Course 221: Hilary Term 2007
Section 9: Signed Measures and the Radon-Nikodym Theorem

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Contents

9 Signed Measures and the Radon-Nikodym Theorem

9.1 Signed Measures

Definition Let X be a set, and let $\mathcal{A}$ be a σ-algebra of subsets of X. A *signed measure* is a function $\nu: \mathcal{A} \to \mathbb{R}$ that maps elements of $\mathcal{A}$ to real numbers and is countably additive, so that

$$\nu\left(\bigcup_{E \in \mathcal{E}} E\right) = \sum_{E \in \mathcal{E}} \nu(E)$$

for any pairwise disjoint countable collection $\mathcal{E}$ of subsets of X satisfying $\mathcal{E} \subset \mathcal{A}$.

Thus (non-negative) measures and signed measures are countably additive functions defined on a σ-algebra $\mathcal{A}$ of subsets of some set X. Non-negative measures take values in the set $[0, +\infty]$ of non-negative extended real numbers, whereas signed measures take values in the field $\mathbb{R}$ of real numbers. In particular, if ν is a signed measure on a σ-algebra of subsets of some set X, and if $E \in \mathcal{A}$, then $\nu(E)$ never takes on either of the values $+\infty$ or $-\infty$.

We shall prove that any signed measure may be represented as the difference of two non-negative measures.

Definition Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, and let ν be a signed measure defined on the σ-algebra $\mathcal{A}$. A subset P of X is said to be a *positive set* if $P \in \mathcal{A}$ and $\nu(S) \geq 0$ for all subsets S of P that belong to $\mathcal{A}$. A subset N of X is said to be a *negative set* if $N \in \mathcal{A}$ and $\nu(S) \leq 0$ for all subsets S of P that belong to $\mathcal{A}$.

9.2 The Hahn Decomposition Theorem

Proposition 9.1 *Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, and let ν be a signed measure defined on $\mathcal{A}$. Suppose that $\nu(N) = 0$ for all negative subsets N of X. Then $\nu(S) \geq 0$ for all $S \in \mathcal{A}$.*

Proof We define the measurable subsets of X to be those that belong to the σ-algebra $\mathcal{A}$. Suppose that there existed a measurable subset S of X for which $\nu(S) < 0$. The S could not be a negative set, and therefore there would exist some measurable subset F of S for which $\nu(F) > 0$. There would then exist some positive integer k_1 and a measurable subset F_1 of S such that $\nu(F_1) \geq 1/k_1$, and such that k_1 is the smallest positive integer for which there exists a measurable subset F_1 of S satisfying $\nu(F_1) \geq 1/k_1$. Let $S_1 = S \setminus F_1$. Then $\nu(S_1) = \nu(S) - \nu(F_1) < 0$. It would therefore follow that there would exist some positive integer k_2 and a measurable subset F_2 of S_1 such that $\nu(F_2) \geq 1/k_2$, and such that k_2 is the smallest positive integer for which there exists a measurable subset F_2 of S_1 satisfying $\nu(F_2) \geq 1/k_2$. Then F_1 and F_2 would be disjoint subsets of S. We could continue in this fashion to obtain an infinite sequence of positive integers $k_1, k_2, k_3, \ldots$ and an infinite sequence $F_1, F_2, F_3, \ldots$ of pairwise disjoint measurable subsets of S such that $\nu(F_m) \geq 1/k_m$ for all positive integers m, and such that, for each positive integer m, the positive integer k_m is the smallest positive integer for which there exists a corresponding subset F_m of $S \setminus \bigcup_{j=1}^{m-1} F_j$ satisfying $\nu(F_m) \geq 1/k_m$. Indeed suppose that positive integers $k_1, k_2, \ldots, k_{m-1}$ and pairwise disjoint measurable subsets $F_1, F_2, \ldots, F_{m-1}$ have been found with the required properties. Let $S_m = S \setminus \bigcup_{j=1}^{m-1} F_j$. Then $\nu(S_m) = \nu(S) - \sum_{j=1}^{m-1} \nu(F_j) < 0$. But then S_m would not be a negative set, as X contains no measurable sets N satisfying $\nu(N) < 0$. Therefore there would exist a subset G of S_m satisfying $\nu(G) > 0$, and therefore there would exist a positive integer k_m and a measurable subset F_m of S_m such that $\nu(F_m) \geq 1/k_m$, and such that k_m is the smallest positive integer for which there exists a measurable subset F_m of S_m satisfying $\nu(F_m) \geq 1/k_m$.

