Course 221: Michaelmas Term 2006
Section 2: Metric Spaces

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Contents

2 Metric Spaces

2.1 Euclidean Spaces

We denote by $\mathbb{R}^n$ the set consisting of all n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers. The set $\mathbb{R}^n$ represents n-dimensional *Euclidean space* (with respect to the standard Cartesian coordinate system). Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$, where

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \quad \mathbf{y} = (y_1, y_2, \ldots, y_n),$$

and let λ be a real number. We define

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n), \\ \mathbf{x} - \mathbf{y} &= (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n), \\ \lambda\mathbf{x} &= (\lambda x_1, \lambda x_2, \ldots, \lambda x_n), \\ \mathbf{x} \cdot \mathbf{y} &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n, \\ |\mathbf{x}| &= \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}. \end{aligned}$$

The quantity $\mathbf{x} \cdot \mathbf{y}$ is the *scalar product* (or *inner product*) of $\mathbf{x}$ and $\mathbf{y}$, and the quantity $|\mathbf{x}|$ is the *Euclidean norm* of $\mathbf{x}$. Note that $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$. The *Euclidean distance* between two points $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{R}^n$ is defined to be the Euclidean norm $|\mathbf{y} - \mathbf{x}|$ of the vector $\mathbf{y} - \mathbf{x}$.

Lemma 2.1 (Schwarz' Inequality) *Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$. Then $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$.*

Proof We note that $|\lambda\mathbf{x} + \mu\mathbf{y}|^2 \geq 0$ for all real numbers λ and μ. But

$$|\lambda\mathbf{x} + \mu\mathbf{y}|^2 = (\lambda\mathbf{x} + \mu\mathbf{y}).(\lambda\mathbf{x} + \mu\mathbf{y}) = \lambda^2|\mathbf{x}|^2 + 2\lambda\mu\mathbf{x} \cdot \mathbf{y} + \mu^2|\mathbf{y}|^2.$$

Therefore $\lambda^2|\mathbf{x}|^2 + 2\lambda\mu\mathbf{x} \cdot \mathbf{y} + \mu^2|\mathbf{y}|^2 \geq 0$ for all real numbers λ and μ. In particular, suppose that $\lambda = |\mathbf{y}|^2$ and $\mu = -\mathbf{x} \cdot \mathbf{y}$. We conclude that

$$|\mathbf{y}|^4|\mathbf{x}|^2 - 2|\mathbf{y}|^2(\mathbf{x} \cdot \mathbf{y})^2 + (\mathbf{x} \cdot \mathbf{y})^2|\mathbf{y}|^2 \geq 0,$$

so that $\left(|\mathbf{x}|^2|\mathbf{y}|^2 - (\mathbf{x} \cdot \mathbf{y})^2\right)|\mathbf{y}|^2 \geq 0$. Thus if $\mathbf{y} \neq \mathbf{0}$ then $|\mathbf{y}| > 0$, and hence

$$|\mathbf{x}|^2|\mathbf{y}|^2 - (\mathbf{x} \cdot \mathbf{y})^2 \geq 0.$$

But this inequality is trivially satisfied when $\mathbf{y} = \mathbf{0}$. Thus $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$, as required. ∎

It follows easily from Schwarz' Inequality that $|\mathbf{x}+\mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. For

$$\begin{aligned} |\mathbf{x}+\mathbf{y}|^2 &= (\mathbf{x}+\mathbf{y}).(\mathbf{x}+\mathbf{y}) = |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2\mathbf{x}\cdot\mathbf{y} \\ &\leq |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2|\mathbf{x}||\mathbf{y}| = (|\mathbf{x}| + |\mathbf{y}|)^2. \end{aligned}$$

It follows that

$$|\mathbf{z}-\mathbf{x}| \leq |\mathbf{z}-\mathbf{y}| + |\mathbf{y}-\mathbf{x}|$$

for all points $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ of $\mathbb{R}^n$. This important inequality is known as the *Triangle Inequality*. It expresses the geometric fact the the length of any triangle in a Euclidean space is less than or equal to the sum of the lengths of the other two sides.

2.2 Metric Spaces

Definition A *metric space* (X, d) consists of a set X together with a *distance function* $d: X \times X \to [0, +\infty)$ on X satisfying the following axioms:

(i) $d(x, y) \geq 0$ for all $x, y \in X$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$,

(iv) $d(x, y) = 0$ if and only if $x = y$.

The quantity $d(x, y)$ should be thought of as measuring the *distance* between the points x and y. The inequality $d(x, z) \leq d(x, y) + d(y, z)$ is referred to as the *Triangle Inequality*. The elements of a metric space are usually referred to as *points* of that metric space.

Note that if X is a metric space with distance function d and if A is a subset of X then the restriction $d|A \times A$ of d to pairs of points of A defines a distance function on A satisfying the axioms for a metric space.

