Course 221: Hilary Term 2007
Section 7: Measure Spaces

David R. Wilkins

Contents

7 Measure Spaces

7.1 Bricks

Definition We define an *n-dimensional brick* to be a subset of $\mathbb{R}^n$ that is a Cartesian product of bounded intervals.

Let B be an n-dimensional brick. Then there exist bounded intervals $I_1, I_2, \ldots, I_n$ such that $B = I_1 \times I_2 \times \cdots \times I_n$. Let a_i and b_i denote the endpoints of the interval I_i for $i = 1, 2, \ldots, n$, where $a_i \leq b_i$. Then the interval I_i must coincide with one of the intervals (a_i, b_i), $(a_i, b_i]$, $[a_i, b_i)$ and $[a_i, b_i]$ determined by its endpoints, where

$$(a_i, b_i) = \{x \in \mathbb{R} : a_i < x < b_i\}, \quad (a_i, b_i] = \{x \in \mathbb{R} : a_i < x \leq b_i\}$$

$$[a_i, b_i) = \{x \in \mathbb{R} : a_i \leq x < b_i\}, \quad [a_i, b_i] = \{x \in \mathbb{R} : a_i \leq x \leq b_i\}.$$

We say that the brick B is *open* if $I_i = (a_i, b_i)$ for $i = 1, 2, \ldots, n$. Similarly we say that the brick B is *closed* if $I_i = [a_i, b_i]$ for $i = 1, 2, \ldots, n$.

Definition Let B be an n-dimensional brick that is the Cartesian product $I_1 \times I_2 \times \cdots \times I_n$ of bounded intervals $I_1, I_2, \ldots, I_n$, and let a_i and b_i denote the endpoints of the interval I_i, where $a_i \leq b_i$. The *content* $m(B)$ of the brick B is then defined to be the product $\prod_{i=1}^{n} (b_i - a_i)$ of the lengths of the intervals $I_1, I_2, \ldots, I_n$.

Note that a one-dimensional brick is a bounded interval in the real line, and the content of the brick is the length of the interval. A two-dimensional brick is a rectangle in $\mathbb{R}^2$ with sides parallel to the coordinate axes, and the content of the brick is the area of the rectangle. The content of a three-dimensional brick is the volume of that brick.

Let B be an n-dimensional brick, and let $B_1, B_2, \ldots, B_s$ be a finite collection of n-dimensional bricks. We shall show that if $B \subset \bigcup_{k=1}^{s} B_k$ then $m(B) \leq \sum_{k=1}^{s} m(B_k)$. We shall also show that if the interiors of the bricks $B_1, B_2, \ldots, B_s$ are disjoint and are contained in B then $m(B) \geq \sum_{k=1}^{s} m(B_k)$. These results are of course fairly intuitive, and may at first sight seem to be obvious.

Now the brick B is the Cartesian product $I_1 \times I_2 \times \times I_n$ of bounded intervals $I_1, I_2, \ldots I_n$ in the real line. Let a_i and b_i denote the endpoints of the interval I_i for $i = 1, 2, \ldots, n$, where a_i and b_i are real numbers satisfying $a_i \leq b_i$. Similarly each brick B_k is a Cartesian product $I_{k,1} \times I_{k,2} \times \times I_{k,n}$ of bounded intervals $I_{k,1}, I_{k,2}, \ldots I_{k,n}$ in the real line. Let $a_{k,i}$ and $b_{k,i}$ denote the endpoints of the interval $I_{k,i}$ for $i = 1, 2, \ldots, n$, where $a_{k,i}$ and $b_{k,i}$ are real numbers satisfying $a_{k,i} \leq b_{k,i}$. Then $m(B) = \prod_{i=1}^{n}(b_i - a_i)$, and $m(B_k) = \prod_{i=1}^{n}(b_{k,i} - a_{k,i})$ for $k = 1, 2, \ldots, s$.