We see therefore that if S were a measurable subset of X satisfying $\nu(S) < 0$ then there would exist an infinite sequence $k_1, k_2, k_3, \ldots$ of positive integers and an infinite sequence $F_1, F_2, F_3, \ldots$ of pairwise disjoint subsets of S such that $\nu(F_m) \geq 1/k_m$ for all positive integers m, and such that, for each positive integer m, the positive integer k_m is the smallest positive integer for which there exists a subset F_m of $S \setminus \bigcup_{j=1}^{m-1} F_j$ satisfying $\nu(F_m) \geq 1/k_m$. Clearly $k_m \leq k_{m+1}$ for all positive integers m. Let $F = \bigcup_{m=1}^{+\infty} F_m$. The countable additivity of the signed measure ν would ensure that

$$\sum_{m=1}^{+\infty} 1/k_m \leq \sum_{m=1}^{+\infty} \nu(F_m) = \nu(F) < +\infty,$$

Therefore $\lim_{m \to +\infty} k_m^{-1} = 0$, and thus $\lim_{m \to +\infty} k_m = +\infty$. Also $\nu(S \setminus F) = \nu(S) - \nu(F) \leq \nu(S) < 0$, and moreover if G were a measurable subset of $S \setminus F$ then $\nu(G) < 1/(k_m - 1)$ for all positive integers k_m for which $k_m > 1$, and therefore $\nu(G) \leq 0$. Thus $S \setminus F$ would be a negative subset of X, and $\nu(S \setminus F) < 0$. But X contains no negative set N satsifying $\nu(N) < 0$. Thus the existence of a measurable subset S of X satisfying $\nu(S) < 0$ would lead to a contradiction. We conclude therefore that $\nu(S) \geq 0$ for all measurable subsets S of X, as required. ∎

Theorem 9.2 (Hahn Decomposition Theorem) *Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, and let ν be a signed measure defined on the σ-algebra $\mathcal{A}$. Then there exist subsets N and P of X, such that N is a negative set, P is a positive set, $X = N \cup P$, $N \cap P = \emptyset$.*

Proof We define the measurable subsets of X to be those that belong to the σ-algebra $\mathcal{A}$. The empty set is a negative set, and therefore the set X has negative subsets. Let the extended real number α be the infimum (or greatest lower bound) of the values of $\nu(Z)$ for all negative subsets Z of X, let $N_1, N_2, N_3, \ldots$ be an infinite sequence of negative sets in X with the property that $\nu(N_j) \to \alpha$ in $[-\infty, 0]$ as $j \to +\infty$, and let $N = \bigcup_{j=1}^{+\infty} N_j$. Let S be a subset of N, let $S_1 = S \cap N_1$ and let $S_m = (S \cap N_m) \setminus \bigcup_{j=1}^{m-1} N_j$ for all integers m satisfying $m > 1$. Then the sets $S_1, S_2, S_3, \ldots$ are pairwise disjoint, $S_m \subset N_m$ for all positive integers m, and $\bigcup_{m=1}^{\infty} S_m = S$. It follows from the countable additivity of the signed measure ν that $\nu(S) = \sum_{m=1}^{+\infty} \nu(S_m)$. Also $\nu(S_m) \leq 0$ for all positive integers m, as S_m is a measurable subset of the negative set N_m. It follows that $\nu(S) \leq 0$. Thus the set N is a measurable set. It follows from this that $\nu(N \setminus N_m) \leq 0$, and therefore that $\nu(N) = \nu(N \setminus N_m) + \nu(N_m) \leq \nu(N_m)$ for all positive integers m. But then $\nu(N) \leq \lim_{m \to +\infty} \nu(N_m) = \alpha$, and therefore $\nu(N) = \alpha$. In particular $\alpha > -\infty$, as the values of the signed measure ν are by definition real numbers.