The set $\mathbb{R}$ of real numbers becomes a metric space with distance function d given by $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. Similarly the set $\mathbb{C}$ of complex numbers becomes a metric space with distance function d given by $d(z, w) = |z - w|$ for all $z, w \in \mathbb{C}$, and n-dimensional Euclidean space $\mathbb{R}^n$ is a metric space with with respect to the *Euclidean distance function* d, given by

$$d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Any subset X of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{R}^n$ may be regarded as a metric space whose distance function is the restriction to X of the distance function on $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{R}^n$ defined above.

Example The *n-sphere* S^n is defined to be the subset of $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ consisting of all elements $\mathbf{x}$ of $\mathbb{R}^{n+1}$ for which $|\mathbf{x}| = 1$. Thus

$$S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}.$$

(Note that S^2 is the standard (2-dimensional) unit sphere in 3-dimensional Euclidean space.) The *chordal distance* between two points $\mathbf{x}$ and $\mathbf{y}$ of S^n is defined to be the length $|\mathbf{x} - \mathbf{y}|$ of the line segment joining $\mathbf{x}$ and $\mathbf{y}$. The n-sphere S^n is a metric space with respect to the chordal distance function.

2.3 Convergence of Sequences in a Metric Space

Definition Let X be a metric space with distance function d. A sequence $x_1, x_2, x_3, \ldots$ of points in X is said to *converge* to a point p in X if, given any strictly positive real number ε, there exists some natural number N such that $d(x_j, p) < \varepsilon$ whenever $j \geq N$.

We refer to p as the *limit* $\lim\limits_{j \to +\infty} x_j$ of the sequence $x_1, x_2, x_3, \ldots$.

Example The set $\mathbb{R}$ of real numbers is considered to be a metric space, whose distance function d is defined such that $d(u, v) = |u - v|$ for all real numbers u and v. An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers converges in this metric space to some real number p if and only if, given any strictly positive real number ε, there exists some positive integer N such that $|x_j - p| < \varepsilon$ whenever $j \geq N$. This criterion reproduces the standard definition of convergence for an infinite sequence of real numbers. We conclude therefore that the definition of convergence for sequences of points in metric spaces generalizes the standard definition of convergence for infinite sequences of real numbers.

Example Let $z_1, z_2, z_3, \ldots$ be an infinite sequence of complex numbers. The set $\mathbb{C}$ of complex numbers is considered to be a metric space, whose distance function d is defined such that $d(z, w) = |z - w|$ for all complex numbers z and w. It follows that the infinite sequence $z_1, z_2, z_3, \ldots$ of complex numbers converges in this metric space to the complex number w if and only if, given any strictly positive real number ε, there exists some positive integer N such that $|z_j - w| < \varepsilon$ whenever $j \geq N$. This is the standard criterion for the convergence of an infinite sequence of complex numbers.

Example Let n be a positive integer, and let $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots$ be an infinite sequence of points in n-dimensional Euclidean space $\mathbb{R}^n$. This sequence of points converges to some point $\mathbf{r}$ of $\mathbb{R}^n$ if and only if, given any strictly positive real number ε, there exists some positive integer N such that $|\mathbf{p}_j - \mathbf{r}| < \varepsilon$ whenever $j \geq N$.

If a sequence of points in a metric space is convergent then the limit of that sequence is unique. Indeed let $x_1, x_2, x_3, \ldots$ be a sequence of points in a metric space (X, d) which converges to points p and p' of X. We show that $p = p'$. Now, given any $\varepsilon > 0$, there exist natural numbers N_1 and N_2 such that $d(x_j, p) < \varepsilon$ whenever $j \geq N_1$ and $d(x_j, p') < \varepsilon$ whenever $j \geq N_2$. On choosing j so that $j \geq N_1$ and $j \geq N_2$ we see that

$$0 \leq d(p, p') \leq d(p, x_j) + d(x_j, p') < 2\varepsilon$$

by a straightforward application of the metric space axioms (i)–(iii). Thus $0 \leq d(p, p') < 2\varepsilon$ for every $\varepsilon > 0$, and hence $d(p, p') = 0$, so that $p = p'$ by Axiom (iv).

Lemma 2.2 *Let (X, d) be a metric space, and let $x_1, x_2, x_3, \ldots$ be a sequence of points of X which converges to some point p of X. Then, for any point y of X, $d(x_j, y) \to d(p, y)$ as $j \to +\infty$.*

Proof Let $\varepsilon > 0$ be given. We must show that there exists some natural number N such that $|d(x_j, y) - d(p, y)| < \varepsilon$ whenever $j \geq N$. However N can be chosen such that $d(x_j, p) < \varepsilon$ whenever $j \geq N$. But

$$d(x_j, y) \leq d(x_j, p) + d(p, y), \qquad d(p, y) \leq d(p, x_j) + d(x_j, y)$$

for all j, hence

$$-d(x_j, p) \leq d(x_j, y) - d(p, y) \leq d(x_j, p)$$

for all j, and hence $|d(x_j, y) - d(p, y)| < \varepsilon$ whenever $j \geq N$, as required. ∎

2.4 Continuity of Functions between Metric Spaces

Definition Let X and Y be metric spaces with distance functions d_X and d_Y respectively. A function $f\colon X \to Y$ from X to Y is said to be *continuous* at a point p of X if and only if the following criterion is satisfied:—

- given any strictly positive real number ε, there exists some strictly positive real number δ such that $d_Y(f(x), f(p)) < \varepsilon$ for all points x of X satisfying $d_X(x, p) < \delta$.

