Course 223, 1986–87, Annual Examination (SF Trinity Term)

1. (a) State and prove Rolle's theorem. (You may assume the theorem that states that if a and b are real numbers with $a < b$ and if $f: [a, b] \to \mathbb{R}$ is a continuous real-valued function defined on the closed bounded interval $[a, b]$ then there exist $t_1, t_2 \in [a, b]$ such that $f(t_1) \leq f(x) \leq f(t_2)$ for all $x \in [a, b]$.)

   (b) Let $f: \mathbb{R} \to \mathbb{R}$ be a 5 times differentiable real-valued function on $\mathbb{R}$. Suppose that $f(0) = 0$, $f'(0) = 0$, $f''(0) = 0$, $f(1) = 0$, $f'(1) = 0$ and $f(2) = 0$. Show that there exists t such that $0 < t < 2$ and $f^{(5)}(t) = 0$, where $f^{(5)}(t)$ is the fifth derivative of f at t.

2. (a) Let a and b be real numbers satisfying $a < b$ and let $f: [a, b] \to \mathbb{R}$ be a real-valued function on the closed bounded interval $[a, b]$ which is bounded above and below on $[a, b]$. State precisely what is meant by saying that f is *Riemann-integrable*, and define the *Riemann integral* of a Riemann-integrable function.(If you utilise the lower sum $L(P, f)$ and the upper sum $U(P, f)$ used in lectures then you should define these quantities.)

   (b) Consider the function $f: [0, 2] \to \mathbb{R}$ defined by

   $$f(x) = \begin{cases} 2 & \text{if } 0 \leq x < 1; \\ 2 - x & \text{if } 1 \leq x \leq 2. \end{cases}$$

   Show from first principles that f is Riemann-integrable, and calculate the Riemann integral of f.

3. (a) Prove that if a and b are real numbers satisfying $a < b$ and if $f: [a, b] \to \mathbb{R}$ is a continuous real-valued function defined on the closed interval $[a, b]$ then

   $$\frac{d}{dx} \int_a^x f(t)\,dt = f(x) \text{ and } \frac{d}{dx} \int_x^b f(t)\,dt = -f(x)$$

   for all $x \in (a, b)$.

   (b) Evaluate

   $$\frac{d}{dx} \int_{\cos x - 5}^{\sin x + 2} t^5 e^{-t}\,dt.$$

4. (a) Let a and b be real numbers satisfying $a < b$ and let $f_1, f_2, \ldots$ be a sequence of real-valued functions defined on $[a, b]$. Let $f: [a, b] \to$

$\mathbb{R}$ be a real-valued function on $[a, b]$. State precisely what is meant by saying that the sequence $(f_n : n \in \mathbb{N})$ of functions *converges uniformly* to f on $[a, b]$.

(b) Prove that if $(f_n : n \in \mathbb{N})$ is a sequence of real-valued continuous functions on $[a, b]$ which converges uniformly to a function $f : [a, b] \to \mathbb{R}$ then f is continuous.

(c) Define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) = \frac{2n^2 x}{(1 + n^2 x^2)^2}.$$

Show that $f_n(x) \to 0$ as $n \to +\infty$ for all $x \in [0, 1]$. Does the sequence $(f_n : n \in \mathbb{N})$ of functions converge to 0 uniformly on $[0, 1]$? What is

$$\lim_{n \to +\infty} \int_0^1 f_n(x)\, dx?$$

5. (a) Let a and h be real numbers and let $f : D \to \mathbb{R}$ be a real-valued function defined an open subset D of $\mathbb{R}$ containing a, $a + h$ and all real numbers lying between a and $a + h$. Suppose that f is C^{k+1} on D for some non-negative integer k. Prove that

$$f(a + h) = \sum_{j=0}^{k} \frac{h^j}{j!} f^{(j)}(a) + h^{k+1} \int_0^1 \frac{(1 - t)^k}{k!} f^{(k+1)}(a + th)\, dt.$$

(b) Show that if $-1 < h < 1$ then

$$\left| \log(1 + h) - \sum_{j=1}^{k} (-1)^{j-1} \frac{h^j}{j} \right| \leq h^k \log(1 + h).$$

Hence prove that if $-1 < h < 1$ then

$$\log(1 + h) = \sum_{j=1}^{+\infty} \frac{(-1)^{j-1}}{j} h^j.$$

6. (a) Let D be an open subset of $\mathbb{R}^n$ for some $n \in \mathbb{N}$ and let $f : D \to \mathbb{R}^m$ be a function mapping D into $\mathbb{R}^m$. Let $\mathbf{a}$ be an element of D. State precisely what is meant by saying that the function f is *differentiable* at $\mathbf{a}$ and define the *derivative* (also known as the *total derivative*) of f at $\mathbf{a}$

(b) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} x^2 \cos y \sin\left(\dfrac{1}{x}\right), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Prove that f is differentiable everywhere in $\mathbb{R}^2$. Calculate the Jacobian matrix representing the derivative of f at every point of $\mathbb{R}^2$. Are the partial derivatives $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ continuous everywhere in $\mathbb{R}^2$? (Give reasons for your answer.)