Let $P = X \setminus N$. If the set P were to contain a negative set S satisfying $\nu(S) < 0$ then $N \cup S$ would be a negative set in X satisfying $\nu(N \cup S) < \alpha$. But this is impossible, as α is by definition the infimum of the values $\nu(Z)$ as Z ranges over all negative subsets of X. Therefore P cannot contain any negative set S satisfying $\nu(S) < 0$. We may therefore apply Proposition 9.1 to the restriction of the signed measure ν to the measurable subsets of P, concluding that $\nu(S) \geq 0$ for all measurable subsets S of P. Thus P is a positive set. We see therefore that N is a negative set, P is a positive set, $X = N \cup P$ and $N \cap P = \emptyset$, as required. ∎

Definition Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, and let ν be a signed measure defined on the σ-algebra $\mathcal{A}$. A *Hahn decomposition* of

X with respect to the signed measure ν is a pair (N, P) of subsets of X such that N is a negative set for the signed measure ν, P is a positive set for ν, $X = N \cup P$ and $N \cap P = \emptyset$.

The Hahn Decomposition Theorem thus guarantees the existence of a Hahn decomposition for any signed measure.

Lemma 9.3 *Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, let ν be a signed measure defined on the σ-algebra $\mathcal{A}$, and let (N_1, P_1) and (N_2, P_2) be Hahn decompositions of X with respect to the signed measure ν, where the sets N_1 and N_2 are negative, and the sets P_1 and P_2 are positive. Then $\nu(S \cap N_1) = \nu(S \cap N_2)$ and $\nu(S \cap P_1) = \nu(S \cap P_2)$ for all $S \in \mathcal{A}$.*

Proof Let $S \in \mathcal{A}$. Then

$$S \cap N_1 = (S \cap N_1 \cap N_2) \cup (S \cap N_1 \cap P_2).$$

Now $S \cap N_1 \cap P_2 \subset N_1$, and therefore $\nu(S \cap N_1 \cap P_2) \leq 0$. Also $S \cap N_1 \cap P_2 \subset P_2$, and therefore $\nu(S \cap N_1 \cap P_2) \geq 0$. It follows that $\nu(S \cap N_1 \cap P_2) = 0$, and therefore

$$\nu(S \cap N_1) = \nu(S \cap N_1 \cap N_2) + \nu(S \cap N_1 \cap P_2) = \nu(S \cap N_1 \cap N_2).$$

Similarly $\nu(S \cap N_2) = \nu(S \cap N_1 \cap N_2)$. It follows that $\nu(S \cap N_1) = \nu(S \cap N_2)$. Moreover

$$\nu(S \cap P_1) = \nu(S) - \nu(S \cap N_1) = \nu(S) - \nu(S \cap N_2) = \nu(S \cap P_2),$$

as required. ∎

9.3 The Jordan Decomposition of a Signed Measure

Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, and let ν be a signed measure defined on the σ-algebra $\mathcal{A}$. Then there exists a Hahn decomposition of X as the disjoint union of a negative set N and a positive set P. Let $\nu_+(S) = \nu(S \cap P)$ and $\nu_-(S) = -\nu(S \cap N)$ for all $S \in \mathcal{S}$. Then ν_+ and ν_- are countably additive functions defined on $\mathcal{A}$ and are thus (non-negative) measures on X. Moreover

$$\nu(S) = \nu(S \cap P) + \nu(S \cap N) = \nu_+(S) - \nu_-(S)$$

for all subsets S of X that belong to $\mathcal{A}$. Moreover it follows from Lemma 9.3 that the values of $\nu_+(S)$ and $\nu_-(S)$ are determined by the signed measure ν and the measurable set S, and do not depend on the choice of the Hahn decomposition (N, P). It follows that every signed measure ν defined on $\mathcal{A}$ may be expressed as the difference $\nu_+ - \nu_-$ of two (non-negative) measures defined on $\mathcal{A}$.