The function $f: X \to Y$ is said to be continuous on X if and only if it is continuous at p for every point p of X.

Example Let X be a subset of the set $\mathbb{R}$ of real numbers. We can regard X and $\mathbb{R}$ as metric spaces whose distance function d is defined such that $d(u, v) = |u - v|$ for all real numbers u and v belonging to the relevant set. A real-valued function $f: X \to \mathbb{R}$ satisfies the above definition of continuity at an element p of X if and only if, given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(x) - f(p)| < \varepsilon$ for all elements x of X satisfying $|x - p| < \delta$. We see from this that the definition of continuity for functions between metric spaces generalizes the standard definition of continuity for functions of a real variable.

Example Let D be a subset of the set $\mathbb{C}$ of complex numbers. We can regard D and $\mathbb{C}$ as metric spaces whose distance function d is defined such that $d(z, w) = |z - w|$ for all complex numbers z and w belonging to the relevant set. A complex-valued function $f: D \to \mathbb{R}$ satisfies the above definition of continuity at an element w of D if and only if, given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(z) - f(w)| < \varepsilon$ for all elements z of D satisfying $|z - w| < \delta$. This is the standard definition of continuity for functions of a complex variable.

Example Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. A function $f: X \to Y$ from X to Y is continuous at a point $\mathbf{p}$ of X if and only if the following criterion is satisfied:—

> given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all points $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta$.

Lemma 2.3 *Let X, Y and Z be metric spaces, and let $f: X \to Y$ and $g: Y \to Z$ be continuous functions. Then the composition function $g \circ f: X \to Z$ is continuous.*

Proof We denote by d_X, d_Y and d_Z the distance functions on X, Y and Z respectively. Let p be any point of X. We show that $g \circ f$ is continuous at p. Let $\varepsilon > 0$ be given. Now the function g is continuous at $f(p)$. Hence there exists some $\eta > 0$ such that $d_Z(g(y), g(f(p))) < \varepsilon$ for all $y \in Y$ satisfying $d_Y(y, f(p)) < \eta$. But then there exists some $\delta > 0$ such that $d_Y(f(x), f(p)) < \eta$ for all $x \in X$ satisfying $d_X(x, p) < \delta$. Thus $d_Z(g(f(x)), g(f(p))) < \varepsilon$ for all $x \in X$ satisfying $d_X(x, p) < \delta$, showing that $g \circ f$ is continuous at p, as required. ∎

Lemma 2.4 *Let $f\colon X \to Y$ be a continuous function between metric spaces X and Y, and let $x_1, x_2, x_3, \ldots$ be a sequence of points in X which converges to some point p of X. Then the sequence $f(x_1), f(x_2), f(x_3), \ldots$ converges to $f(p)$.*

Proof We denote by d_X and d_Y the distance functions on X and Y respectively. Let $\varepsilon > 0$ be given. We must show that there exists some natural number N such that $d_Y(f(x_n), f(p)) < \varepsilon$ whenever $n \geq N$. However there exists some $\delta > 0$ such that $d_Y(f(x), f(p)) < \varepsilon$ for all $x \in X$ satisfying $d_X(x, p) < \delta$, since the function f is continuous at p. Also there exists some natural number N such that $d_X(x_n, p) < \delta$ whenever $n \geq N$, since the sequence $x_1, x_2, x_3, \ldots$ converges to p. Thus if $n \geq N$ then $d_Y(f(x_n), f(p)) < \varepsilon$, as required. ∎

2.5 Continuity of Functions with Values in Euclidean Spaces

Let $f\colon X \to \mathbb{R}^n$ be a function mapping a mapping a set X into n-dimensional Euclidean space $\mathbb{R}^n$. Then

$$f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$$

for all $x \in X$, where $f_1, f_2, \ldots, f_n$ are functions from X to $\mathbb{R}$, referred to as the *components* of the function f.

Proposition 2.5 *Let X be a metric space, and let p be a point of X. A function $f\colon X \to \mathbb{R}^n$ mapping X into the Euclidean space $\mathbb{R}^n$ is continuous at p if and only if its components are continuous at p.*

Proof Note that the ith component f_i of f is given by $f_i = p_i \circ f$, where $p_i\colon \mathbb{R}^n \to \mathbb{R}$ is the continuous function which maps $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ onto its ith coordinate y_i. It therefore follows immediately from Lemma 2.3 that if f is continuous the point p, then so are the components of f.