7. (a) Define the *limit* $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ of a real-valued function $f\colon \mathbb{R}^n \to \mathbb{R}$ on $\mathbb{R}^n$, (when the limit exists).

(b) Prove from first principles that if $u\colon \mathbb{R}^n \to \mathbb{R}$ and $v\colon \mathbb{R}^n \to \mathbb{R}$ are real-valued functions on $\mathbb{R}^n$ and if $\lim_{\mathbf{x}\to\mathbf{a}} u(\mathbf{x})$ and $\lim_{\mathbf{x}\to\mathbf{a}} v(\mathbf{x})$ exist then $\lim_{\mathbf{x}\to\mathbf{a}} u(\mathbf{x})v(\mathbf{x})$ exists and

$$\lim_{\mathbf{x}\to\mathbf{a}} u(\mathbf{x})v(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{a}} u(\mathbf{x}) \lim_{\mathbf{x}\to\mathbf{a}} v(\mathbf{x}).$$

(c) Prove that if $f\colon \mathbb{R}^n \to \mathbb{R}$ and $g\colon \mathbb{R}^n \to \mathbb{R}$ are differentiable real-valued functions on $\mathbb{R}^n$ then the product $f.g\colon \mathbb{R}^n \to \mathbb{R}$ is differentiable (where $(f.g)(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x})$).

8. (a) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous real-valued function on $\mathbb{R}^2$. Suppose that $\dfrac{\partial f}{\partial x}$ exists and is continuous on $\mathbb{R}^2$ and that $\dfrac{\partial^2 f}{\partial y\,\partial x}$ exists everywhere on $\mathbb{R}^2$. Let (a,b) be a point of $\mathbb{R}^2$. Prove that given any real numbers h and k there exist p between a and $a+h$ and q between b and $b+k$ such that

$$f(a+h,b+k) - f(a+h,b) - f(a,b+k) + f(a,b) = hk \left.\frac{\partial^2 f}{\partial y\,\partial x}\right|_{(p,q)}.$$

(b) Hence or otherwise, prove that if $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ and $\dfrac{\partial^2 f}{\partial y\,\partial x}$ exist and are continuous everywhere on $\mathbb{R}^2$ then $\dfrac{\partial^2 f}{\partial x\,\partial y}$ exists everywhere on $\mathbb{R}^2$ and $\dfrac{\partial^2 f}{\partial x\,\partial y} = \dfrac{\partial^2 f}{\partial y\,\partial x}$.

9. (a) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous real-valued function on $\mathbb{R}^n$. State precisely what is meant by saying that f is *uniformly continuous* on $\mathbb{R}^n$.

(b) Let $g: \mathbb{R} \to \mathbb{R}$ be a differentiable real-valued function on $\mathbb{R}$, and let K be a positive constant. Prove that if $|g'(t)| \leq K$ for all $t \in \mathbb{R}$ then

$$|g(u) - g(v)| \leq K|u - v|$$

for all $u, v \in \mathbb{R}$.

(c) Let $f: \mathbb{R}^n \to \mathbb{R}$ be a differentiable real-valued function on $\mathbb{R}^n$ with the property that $|f'(\mathbf{a})\mathbf{h}| \leq K|\mathbf{h}|$ for all $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{h} \in \mathbb{R}^n$, where K is some positive constant. Using the answer to part (b), or otherwise, prove that

$$|f(\mathbf{a}) - f(\mathbf{b})| \leq K|\mathbf{a} - \mathbf{b}|$$

for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. Hence prove that $f: \mathbb{R}^n \to \mathbb{R}$ is uniformly continuous on $\mathbb{R}^n$.

10. (a) Let a and b be real numbers satisfying $a < b$ and let $\gamma: [a, b] \to \mathbb{R}^n$ be a continuously differentiable curve mapping the interval $[a, b]$ into an open subset D of $\mathbb{R}^n$. Let $f_1, f_2, \ldots, f_n$ be continuous functions that map D into $\mathbb{R}$. Explain the procedure for evaluating the line integral

$$\int_\gamma (f_1\, dx_1 + f_2\, dx_2 + \cdots + f_n\, dx_n)$$

taken along the curve γ.

(b) Let $\gamma: [0, 1] \to \mathbb{R}^2$ be the smooth curve parameterizing the unit circle, given by

$$\gamma(t) = (\cos(2\pi t), \sin(2\pi t)).$$

Evaluate

$$\int_\gamma \frac{x\, dy - y\, dx}{x^2 + y^2}.$$

(c) State Green's Theorem in the Plane.

(d) Using Green's theorem, or otherwise, show that

$$\int_\sigma \frac{x\, dy - y\, dx}{x^2 + y^2} = 2\pi,$$

where $\sigma: [0, 1] \to \mathbb{R}^2$ is a smooth curve parameterizing an ellipse, given by

$$\sigma(t) = (4\cos(2\pi t), 3\sin(2\pi t)).$$