Now there exist finite sets $P_1, P_2, \ldots, P_n$ such that $a_i \in P_i$, $b_i \in P_i$, $a_{k,i} \in P_i$ and $b_{k,i} \in P_i$ for $i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, s$. Let

$$P_i = \{t_{i,0}, t_{i,1}, t_{i,2}, \ldots, t_{i,m_i}\}$$

for $i = 1, 2, \ldots, m_i$, where

$$t_{i,0} < t_{i,1} < t_{i,2} < \cdots < t_{i,m_i}.$$

Also let J denote the set consisting of all n-tuples $(j_1, j_2, j_3, \ldots, j_n)$ with $1 \leq j_i \leq m_i$ for $i = 1, 2, \ldots, n$, and, for each $(j_1, j_2, \ldots, j_n) \in J$, let $V_{j_1,j_2,\ldots,j_n}$ denote the open brick consisting of all points $(x_1, x_2, \ldots, x_n)$ of $\mathbb{R}^n$ that satisfy $t_{i,j_i-1} < x_i < t_{i,j_i}$ for $i = 1, 2, \ldots, n$. Then the content $m(V_{j_1,j_2,\ldots,j_n})$ of the brick $V_{j_1,j_2,\ldots,j_n}$ is the product $\prod_{i=1}^{n}(t_{i,j_i} - t_{i,j_i-1})$ of the lengths $t_{i,j_i} - t_{i,j_i-1}$ of the intervals (t_{i,j_i-1}, t_{i,j_i}).

Now, given any integer i between 1 and n, the endpoints a_i and b_i of the interval I_i belong to the set P_i, and therefore there exist integers p_i and q_i satisfying $1 \leq p_i \leq q_i \leq m_i$ such that $a = t_{i,p_i}$ and $b = t_{i,q_i}$. Then

$$b_i - a_i = \sum_{p_i < j_i \leq q_i} (t_{i,j_i} - t_{i,j_i-1}).$$

(The sum on the right hand side of the above equality has the value zero when $p_i = q_i$.) It follows from this that

$$\begin{aligned} m(B) &= \prod_{i=1}^{n}(b_i - a_i) = \sum_{(j_1,j_2,\ldots,j_n)\in J(B)} \prod_{i=1}^{n}(t_{i,j_i} - t_{i,j_i-1}) \\ &= \sum_{(j_1,j_2,\ldots,j_n)\in J(B)} m(V_{j_1,j_2,\ldots,j_n}), \end{aligned}$$

where

$$\begin{aligned} J(B) &= \{(j_1, j_2, \ldots, j_n) \in J : p_i < j_i \leq q_i \text{ for } i = 1, 2, \ldots, n\} \\ &= \{(j_1, j_2, \ldots, j_n) \in J : V_{j_1,j_2,\ldots,j_n} \subset B\}. \end{aligned}$$

Now $(j_1, j_2, \ldots, j_n) \in J(B)$ if and only if $V_{j_1,j_2,\ldots,j_n} \subset B$. We conclude therefore that that the content $m(B)$ of the brick B is the sum of the contents of those open bricks $V_{j_1,j_2,\ldots,j_n}$ for which $(j_1, j_2, \ldots, j_n) \in J$ and $V_{j_1,j_2,\ldots,j_n} \subset B$. Similarly

$$m(B_k) = \sum_{(j_1,j_2,\ldots,j_n)\in J(B_k)} m(V_{j_1,j_2,\ldots,j_n})$$

for $k = 1, 2, \ldots, s$, where

$$J(B_k) = \{(j_1, j_2, \ldots, j_n) \in J : V_{j_1,j_2,\ldots,j_n} \subset B_k\}.$$

Now suppose that $B \subset \bigcup_{k=1}^{s} B_k$. Then $J(B) \subset \bigcup_{k=1}^{s} J(B_k)$, and therefore

$$\begin{aligned} m(B) &= \sum_{(j_1,j_2,\ldots,j_n)\in J(B)} m(V_{j_1,j_2,\ldots,j_n}) \\ &\leq \sum_{k=1}^{s} \sum_{(j_1,j_2,\ldots,j_n)\in J(B_k)} m(V_{j_1,j_2,\ldots,j_n}) = \sum_{k=1}^{s} m(B_k) \end{aligned}$$

On the other hand, suppose that the interiors of the bricks $B_1, B_2, \ldots, B_s$ are disjoint and contained in B. Then $\bigcup_{k=1}^{s} J(B_k) \subset J(B)$, and moreover each n-tuple $(j_1, j_2, \ldots, j_n)$ of integers in $J(B)$ belongs to at most one of the sets $J(B_1), J(B_2), \ldots, J(B_s)$. Therefore

$$m(B) \geq \sum_{k=1}^{s} \sum_{(j_1,j_2,\ldots,j_n)\in J(B_k)} m(V_{j_1,j_2,\ldots,j_n}) = \sum_{k=1}^{s} m(B_k).$$

We have therefore proved the following two results.