Definition Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, and let ν be a signed measure defined on the σ-algebra $\mathcal{A}$. The *Jordan decomposition* of ν on X is the representation of ν as the difference $\nu_+ - \nu_-$ of two (non-negative) measures ν_+ and ν_- defined on $\mathcal{A}$, where, for each $S \in \mathcal{A}$, the values $\nu_+(S)$ and $\nu_-(S)$ of the measures ν_+ and ν_- on S are characterized by the property that $\nu_+(S) = \nu(S \cap P)$ and $\nu_-(S) = -\nu(S \cap N)$ for any Hahn decomposition of X as the disjoint union of a negative set N and a positive set P.

Definition Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, and let ν be a signed measure defined on the σ-algebra $\mathcal{A}$. The *total variation* of ν is the (non-negative) measure $|\nu|$ defined on $\mathcal{A}$ such that $|\nu|(E) = \nu_+(E) + \nu_-(E)$ for all $E \in \mathcal{A}$.

Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, and let ν be a signed measure defined on the σ-algebra $\mathcal{A}$. Then

$$\nu_+ = \tfrac{1}{2}(|\nu| + \nu) \quad \text{and} \quad \nu_- = \tfrac{1}{2}(|\nu| - \nu),$$

9.4 Absolute Continuity

Definition Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, let μ be a (non-negative) measure defined on $\mathcal{A}$, and let ν be a measure or signed measure defined on $\mathcal{A}$. The measure ν is said to be *absolutely continuous* with respect to the measure μ if $\nu(E) = 0$ for all $E \in \mathcal{A}$ satisfying $\mu(E) = 0$. If ν is absolutely continuous with respect to the measure μ then we denote this fact by writing $\nu << \mu$.

Lemma 9.4 *Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, let μ be a (non-negative) measure defined on $\mathcal{A}$, and let ν be a measure or signed measure defined on $\mathcal{A}$. Then $\nu << \mu$ if and only if $\nu_+ << \nu$ and $\nu_- << \nu$. Also $\nu << \mu$ if and only if $|\nu| << \mu$.*

Proof Let $X = N \cup P$ where N is the negative set and P is the positive set determined by a Hahn Decomposition of X, so that $P = X \setminus N$. Then $\nu - +(E) = \nu(E \cap P)$ and $\nu_-(E) = -\nu(E \cap N)$ for all $E \in \mathcal{A}$.

If $\nu_+ << \mu$ and $\nu_- << \mu$ then $\nu(E) = \nu_+(E) - \nu_-(E) = 0$ for all $E \in \mathcal{A}$ satisfying $\mu(E) = 0$, and therefore $\nu << \mu$. Conversely suppose that $\nu << \mu$. Let $E \in \mathcal{A}$ satisfy $\mu(E) = 0$. Then $\mu(E \cap P) = 0$ and $\mu(E \cap N) = 0$, and therefore $\nu_+(E) = \nu(E \cap P) = 0$ and $\nu_-(E) = -\nu(E \cap N) = 0$. Thus $\nu_+ << \mu$ and $\nu_- << \mu$ if and only if $\nu << \mu$.

Now $0 \leq \nu_+(E) \leq |\nu|(E)$, $0 \leq \nu_-(E) \leq |\nu|(E)$ and $|\nu|(E) = \nu_+(E) + \nu_-(E)$ for all $E \in \mathcal{A}$. Therefore $|\nu|(E) = 0$ if and only if $\nu_+(E) = 0$ and

$\nu_-(E) = 0$. It follows that $|\nu| << \mu$ if and only if $\nu_+ << \mu$ and $\nu_- << \mu$. These results ensure that $\nu << \mu$ if and only if $|\nu| << \mu$, as required. ∎