Conversely suppose that the components of f are continuous at $p \in X$. Let $\varepsilon > 0$ be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots, \delta_n$ such that $|f_i(x) - f_i(p)| < \varepsilon/\sqrt{n}$ for $x \in X$ satisfying $d(x, p) < \delta_i$, where d denotes the distance function on the metric space X. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_n$. If $x \in X$ satisfies $d(x, p) < \delta$ then

$$|f(x) - f(p)|^2 = \sum_{i=1}^{n} |f_i(x) - f_i(p)|^2 < \varepsilon^2,$$

and hence $|f(x) - f(p)| < \varepsilon$. Thus the function f is continuous at p, as required. ∎

2.6 Open Sets in Euclidean Spaces

Let X be a subset of $\mathbb{R}^n$. Given a point $\mathbf{p}$ of X and a non-negative real number r, the *open ball* $B_X(\mathbf{p}, r)$ in X of *radius* r about $\mathbf{p}$ is defined to be the subset of X given by

$$B_X(\mathbf{p}, r) \equiv \{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < r\}.$$

(Thus $B_X(\mathbf{p}, r)$ is the set consisting of all points of X that lie within a sphere of radius r centred on the point $\mathbf{p}$.)

Definition Let X be a subset of $\mathbb{R}^n$. A subset V of X is said to be *open* in X if and only if, given any point $\mathbf{p}$ of V, there exists some $\delta > 0$ such that $B_X(\mathbf{p}, \delta) \subset V$.

By convention, we regard the empty set $\emptyset$ as being an open subset of X. (The criterion given above is satisfied vacuously in the case when V is the empty set.)

In particular, a subset V of $\mathbb{R}^n$ is said to be an *open set* (in $\mathbb{R}^n$) if and only if, given any point $\mathbf{p}$ of V, there exists some $\delta > 0$ such that $B(\mathbf{p}, \delta) \subset V$, where $B(\mathbf{p}, r) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < r\}$.

Example Let $H = \{(x, y, z) \in \mathbb{R}^3 : z > c\}$, where c is some real number. Then H is an open set in $\mathbb{R}^3$. Indeed let $\mathbf{p}$ be a point of H. Then $\mathbf{p} = (u, v, w)$, where $w > c$. Let $\delta = w - c$. If the distance from a point (x, y, z) to the point (u, v, w) is less than δ then $|z - w| < \delta$, and hence $z > c$, so that $(x, y, z) \in H$. Thus $B(\mathbf{p}, \delta) \subset H$, and therefore H is an open set.

Lemma 2.6 *Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any positive real number r, the open ball $B_X(\mathbf{p}, r)$ in X of radius r about $\mathbf{p}$ is open in X.*

Proof Let $\mathbf{x}$ be an element of $B_X(\mathbf{p}, r)$. We must show that there exists some $\delta > 0$ such that $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{p}, r)$. Let $\delta = r - |\mathbf{x} - \mathbf{p}|$. Then $\delta > 0$, since $|\mathbf{x} - \mathbf{p}| < r$. Moreover if $\mathbf{y} \in B_X(\mathbf{x}, \delta)$ then

$$|\mathbf{y} - \mathbf{p}| \leq |\mathbf{y} - \mathbf{x}| + |\mathbf{x} - \mathbf{p}| < \delta + |\mathbf{x} - \mathbf{p}| = r,$$

by the Triangle Inequality, and hence $\mathbf{y} \in B_X(\mathbf{p}, r)$. Thus $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{p}, r)$. This shows that $B_X(\mathbf{p}, r)$ is an open set, as required. ∎

Lemma 2.7 *Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any non-negative real number r, the set $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| > r\}$ is an open set in X.*

Proof Let $\mathbf{x}$ be a point of X satisfying $|\mathbf{x} - \mathbf{p}| > r$, and let $\mathbf{y}$ be any point of X satisfying $|\mathbf{y} - \mathbf{x}| < \delta$, where $\delta = |\mathbf{x} - \mathbf{p}| - r$. Then

$$|\mathbf{x} - \mathbf{p}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{p}|,$$

by the Triangle Inequality, and therefore

$$|\mathbf{y} - \mathbf{p}| \geq |\mathbf{x} - \mathbf{p}| - |\mathbf{y} - \mathbf{x}| > |\mathbf{x} - \mathbf{p}| - \delta = r.$$

Thus $B_X(\mathbf{x}, \delta)$ is contained in the given set. The result follows. ∎

2.7 Open Sets in Metric Spaces

Definition Let (X, d) be a metric space. Given a point p of X and $r \geq 0$, the *open ball* $B_X(p, r)$ of *radius* r about p in X is defined by

$$B_X(x, r) = \{x \in X : d(x, p) < r\}.$$

Definition Let (X, d) be a metric space. A subset V of X is said to be an *open set* if and only if the following condition is satisfied:

- given any point v of V there exists some $\delta > 0$ such that $B_X(v, \delta) \subset V$.