Proposition 7.1 *Let B be a brick in n-dimensional Euclidean space $\mathbb{R}^n$, and let $B_1, B_2, \ldots, B_s$ be a finite collection of bricks in $\mathbb{R}^n$. Suppose that $B \subset \bigcup_{k=1}^{s} B_k$. Then $m(B) \leq \sum_{k=1}^{s} m(B_k)$.*

Proposition 7.2 *Let B be a brick in n-dimensional Euclidean space $\mathbb{R}^n$, and let $B_1, B_2, \ldots, B_s$ be a finite collection of bricks in $\mathbb{R}^n$. Suppose that the interiors of the bricks $B_1, B_2, \ldots, B_s$ are disjoint and are contained in B. Then $m(B) \geq \sum_{k=1}^{s} m(B_k)$.*

The following corollary follows immediately from the inequalities proved above.

Corollary 7.3 *Let B be a brick in n-dimensional Euclidean space $\mathbb{R}^n$, and let $B_1, B_2, \ldots, B_s$ be a finite collection of bricks in $\mathbb{R}^n$. Suppose that the interiors of the bricks $B_1, B_2, \ldots, B_s$ are disjoint and $B = \bigcup_{k=1}^{s} B_k$. Then $m(B) = \sum_{k=1}^{s} m(B_k)$.*

Lemma 7.4 *Let B be an brick in $\mathbb{R}^n$, and let ε be any positive real number. Then there exist a closed brick F and and open brick V such that $F \subset B \subset V$, $m(F) > m(B) - \varepsilon$ and $m(V) < m(B) + \varepsilon$.*

Proof Suppose that $B = I_1 \times I_2 \times \cdots \times I_n$, where $I_1, I_2, \ldots, I_n$ are bounded intervals. Now

$$\lim_{h \to 0} \prod_{i=1}^{n} (m(I_i) + h) = \prod_{i=1}^{n} m(I_i) = m(B).$$

It follows that, given any positive real number ε, we can choose the positive real number δ small enough to ensure that

$$\prod_{i=1}^{n} (m(I_i) - \delta) > m(B) - \varepsilon, \quad \prod_{i=1}^{n} (m(I_i) + \delta) < m(B) + \varepsilon.$$

Let $F = J_1 \times J_2 \times \cdots \times J_n$ and $V = K_1 \times K_2 \times \cdots \times K_n$, where $J_1, J_2, \ldots, J_n$ are closed bounded intervals chosen such that $J_i \subset I_i$ and $m(J_i) > m(I_i) - \delta$ for $i = 1, 2, \ldots, n$, and $K_1, K_2, \ldots, K_n$ are open bounded intervals chosen such that $I_i \subset K_i$ and $m(K_i) < m(I_i) + \delta$ for $i = 1, 2, \ldots, n$. Then F is a closed brick, V is an open brick, $F \subset B \subset V$, $m(F) > m(B) - \varepsilon$ and $m(V) < m(B) + \varepsilon$, as required. ∎

Any closed n-dimensional brick F is a compact subset of $\mathbb{R}^n$. This means that, given any collection $\mathcal{V}$ of open sets in $\mathbb{R}^n$ that covers F (so that each point of F belongs to at least one of the open sets in the collection), there exists some finite collection $V_1, V_2, \ldots, V_s$ of open sets belonging to the collection $\mathcal{V}$ such that

$$F \subset V_1 \cup V_2 \cup \cdots \cup V_s.$$

We shall use this property of closed bricks in order to generalize Proposition 7.1 to countable infinite unions of bricks.

Proposition 7.5 *Let A be a brick in n-dimensional Euclidean space $\mathbb{R}^n$, and let $\mathcal{C}$ be a countable collection of bricks in $\mathbb{R}^n$. Suppose that $A \subset \bigcup_{B \in \mathcal{C}} B$. Then $m(A) \leq \sum_{B \in \mathcal{C}} m(B)$.*

Proof There is nothing to prove if $\sum_{B\in\mathcal{C}} m(B) = +\infty$. We may therefore restrict our attention to the case where $\sum_{B\in\mathcal{C}} m(B) < +\infty$. Moreover the result is an immediate consequence of Proposition 7.1 if the collection $\mathcal{C}$ is finite. It therefore only remains to prove the result in the case where the collection $\mathcal{C}$ is infinite, but countable. In that case there exists an infinite sequence $B_1, B_2, B_3, \ldots$ of bricks with the property that each brick in the collection $\mathcal{C}$ occurs exactly once in the sequence.