Proposition 9.5 *Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, let μ and ν be (non-negative) measures defined on $\mathcal{A}$, where $\nu(X) < +\infty$. Then $\nu << \mu$ if and only if, given any strictly positive real number ε, there exists some strictly positive real number δ such that $\nu(E) < \varepsilon$ for all $E \in \mathcal{A}$ satisfying $\mu(E) < \delta$.*

Proof Let ν be a non-negative measure satisfying $\nu(X) < +\infty$, and suppose that, given any strictly positive real number ε, there exists some strictly positive real number δ such that $\nu(E) < \varepsilon$ for all $E \in \mathcal{A}$ satisfying $\mu(E) < \delta$. Let $E \in \mathcal{A}$ satisfy $\mu(E) = 0$. Then $\nu(E) < \varepsilon$ for all $\varepsilon > 0$, and therefore $\nu(E) = 0$.

We must prove the converse result. Suppose therefore that ν is a non-negative measure for which $\nu(X) < +\infty$, and suppose also that it is not the case that, given any strictly positive real number ε, there exists some strictly positive real number δ such that $\nu(E) < \varepsilon$ for all $E \in \mathcal{A}$ satisfying $\mu(E) < \delta$. Then there exists some strictly positive real number ε_0 and an infinite sequence $E_1, E_2, E_3, \ldots$ of subsets of X belonging to $\mathcal{A}$ such that $\nu(E_j) \geq \varepsilon_0$ and $\mu(E_j) < 1/2^j$ for all positive integers j. Let $F_j = \bigcup_{k=j}^{+\infty} E_k$. Then

$$\mu(F_j) \leq \sum_{k=j}^{+\infty} \mu(E_k) < \sum_{k=j}^{+\infty} \frac{1}{2^k} = \frac{1}{2^{j-1}}$$

for all $j \in \mathbb{N}$. It follows that $\lim_{j \to +\infty} \mu(F_j) = 0$. Now $F_{j+1} \subset F_j$ and $\nu(F_j) \geq \nu(E_j) \geq \varepsilon_0$ for all $j \in \mathbb{N}$. Let $F = \bigcap_{j=1}^{+\infty} \mu(F_j)$. It follows from Lemma 7.21 that $\mu(F) = \lim_{j \to +\infty} \mu(F_j) = 0$ and $\nu(F) = \lim_{j \to +\infty} \mu(F_j) \geq \varepsilon_0$. Thus the measure ν is not absolutely continuous with respect to ν. We conclude that if ν is a measure that is absolutely continuous with respect to μ, and if $\nu(X) < +\infty$, then, given any strictly positive real number ε, there exists some strictly positive real number δ such that $\nu(E) < \varepsilon$ for all $E \in \mathcal{A}$ satisfying $\mu(E) < \delta$, as required. ∎

Corollary 9.6 *Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, let μ be a (non-negative) measure defined on $\mathcal{A}$, and let ν be a signed measure defined on $\mathcal{A}$. Then $\nu << \mu$ if and only if, given any strictly positive real number ε, there exists some strictly positive real number δ such that $|\nu(E)| < \varepsilon$ for all $E \in \mathcal{A}$ satisfying $\mu(E) < \delta$.*

Proof If ν is a signed measure then ν is absolutely continuous with respect to μ if and only if its total variation $|\nu|$ is absolutely continuous with respect to μ (Lemma 9.4). Moreover $|\nu(X)| = \nu_+(X) + \nu_-(X) < +\infty$. The result therefore follows immediately on applying Proposition 9.5 to the measure $|\nu|$. ∎

9.5 The Radon-Nikodym Theorem

The following theorem is a special case of the Radon-Nikodym Theorem.

Theorem 9.7 *Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, and let μ and ν be (non-negative) measures defined on $\mathcal{A}$. Suppose that $\mu(A) < +\infty$ and $\nu(A) < +\infty$, and that $\nu << \mu$. Then there exists a non-negative function $f\colon X \to [0, +\infty]$ on X that is integrable with respect to the measure μ and that satisfies*

$$\nu(E) = \int_E f\,d\mu$$

for all $E \in \mathcal{A}$.