By convention, we regard the empty set $\emptyset$ as being an open subset of X. (The criterion given above is satisfied vacuously in this case.)

Lemma 2.8 *Let X be a metric space with distance function d, and let p be a point of X. Then, for any $r > 0$, the open ball $B_X(p, r)$ of radius r about p is an open set in X.*

Proof Let $q \in B_X(p, r)$. We must show that there exists some $\delta > 0$ such that $B_X(q, \delta) \subset B_X(p, r)$. Now $d(q, p) < r$, and hence $\delta > 0$, where $\delta = r - d(q, p)$. Moreover if $x \in B_X(q, \delta)$ then

$$d(x, p) \leq d(x, q) + d(q, p) < \delta + d(q, p) = r,$$

by the Triangle Inequality, hence $x \in B_X(p, r)$. Thus $B_X(q, \delta) \subset B_X(p, r)$, showing that $B_X(p, r)$ is an open set, as required. ∎

Lemma 2.9 *Let X be a metric space with distance function d, and let p be a point of X. Then, for any $r \geq 0$, the set $\{x \in X : d(x, p) > r\}$ is an open set in X.*

Proof Let q be a point of X satisfying $d(q,p) > r$, and let x be any point of X satisfying $d(x,q) < \delta$, where $\delta = d(q,p) - r$. Then

$$d(q,p) \leq d(q,x) + d(x,p),$$

by the Triangle Inequality, and therefore

$$d(x,p) \geq d(q,p) - d(x,q) > d(q,p) - \delta = r.$$

Thus $B_X(x,\delta) \subset \{x \in X : d(x,p) > r\}$, as required. ∎

Proposition 2.10 *Let X be a metric space. The collection of open sets in X has the following properties:—*

(i) *the empty set $\emptyset$ and the whole set X are both open sets;*

(ii) *the union of any collection of open sets is itself an open set;*

(iii) *the intersection of any* finite *collection of open sets is itself an open set.*

Proof The empty set $\emptyset$ is an open set by convention. Moreover the definition of an open set is satisfied trivially by the whole set X. Thus (i) is satisfied.

Let $\mathcal{A}$ be any collection of open sets in X, and let U denote the union of all the open sets belonging to $\mathcal{A}$. We must show that U is itself an open set. Let $x \in U$. Then $x \in V$ for some open set V belonging to the collection $\mathcal{A}$. Therefore there exists some $\delta > 0$ such that $B_X(x,\delta) \subset V$. But $V \subset U$, and thus $B_X(x,\delta) \subset U$. This shows that U is open. Thus (ii) is satisfied.

Finally let $V_1, V_2, V_3, \ldots, V_k$ be a *finite* collection of open sets in X, and let $V = V_1 \cap V_2 \cap \cdots \cap V_k$. Let $x \in V$. Now $x \in V_j$ for all j, and therefore there exist strictly positive real numbers $\delta_1, \delta_2, \ldots, \delta_k$ such that $B_X(x,\delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. (This is where we need the fact that we are dealing with a finite collection of open sets.) Moreover $B_X(x,\delta) \subset B_X(x,\delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$, and thus $B_X(x,\delta) \subset V$. This shows that the intersection V of the open sets $V_1, V_2, \ldots, V_k$ is itself open. Thus (iii) is satisfied. ∎

Example The set $\{(x,y,z) \in \mathbb{R}^3 : (x-1)^2 + (y-2)^2 + z^2 < 9 \text{ and } z > 1\}$ is an open set in $\mathbb{R}^3$, since it is the intersection of the open ball of radius 3 about the point $(1,2,0)$ with the open set $\{(x,y,z) \in \mathbb{R}^3 : z > 1\}$.

Example The set $\{(x,y,z) \in \mathbb{R}^3 : (x-1)^2 + (y-2)^2 + z^2 < 4 \text{ or } z > 1\}$ is an open set in $\mathbb{R}^3$, since it is the union of the open ball of radius 2 about the point $(1,2,0)$ with the open set $\{(x,y,z) \in \mathbb{R}^3 : z > 1\}$.

Example For each natural number k, let

$$V_k = \{(x, y, z) \in \mathbb{R}^3 : k^2(x^2 + y^2 + z^2) < 1\}.$$

Now each set V_k is an open ball of radius $1/k$ about the origin, and is therefore an open set in $\mathbb{R}^3$. However the intersection of the sets V_k for all natural numbers k is the set $\{(0, 0, 0)\}$, and thus the intersection of the sets V_k for all natural numbers k is not itself an open set in $\mathbb{R}^3$. This example demonstrates that infinite intersections of open sets need not be open.

