)}{\partial x}$$

then there exists a smooth function f on $\mathbb{R}^2$ with the property that

$$P(x, y) = \frac{\partial f(x, y)}{\partial x}, \qquad Q(x, y) = \frac{\partial f(x, y)}{\partial y}.$$

5. Let D be an open set in $\mathbb{R}^3$ which is star-shaped with respect to some point of D. Let $\mathbf{V}$ be a smooth vector field on D. Use the Poincaré Lemma to prove the following results:

   (a) (a) if $\operatorname{curl} \mathbf{V} = 0$ then there exists a smooth real-valued function f on D such that $\mathbf{V} = \operatorname{grad} f$,

   (b) (b) if $\operatorname{div} \mathbf{V} = 0$ then there exists a smooth vector field $\mathbf{A}$ on D such that $\mathbf{V} = \operatorname{curl} A$.

Course 223, 1987–88, Assignment 8 (SF Trinity Term)

Assignment due 2nd May 1988

1. Let $f_1, f_2, f_3, \ldots$ be a sequence of continuous real-valued functions defined on an n-rectangle C in $\mathbb{R}^n$. Suppose that $f_j \to f$ *uniformly* on C as $j \to +\infty$, where f is a continuous real-valued function defined on C. Prove that

$$\lim_{j \to +\infty} \int_C f_j(\mathbf{x})\, dx_1\, dx_2 \ldots dx_n = \int_C f(\mathbf{x})\, dx_1\, dx_2 \ldots dx_n.$$

   [Hint: A corresponding result was proved in the Michaelmas Term for functions defined over some closed bounded interval in $\mathbb{R}$.]

2. Let $f\colon C \to \mathbb{R}$ be a continuous real-valued function defined over a rectangle C in $\mathbb{R}^2$, where

$$C = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, \quad c \leq y \leq d\}$$

   for some real numbers a, b, c and d satisfying $a < b$ and $c < d$. Suppose that $f(u, v) > 0$ at some point (u, v) in the interior of C, where $a < u < b$ and $c < v < d$.

   (a) Using the formal definition of continuity, prove that there exist real numbers l, p, q, r and s, where $l > 0$, $a < p < u < q < b$ and $c < r < v < s < d$, such that $f(x, y) \geq l$ for all points (x, y) which satisfy $p \leq x \leq q$ and $r \leq y \leq s$.

   (b) Let $\chi\colon C \to \mathbb{R}$ be defined by

$$\chi(x, y) = \begin{cases} l & \text{if } p \leq x \leq q \text{ and } r \leq y \leq s, \\ 0 & \text{otherwise,} \end{cases}$$

   Show that

$$\int_C \chi(x, y)\, dx\, dy = l(s - r)(q - p).$$

   (c) Let g and h be Riemann-integrable functions on the rectangle C in $\mathbb{R}^2$. Suppose that $g(x, y) \leq h(x, y)$ for all points (x, y) of C. Explain briefly why

$$\int_C g(x, y)\, dx\, dy \leq \int_C h(x, y)\, dx\, dy.$$

(d) Using the answers to parts (a), (b) and (c), or otherwise, prove that if $f\colon C \to \mathbb{R}$ is a *continuous* real-valued function defined on the rectangle C, where

$$C = \{(x,y) \in \mathbb{R}^2 : a \leq x \leq b, \quad c \leq y \leq d\},$$

if $f(x,y) \geq 0$ for all points (x,y) of C, if $f(u,v) > 0$ at some point (u,v) of the interior of C, where $a < u < b$ and $c < v < d$ then

$$\int_C f(x,y)\,dx\,dy > 0.$$

3. Let (r,θ,φ) be spherical polar coordinates defined over some appropriately chosen open set U in $\mathbb{R}^3$. Thus

$$x = r\sin\theta\,\cos\varphi, \qquad y = r\sin\theta\,\sin\varphi, \qquad z = r\cos\theta$$

at each point (x,y,z) of U. Let ω and η be the differential forms on U defined by

$$\omega = r\,dr, \qquad \eta = r^3\sin\theta d\theta\wedge d\varphi.$$

Calculate $d\omega$ and $d\eta$ and show that $3\omega\wedge\eta = r^2 d\eta$. Show that $\omega = d(\frac{1}{2}r^2)$. [Use spherical polar coordinates throughout this calculation.]