Proof We define the measurable subsets of X to be those subsets that belong to the σ-algebra $\mathcal{A}$ on which the measures μ and ν are defined.

Let $\mathcal{G}$ denote the collection of all non-negative functions $g\colon X \to [0, +\infty]$ on X that are integrable with respect to the measure μ and that satisfy

$$\int_E g\,d\mu \leq \nu(E)$$

for all $E \in \mathcal{A}$. The set $\mathcal{G}$ is non-empty, as it contains the zero function. Moreover $\int_X g\,d\mu \leq \nu(X)$ for all $g \in \mathcal{G}$. It follows that the set

$$\left\{ \int_X g\,d\mu : g \in \mathcal{G} \right\}$$

is a non-empty set of real numbers which is bounded above, and therefore has a least upper bound in the set $\mathbb{R}$ of real numbers. Let

$$M = \sup \left\{ \int_X g\,d\mu : g \in \mathcal{G} \right\}.$$

We shall prove that there exists a function f in the collection $\mathcal{G}$ that satisfies $\int_X f\,d\mu = M$. We shall also prove that such a function f satisfies $\nu(E) = \int_E f\,d\mu$ for all $E \in \mathcal{A}$.

Now the definition of M as the least upper bound of the integrals of the functions in $\mathcal{G}$ ensures that there exists an infinite sequence $g_1, g_2, g_3, \ldots$ of functions belonging to $\mathcal{G}$, where

$$\int_X g_m \, d\mu > M - \frac{1}{m}$$

for all $m \in \mathbb{N}$. For each positive integer m, let $f_m\colon X \to [0, +\infty]$ be defined such that

$$f_m(x) = \max\{g_j(x) : j = 1, 2, \ldots, m\},$$

and let $E_{m,1}, E_{m,2}, \ldots, E_{m,m} \in \mathcal{A}$ be defined such that

$$E_{m,j} = \{x \in X : f_m(x) = g_j(x) \text{ and } f_m(x) > g_k(x) \text{ for all } k < j\}.$$

(Thus a point x of X belongs to the set $E_{m,j}$ for some positive integers m and j with $j \leq m$ if and only if j is the smallest positive integer for which $f_m(x) = g_j(x)$.) Then, for each positive integer m, the function f_m is measurable with respect to the measure μ, the sets $E_{m,1}, E_{m,2}, \ldots, E_{m,j}$ are pairwise disjoint, and $X = \bigcup_{j=1}^{m} E_{m,j}$. It follows that

$$\int_E f_m \, d\mu = \sum_{j=1}^{m} \int_{E \cap E_{m,j}} f_m \, d\mu = \sum_{j=1}^{m} \int_{E \cap E_{m,j}} g_j \, d\mu \leq \sum_{j=1}^{m} \nu(E \cap E_{m,j}) = \nu(E)$$

for all $E \in \mathcal{A}$, and therefore $f_m \in \mathcal{G}$. Now $f_m(x) \geq g_m(x)$, and therefore

$$M - \frac{1}{m} < \int_X g_m \, d\mu \leq \int_X f_m \, d\mu \leq M$$

for all positive integers m. Also $f_m(x) \leq f_{m+1}(x)$ for all $m \in \mathbb{N}$ and $x \in X$. Let the function $f\colon X \to [0, +\infty]$ be defined such that $f(x) = \lim_{m \to +\infty} f_m(x)$ for all $x \in X$. Then the function f is measurable, and it follows from the Monotone Convergence Theorem (Theorem 8.19) that

$$\int_E f \, d\mu = \lim_{m \to +\infty} \int_E f_m \, d\mu \leq \nu(E)$$

for all $E \in \mathcal{E}$. Thus $f \in \mathcal{G}$. Moreover

$$\int_X f \, d\mu = \lim_{m \to +\infty} \int_X f_m \, d\mu = M.$$

Finally we prove that $\int_E f \, d\mu = \nu(E)$ for all $E \in \mathcal{A}$. Suppose that this were not the case. Then there would exist some $E \in \mathcal{A}$ for which