Lemma 2.11 *A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ converges to a point $\mathbf{p}$ if and only if, given any open set U which contains $\mathbf{p}$, there exists some natural number N such that $\mathbf{x}_j \in U$ for all j satisfying $j \geq N$.*

Proof Suppose that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ has the property that, given any open set U which contains $\mathbf{p}$, there exists some natural number N such that $\mathbf{x}_j \in U$ whenever $j \geq N$. Let $\varepsilon > 0$ be given. The open ball $B(\mathbf{p}, \varepsilon)$ of radius ε about $\mathbf{p}$ is an open set by Lemma 2.6. Therefore there exists some natural number N such that $\mathbf{x}_j \in B(\mathbf{p}, \varepsilon)$ whenever $j \geq N$. Thus $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$. This shows that the sequence converges to $\mathbf{p}$.

Conversely, suppose that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to $\mathbf{p}$. Let U be an open set which contains $\mathbf{p}$. Then there exists some $\varepsilon > 0$ such that the open ball $B(\mathbf{p}, \varepsilon)$ of radius ε about $\mathbf{p}$ is a subset of U. Thus there exists some $\varepsilon > 0$ such that U contains all points $\mathbf{x}$ of X that satisfy $|\mathbf{x} - \mathbf{p}| < \varepsilon$. But there exists some natural number N with the property that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$, since the sequence converges to $\mathbf{p}$. Therefore $\mathbf{x}_j \in U$ whenever $j \geq N$, as required. ∎

Lemma 2.12 *Let X be a metric space. A sequence $x_1, x_2, x_3, \ldots$ of points in X converges to a point p if and only if, given any open set U which contains p, there exists some natural number N such that $x_j \in U$ for all $j \geq N$.*

Proof Let $x_1, x_2, x_3, \ldots$ be a sequence satisfying the given criterion, and let $\varepsilon > 0$ be given. The open ball $B_X(p, \varepsilon)$ of radius ε about p is an open set (see Lemma 2.8). Therefore there exists some natural number N such that, if $j \geq N$, then $x_j \in B_X(p, \varepsilon)$, and thus $d(x_j, p) < \varepsilon$. Hence the sequence (x_j) converges to p.

Conversely, suppose that the sequence (x_j) converges to p. Let U be an open set which contains p. Then there exists some $\varepsilon > 0$ such that $B_X(p, \varepsilon) \subset U$. But $x_j \to p$ as $j \to +\infty$, and therefore there exists some natural number N such that $d(x_j, p) < \varepsilon$ for all $j \geq N$. If $j \geq N$ then $x_j \in B_X(p, \varepsilon)$ and thus $x_j \in U$, as required. ∎

Definition Let (X, d) be a metric space, and let x be a point of X. A subset N of X is said to be a *neighbourhood* of x (in X) if and only if there exists some $\delta > 0$ such that $B_X(x, \delta) \subset N$, where $B_X(x, \delta)$ is the open ball of radius δ about x.

It follows directly from the relevant definitions that a subset V of a metric space X is an open set if and only if V is a neighbourhood of v for all $v \in V$.

2.8 Closed Sets in a Metric Space

A subset F of a metric space X is said to be a *closed set* in X if and only if its complement $X \setminus F$ is open. (Recall that the *complement* $X \setminus F$ of F in X is, by definition, the set of all points of the metric space X that do not belong to F.) The following result follows immediately from Lemma 2.8 and Lemma 2.9.

Example The sets $\{(x, y, z) \in \mathbb{R}^3 : z \geq c\}$, $\{(x, y, z) \in \mathbb{R}^3 : z \leq c\}$, and $\{(x, y, z) \in \mathbb{R}^3 : z = c\}$ are closed sets in $\mathbb{R}^3$ for each real number c, since the complements of these sets are open in $\mathbb{R}^3$.

Lemma 2.13 *Let X be a metric space with distance function d, and let $x_0 \in X$. Given any $r \geq 0$, the sets*

$$\{x \in X : d(x, x_0) \leq r\}, \qquad \{x \in X : d(x, x_0) \geq r\}$$

are closed. In particular, the set $\{x_0\}$ consisting of the single point x_0 is a closed set in X.

Let $\mathcal{A}$ be some collection of subsets of a set X. Then

$$X \setminus \bigcup_{S \in \mathcal{A}} S = \bigcap_{S \in \mathcal{A}} (X \setminus S), \qquad X \setminus \bigcap_{S \in \mathcal{A}} S = \bigcup_{S \in \mathcal{A}} (X \setminus S)$$

(i.e., the complement of the union of some collection of subsets of X is the intersection of the complements of those sets, and the complement of the intersection of some collection of subsets of X is the union of the complements of those sets, so that the operation of taking complements converts unions into intersections and intersections into unions). The following result therefore follows directly from Proposition 2.10.