Let some positive real number ε be given. It follows from Lemma 7.4 that there exists a closed brick F such that $F \subset A$ and $m(F) \geq m(A) - \varepsilon$. Also, for each $k \in \mathbb{N}$, there exists an open brick V_k such that $B_k \subset V_k$ and $m(V_k) < m(B_k) + 2^{-k}\varepsilon$. Then $F \subset \bigcup_{k=1}^{+\infty} V_k$, and thus $\{V_1, V_2, V_3, \ldots\}$ is a collection of open sets in $\mathbb{R}^n$ which covers the closed bounded set F. It follows from the compactness of F that there exists a finite collection $k_1, k_2, \ldots, k_s$ of positive integers such that $F \subset V_{k_1} \cup V_{k_2} \cup \cdots \cup V_{k_s}$. It then follows from Proposition 7.1 that

$$m(F) \leq m(V_{k_1}) + m(V_{k_2}) + \cdots + m(V_{k_s}).$$

Now

$$\frac{1}{2^{k_1}} + \frac{1}{2^{k_2}} + \cdots + \frac{1}{2^{k_s}} \leq \sum_{k=1}^{+\infty} \frac{1}{2^k} = 1,$$

and therefore

$$\begin{aligned} m(F) &\leq m(V_{k_1}) + m(V_{k_2}) + \cdots + m(V_{k_s}) \\ &\leq m(B_{k_1}) + m(B_{k_2}) + \cdots + m(B_{k_s}) + \varepsilon \\ &\leq \sum_{k=1}^{+\infty} m(B_k) + \varepsilon. \end{aligned}$$

Also $m(A) < m(F) + \varepsilon$. It follows that

$$m(A) \leq \sum_{k=1}^{+\infty} m(B_k) + 2\varepsilon.$$

Moreover this inequality holds no matter how small the value of the positive real number ε. It follows that

$$m(A) \leq \sum_{k=1}^{+\infty} m(B_k),$$

as required. ∎

7.2 Lebesgue Outer Measure

We say that a collection $\mathcal{C}$ of n-dimensional bricks *covers* a subset E of $\mathbb{R}^n$ if $E \subset \bigcup_{B\in\mathcal{C}} B$, (where $\bigcup_{B\in\mathcal{C}} B$ denotes the union of all the bricks belonging to the collection $\mathcal{C}$). Given any subset E of $\mathbb{R}^n$, we shall denote by $\mathbf{CCB}_n(E)$ the set of all countable collections of n-dimensional bricks that cover the set E.

Definition Let E be a subset of $\mathbb{R}^n$. We define the *Lebesgue outer measure* $\mu^*(E)$ of E to be the infimum, or greatest lower bound, of the quantities $\sum_{B\in\mathcal{C}} m(B)$, where this infimum is taken over all countable collections $\mathcal{C}$ of n-dimensional bricks that cover the set E. Thus

$$\mu^*(E) = \inf\left\{\sum_{B\in\mathcal{C}} m(B) : \mathcal{C} \in \mathbf{CCB}_n(E)\right\}.$$

The Lebesgue outer measure $\mu^*(E)$ of a subset E of $\mathbb{R}^n$ is thus the greatest extended real number l with the property that $l \leq \sum_{B\in\mathcal{C}} m(B)$ for any countable collection $\mathcal{C}$ of n-dimensional bricks that covers the set E. In particular, $\mu^*(E) = +\infty$ if and only if $\sum_{B\in\mathcal{C}} m(B) = +\infty$ for every countable collection $\mathcal{C}$ of n-dimensional bricks that covers the set E.

Note that $\mu^*(E) \geq 0$ for all subsets E of $\mathbb{R}^n$.