$\int_E f\,d\mu < \nu(E)$. Moreover $\mu(E) < +\infty$ (because $\mu(X) < +\infty$), and therefore there would exist some strictly positive real number ε such that $\int_E (f+\varepsilon)\,d\mu < \nu(E)$. Let $\sigma(F) = \nu(F) - \int_F (f+\varepsilon)\,d\mu$ for all measurable subsets F of E. Then σ would be a signed measure defined on the σ-algebra of measurable subsets of E. It follows from the Hahn Decomposition Theorem (Theorem 9.2) that there would exist a measurable subset P of E such that P is positive with respect to the signed measure σ and $E \setminus P$ is negative with respect to σ. Moreover $\sigma(P) \geq \sigma(P) + \sigma(E \setminus P) = \sigma(E) > 0$, and therefore $\nu(P) > \int_P (f + \epsilon)\,d\mu \geq 0$. The absolute continuity of the measure ν with respect to the measure μ would then ensure that $\mu(P) > 0$. Define $h\colon X \to [0, +\infty]$ such that

$$h(x) = \begin{cases} f(x) + \varepsilon & \text{if } x \in P; \\ f(x) & \text{if } x \in X \setminus P. \end{cases}$$

Then $\sigma(S) \geq 0$ and thus $\int_S h\,d\mu \leq \nu(S)$ for all measurable subsets S of P. Also $\int_S h\,d\mu = \int_S f\,d\mu \leq \nu(S)$ for all measurable subsets S of $X \setminus P$. It would follow that $\int_S h\,d\mu \leq \nu(S)$ for all $S \in \mathcal{A}$, and therefore $h \in \mathcal{G}$. But this is impossible as

$$\int_X h\,d\mu = \int_X f\,d\mu + \varepsilon\mu(P) = M + \varepsilon\mu(P) > M,$$

where M is by definition the least upper bound of the values of the integrals $\int_X g\,d\mu$ for all $g \in \mathcal{G}$. Thus the existence of a measurable set E for which $\int_E f\,d\mu < \nu(E)$ would lead to a contradiction. We conclude therefore that $\int_E f\,d\mu = \nu(E)$ for all $E \in \mathcal{A}$, and thus f is the required function. ∎

Definition Let $(X, \mathcal{A}, \mu)$ be a measure space. The measure μ is said to be *σ-finite* if there exists an infinite sequence $E_1, E_2, E_3, \ldots$ of measurable subsets of X such that $\mu(E_j) < +\infty$ for all j and $X = \bigcup_{j=1}^{+\infty} E_j$.

Example Lebesgue measure is a σ-finite measure defined on the Lebesgue-measurable sets of n-dimensional Euclidean space $\mathbb{R}^n$. Indeed $\mathbb{R}^n$ is the union of the open balls of radius j about the origin for $j = 1, 2, 3, \ldots$, and each such open ball is a Lebesgue-measurable set with finite measure.

Definition Let $(X, \mathcal{A}, \mu)$ be a measure space. A property $P(x)$ that may or may not be satisfied by points x of X is said to hold *almost everywhere* on X if the set of points of X for which the property fails to hold is contained in a measurable set of measure zero.

Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f and g be functions defined on X. The functions f and g are said to be *equal almost everywhere* on X if $\{x \in X : f(x) \neq g(x)\} \subset E$ for some measurable subset E of X satisfying $\mu(E) = 0$.