Proposition 2.14 *Let X be a metric space. The collection of closed sets in X has the following properties:—*

(i) *the empty set* $\emptyset$ *and the whole set* X *are both closed sets;*

(ii) *the intersection of any collection of closed sets in* X *is itself a closed set;*

(iii) *the union of any* finite *collection of closed sets in* X *is itself a closed set.*

Lemma 2.15 *Let* F *be a closed set in a metric space* X *and let* $(x_j : j \in \mathbb{N})$ *be a sequence of points of* F. *Suppose that* $x_j \to p$ *as* $j \to +\infty$. *Then* p *also belongs to* F.

Proof Suppose that the limit p of the sequence were to belong to the complement $X \setminus F$ of the closed set F. Now $X \setminus F$ is open, and thus it would follow from Lemma 2.12 that there would exist some natural number N such that $x_j \in X \setminus F$ for all $j \geq N$, contradicting the fact that $x_j \in F$ for all j. This contradiction shows that p must belong to F, as required. ∎

Definition Let A be a subset of a metric space X. The *closure* $\overline{A}$ of A is the intersection of all closed subsets of X containing A.

Let A be a subset of the metric space X. Note that the closure $\overline{A}$ of A is itself a closed set in X, since the intersection of any collection of closed subsets of X is itself a closed subset of X (see Proposition 2.14). Moreover if F is any closed subset of X, and if $A \subset F$, then $\overline{A} \subset F$. Thus the closure $\overline{A}$ of A is the smallest closed subset of X containing A.

Lemma 2.16 *Let* X *be a metric space with distance function* d, *let* A *be a subset of* X, *and let* x *be a point of* X. *Then* x *belongs to the closure* $\overline{A}$ *of* A *if and only if, given any* $\varepsilon > 0$, *there exists some point* a *of* A *such that* $d(x, a) < \varepsilon$.

Proof Let x be a point of X with the property that, given any $\varepsilon > 0$, there exists some $a \in A$ satisfying $d(x, a) < \varepsilon$. Let F be any closed subset of X containing A. If x did not belong to F then there would exist some $\varepsilon > 0$ with the property that $B_X(x, \varepsilon) \cap F = \emptyset$, where $B_X(x, \varepsilon)$ denotes the open ball of radius ε about x. But this would contradict the fact that $B_X(x, \varepsilon) \cap A$ is non-empty for all $\varepsilon > 0$. Thus the point x belongs to every closed subset F of X that contains A, and therefore $x \in \overline{A}$, by definition of the closure $\overline{A}$ of A.

Conversely let $x \in \overline{A}$, and let $\varepsilon > 0$ be given. Let F be the complement $X \setminus B_X(x, \varepsilon)$ of $B_X(x, \varepsilon)$. Then F is a closed subset of X, and the point x does not belong to F. If $B_X(x, \varepsilon) \cap A = \emptyset$ then A would be contained in F, and hence $x \in F$, which is impossible. Therefore there exists $a \in A$ satisfying $d(x, a) < \varepsilon$, as required. ∎

2.9 Continuous Functions and Open and Closed Sets

Let X and Y be metric spaces, and let $f\colon X \to Y$ be a function from X to Y. We recall that the function f is continuous at a point p of X if and only if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that $d_Y(f(x), f(p)) < \varepsilon$ for all points x of X satisfying $d_X(x,p) < \delta$, where d_X and d_Y denote the distance functions on X and Y respectively. Expressed in terms of open balls, this means that the function $f\colon X \to Y$ is continuous at p if and only if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that f maps $B_X(p,\delta)$ into $B_Y(f(p),\varepsilon)$ (where $B_X(p,\delta)$ and $B_Y(f(p),\varepsilon)$ denote the open balls of radius δ and ε about p and $f(p)$ respectively).

Let $f\colon X \to Y$ be a function from a set X to a set Y. Given any subset V of Y, we denote by $f^{-1}(V)$ the *preimage* of V under the map f, defined by

$$f^{-1}(V) = \{x \in X : f(x) \in V\}.$$

Proposition 2.17 *Let X and Y be metric spaces, and let $f\colon X \to Y$ be a function from X to Y. The function f is continuous if and only if $f^{-1}(V)$ is an open set in X for every open set V of Y.*

Proof Suppose that $f\colon X \to Y$ is continuous. Let V be an open set in Y. We must show that $f^{-1}(V)$ is open in X. Let p be a point belonging to $f^{-1}(V)$. We must show that there exists some $\delta > 0$ with the property that $B_X(p,\delta) \subset f^{-1}(V)$. Now $f(p)$ belongs to V. But V is open, hence there exists some $\varepsilon > 0$ with the property that $B_Y(f(p),\varepsilon) \subset V$. But f is continuous at p. Therefore there exists some $\delta > 0$ such that f maps the open ball $B_X(p,\delta)$ into $B_Y(f(p),\varepsilon)$ (see the remarks above). Thus $f(x) \in V$ for all $x \in B_X(p,\delta)$, showing that $B_X(p,\delta) \subset f^{-1}(V)$. We have thus shown that if $f\colon X \to Y$ is continuous then $f^{-1}(V)$ is open in X for every open set V in Y.