Lemma 7.6 *Let E be a brick in $\mathbb{R}^n$. Then $\mu^*(E) = m(E)$, where $m(E)$ is the content of the brick E.*

Proof It follows from Proposition 7.5 that $m(E) \leq \sum_{B\in\mathcal{C}} m(B)$ for any countable collection of n-dimensional bricks that covers the brick E. Therefore $m(E) \leq \mu^*(E)$. But the collection $\{E\}$ consisting of the single brick E is itself a countable collection of bricks covering E, and therefore $\mu^*(E) \leq m(E)$. It follows that $\mu^*(E) = m(E)$, as required. ∎

Lemma 7.7 *Let E and F be subsets of $\mathbb{R}^n$. Suppose that $E \subset F$. Then $\mu^*(E) \leq \mu^*(F)$.*

Proof Any countable collection of n-dimensional bricks that covers the set F will also cover the set E, and therefore $\mathbf{CCB}_n(F) \subset \mathbf{CCB}_n(E)$. It follows that

$$\begin{aligned} \mu^*(F) &= \inf\left\{\sum_{B\in\mathcal{C}} m(B) : \mathcal{C} \in \mathbf{CCB}_n(F)\right\} \\ &\geq \inf\left\{\sum_{B\in\mathcal{C}} m(B) : \mathcal{C} \in \mathbf{CCB}_n(E)\right\} = \mu^*(E), \end{aligned}$$

as required. ∎

Proposition 7.8 *Let $\mathcal{E}$ be a countable collection of subsets of $\mathbb{R}^n$. Then*

$$\mu^*\left(\bigcup_{E\in\mathcal{E}} E\right) \leq \sum_{E\in\mathcal{E}} \mu^*(E).$$

Proof Let $K = \mathbb{N}$ in the case where the countable collection $\mathcal{E}$ is infinite, and let $K = \{1, 2, \ldots, m\}$ in the case where the collection $\mathcal{E}$ is finite and has m elements. Then there exists a bijective function $\varphi\colon K \to \mathcal{E}$. We define $E_k = \varphi(k)$ for all $k \in K$. Then $\mathcal{E} = \{E_k : k \in K\}$, and any subset of $\mathbb{R}^n$ belonging to the collection $\mathcal{E}$ is of the form E_k for exactly one element k of the indexing set K.

Let some positive real number ε be given. Then corresponding to each element k of K there exists a countable collection $\mathcal{C}_k$ of n-dimensional bricks covering the set E_k for which

$$\sum_{B\in\mathcal{C}_k} m(B) < \mu^*(E_k) + \frac{\varepsilon}{2^k}.$$

Let $\mathcal{C} = \bigcup_{k\in K} \mathcal{C}_k$. Then $\mathcal{C}$ is a collection of n-dimensional bricks that covers the union $\bigcup_{E\in\mathcal{E}} E$ of all the sets in the collection $\mathcal{E}$. Moreover every brick belonging to the collection $\mathcal{C}$ belongs to at least one of the collections $\mathcal{C}_k$, and therefore belongs to exactly one of the collections $\mathcal{D}_k$, where $\mathcal{D}_k = \mathcal{C}_k \setminus \bigcup_{j<k} \mathcal{C}_j$. It follows that

$$\begin{aligned}
\mu^*\left(\bigcup_{E\in\mathcal{E}} E\right) &\leq \sum_{B\in\mathcal{C}} m(B) = \sum_{k\in K}\sum_{B\in\mathcal{D}_k} m(B) \\
&\leq \sum_{k\in K}\sum_{B\in\mathcal{C}_k} m(B) \leq \sum_{k\in K}\left(\mu^*(E_k) + \frac{\varepsilon}{2^k}\right) \\
&\leq \sum_{k\in K} \mu^*(E_k) + \varepsilon
\end{aligned}$$

Thus $\mu^*\left(\bigcup_{E\in\mathcal{E}} E\right) \leq \sum_{k\in K} \mu^*(E_k) + \varepsilon$, no matter how small the value of ε. It follows that $\mu^*\left(\bigcup_{E\in\mathcal{E}} E\right) \leq \sum_{k\in K} \mu^*(E_k)$, as required. ∎

7.3 Outer Measures

Definition Let X be a set, and let $\mathcal{P}(X)$ be the collection of all subsets of X. An *outer measure* $\lambda\colon \mathcal{P}(X) \to [0, +\infty]$ on X is a function, mapping subsets of X to non-negative extended real numbers, which has the following properties:

(i) $\lambda(\emptyset) = 0$;

(ii) $\lambda(E) \leq \lambda(F)$ for all subsets E and F of X that satisfy $E \subset F$;

(iii) $\lambda\left(\bigcup_{E\in\mathcal{E}} E\right) \leq \sum\limits_{E\in\mathcal{E}} \lambda(E)$ for any countable collection $\mathcal{E}$ of subsets of X.