Proposition 9.8 *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let f and g be real-valued measurable functions defined on X. Suppose that $\int_E f\,d\mu = \int_E g\,d\mu$ for all $E \in \mathcal{A}$. Then the functions f and g are equal almost everywhere on X.*

Proof For each positive integer j, let $E_j = \{x \in X : f(x) - g(x) \geq 1/j\}$. Then

$$0 = \int_{E_j} (f - g)\,d\mu \geq \frac{\mu(E_j)}{j} \geq 0,$$

and therefore $\mu(E_j) = 0$ for all positive integers j. Now

$$\{x \in X : f(x) > g(x)\} = \bigcup_{j=1}^{+\infty} E_j,$$

where $E_j \subset E_{j+1}$ for all $j \in \mathbb{N}$. It follows that

$$\mu(\{x \in X : f(x) > g(x)\}) = \lim_{j \to +\infty} \mu(E_j) = 0,$$

Similarly $\mu(\{x \in X : f(x) < g(x)\}) = 0$. It follows that

$$\mu(\{x \in X : f(x) \neq g(x)\}) = 0,$$

and thus the functions f and g are equal almost everywhere on X, as required. ∎

Theorem 9.9 (Radon-Nikodym) *Let X be a set, let $\mathcal{A}$ be a σ-algebra of subsets of X, let μ be a σ-finite measure defined on $\mathcal{A}$, and let ν be a signed measure defined on $\mathcal{A}$. Suppose that $\nu << \mu$. Then there exists a function $f\colon X \to [-\infty, +\infty]$ on X that is integrable with respect to the measure μ and that satisfies*

$$\nu(E) = \int_E f\,d\mu$$

for all $E \in \mathcal{A}$. Moreover any two functions with this property are equal almost everywhere on X.

Proof We define the measurable subsets of X to be those subsets that belong to the σ-algebra $\mathcal{A}$. The measure μ is σ-finite, and therefore there exists an infinite sequence $F_1, F_2, F_3, \ldots$ of measurable subsets of X such that $\mu(F_j) < +\infty$ for all $j \in \mathbb{N}$ and $X = \bigcup_{j=1}^{+\infty} F_j$. Let $E_1 = F_1$, and let $E_j = F_j \setminus \bigcup_{k=1}^{j-1} F_k$ for all integers j satisfying $j > 1$. Then the sets $E_1, E_2, E_3, \ldots$ are pairwise disjoint measurable sets, $\mu(E_j) < +\infty$ for all $j \in \mathbb{N}$, and $X = \bigcup_{j=1}^{+\infty} E_j$.

First let us suppose that the measure ν is non-negative. We may then apply Theorem 9.7 to the restriction of the measure ν to each set E_j. It follows that, for each positive integer j, there exists a integrable function $\tilde{f}_j$ defined on E_j such that $\nu(E) = \int_E \tilde{f}_j \, d\mu$ for all measurable subsets E of E_j. Let f_j be the integrable function on X which is defined such that $f_j(x) = \tilde{f}_j(x)$ for all $x \in E_j$, and $f_j(x) = 0$ for all $x \in X \setminus E_j$. Then $\int_E f_j \, \mu = \nu(E \cap E_j)$ for all measurable subsets E of X. Let $f = \sum_{j=1}^{+\infty} f_j$ (so that $f(x) = f_j(x)$ for all $x \in E_j$). It follows from Proposition 8.21 that

$$\int_E f \, d\mu = \sum_{j=1}^{+\infty} \int_E f_j \, d\mu = \sum_{j=1}^{+\infty} \nu(E \cap E_j) = \nu(E)$$

for all measurable subsets E of X. This proves the existence of the required integrable function f in the case where the signed measure ν is non-negative. If the signed measure ν takes both positive and negative values then we can apply the result for non-negative measures to the measures ν_+ and ν_- in order to deduce the existence of integrable non-negative functions f_+ and f_- on X such that $\nu_+(E) = \int_E f_+ \, d\mu$ and $\nu_-(E) = \int_E f_- \, d\mu$ for all $E \in \mathcal{A}$. Then $\nu(E) = \nu_+(E) - \nu_-(E) = \int_E f \, d\mu$ for all $E \in \mathcal{A}$, where $f = f_+ - f_-$.

It follows from Proposition 9.8 that if f and g are integrable functions on X, and if $\nu(E) = \int_E f \, d\mu = \int_E g \, d\mu$ for all $E \in \mathcal{A}$, then the functions f and g are equal almost everywhere on X, as required. ∎