Conversely suppose that $f\colon X \to Y$ has the property that $f^{-1}(V)$ is open in X for every open set V in Y. Let p be any point of X. We must show that f is continuous at p. Let $\varepsilon > 0$ be given. The open ball $B_Y(f(p),\varepsilon)$ is an open set in Y, by Lemma 2.8, hence $f^{-1}\left(B_Y(f(p),\varepsilon)\right)$ is an open set in X which contains p. It follows that there exists some $\delta > 0$ such that $B_X(p,\delta) \subset f^{-1}\left(B_Y(f(p),\varepsilon)\right)$. We have thus shown that, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that f maps the open ball $B_X(p,\delta)$ into $B_Y(f(p),\varepsilon)$. We conclude that f is continuous at p, as required. ∎

Let $f\colon X \to Y$ be a function between metric spaces X and Y. Then the preimage $f^{-1}(Y \setminus G)$ of the complement $Y \setminus G$ of any subset G of Y is equal to the complement $X \setminus f^{-1}(G)$ of the preimage $f^{-1}(G)$ of G. Indeed

$$x \in f^{-1}\left(Y \setminus G\right) \iff f(x) \in Y \setminus G \iff f(x) \notin G \iff x \notin f^{-1}(G).$$

Also a subset of a metric space is closed if and only if its complement is open. The following result therefore follows directly from Proposition 2.17.

Corollary 2.18 *Let X and Y be metric spaces, and let $f\colon X \to Y$ be a function from X to Y. The function f is continuous if and only if $f^{-1}(G)$ is a closed set in X for every closed set G in Y.*

Let $f\colon X \to Y$ be a continuous function from a metric space X to a metric space Y. Then, for any point y of Y, the set $\{x \in X : f(x) = y\}$ is a closed subset of X. This follows from Corollary 2.18, together with the fact that the set $\{y\}$ consisting of the single point y is a closed subset of the metric space Y.

Let X be a metric space, and let $f\colon X \to \mathbb{R}$ be a continuous function from X to $\mathbb{R}$. Then, given any real number c, the sets

$$\{x \in X : f(x) > c\}, \qquad \{x \in X : f(x) < c\}$$

are open subsets of X, and the sets

$$\{x \in X : f(x) \geq c\}, \qquad \{x \in X : f(x) \leq c\}, \qquad \{x \in X : f(x) = c\}$$

are closed subsets of X. Also, given real numbers a and b satisfying $a < b$, the set

$$\{x \in X : a < f(x) < b\}$$

is an open subset of X, and the set

$$\{x \in X : a \leq f(x) \leq b\}$$

is a closed subset of X.

Similar results hold for continuous functions $f\colon X \to \mathbb{C}$ from X to $\mathbb{C}$. Thus, for example,

$$\{x \in X : |f(x)| < R\}, \qquad \{x \in X : |f(x)| > R\}$$

are open subsets of X and

$$\{x \in X : |f(x)| \leq R\}, \qquad \{x \in X : |f(x)| \geq R\}, \qquad \{x \in X : |f(x)| = R\}$$

are closed subsets of X, for any non-negative real number R.

2.10 Homeomorphisms

Let X and Y be metric spaces. A function $h: X \to Y$ from X to Y is said to be a *homeomorphism* if it is a bijection and both $h: X \to Y$ and its inverse $h^{-1}: Y \to X$ are continuous. If there exists a homeomorphism $h: X \to Y$ from a metric space X to a metric space Y, then the metric spaces X and Y are said to be *homeomorphic*.

The following result follows directly on applying Proposition 2.17 to $h: X \to Y$ and to $h^{-1}: Y \to X$.

Lemma 2.19 *Any homeomorphism $h: X \to Y$ between metric spaces X and Y induces a one-to-one correspondence between the open sets of X and the open sets of Y: a subset V of Y is open in Y if and only if $h^{-1}(V)$ is open in X.*

Let X and Y be metric spaces, and let $h: X \to Y$ be a homeomorphism. A sequence $x_1, x_2, x_3, \ldots$ of points in X is convergent in X if and only if the corresponding sequence $h(x_1), h(x_2), h(x_3), \ldots$ is convergent in Y. (This follows directly on applying Lemma 2.4 to $h: X \to Y$ and its inverse $h^{-1}: Y \to X$.) Let Z and W be metric spaces. A function $f: Z \to X$ is continuous if and only if $h \circ f: Z \to Y$ is continuous, and a function $g: Y \to W$ is continuous if and only if $g \circ h: X \to W$ is continuous.