Lebesgue outer measure is an outer measure on the set $\mathbb{R}^n$. (This follows directly from the definition of Lebesgue outer measure, and from Lemma 7.7 and Proposition 7.8.)

We shall prove that any outer measure on a set X determines a collection of subsets of X with particular properties. The subsets belonging to this collection are known as *measurable sets*. Any countable union or intersection of measurable sets is itself a measurable set. Also any difference of measurable sets is itself a measurable set. We shall also prove that if $\mathcal{C}$ is any countable collection of pairwise disjoint measurable sets then $\lambda\left(\bigcup_{E\in\mathcal{E}} E\right) = \sum\limits_{E\in\mathcal{E}} \lambda(E)$. These results are fundamental to the branch of mathematics known as *measure theory*. Moreover the existence of such collections of measurable sets underlies the powerful and very general theory of integration introduced into mathematics by Lebesgue.

Definition Let λ be an outer measure on a set X. A subset E of X is said to be λ-*measurable* if $\lambda(A) = \lambda(A \cap E) + \lambda(A \setminus E)$ for all subsets A of X.

The above definition of measurable sets may seem at first somewhat strange and unmotivated. Nevertheless it serves to characterize a collection of subsets of X with convenient properties, as we shall see.

The empty set $\emptyset$ and the set X are both λ-measurable subsets of X. This follows directly from the definition of measurable sets.

We say that the sets in some collection are *pairwise disjoint* if the intersection of any two distinct sets belonging to this collection is the empty set.

Lemma 7.9 *Let λ be an outer measure on a set X, let A be a subset of X, and let $E_1, E_2, \ldots, E_m$ be pairwise disjoint λ-measurable sets. Then*

$$\lambda\left(A \cap \bigcup_{k=1}^{m} E_k\right) = \sum_{k=1}^{m} \lambda(A \cap E_k).$$

Proof There is nothing to prove if $m = 1$. Suppose that $m > 1$. It follows from the definition of measurable sets that

$$\lambda\left(A \cap \bigcup_{k=1}^{m} E_k\right) = \lambda\left(\left(A \cap \bigcup_{k=1}^{m} E_k\right) \setminus E_m\right) + \lambda\left(\left(A \cap \bigcup_{k=1}^{m} E_k\right) \cap E_m\right).$$

But $\left(A \cap \bigcup_{k=1}^{m} E_k\right) \setminus E_m = A \cap \bigcup_{k=1}^{m-1} E_k$ and $\left(A \cap \bigcup_{k=1}^{m} E_k\right) \cap E_m = A \cap E_m$, because the sets $E_1, E_2, \ldots, E_m$ are pairwise disjoint. Therefore

$$\lambda\left(A \cap \bigcup_{k=1}^{m} E_k\right) = \lambda\left(A \cap \bigcup_{k=1}^{m-1} E_k\right) + \lambda(A \cap E_m).$$

The required result therefore follows by induction on m. ∎

Proposition 7.10 *Let λ be an outer measure on a set X, and let E and F be λ-measurable subsets of X. Then the complement $X \setminus E$ of E, and the union $E \cup F$, intersection $E \cap F$ and difference $E \setminus F$ of E and F are λ-measurable.*

Proof Let $E^c = X \setminus E$, $F^c = X \setminus F$ and $(E \cup F)^c = X \setminus (E \cup F)$. Then $A \cap E^c = A \setminus E$ and $A \setminus E^c = A \cap E$, and therefore

$$\lambda(A) = \lambda(A \setminus E) + \lambda(A \cap E) = \lambda(A \cap E^c) + \lambda(A \setminus E^c).$$

We conclude that the complement $X \setminus E$ of the λ-measurable subset E of X is itself a λ-measurable subset of X.

Next we show that $E \cup F$ is λ-measurable. Now

$$\lambda(A) = \lambda(A \cap E) + \lambda(A \setminus E) = \lambda(A \cap E) + \lambda(A \cap E^c).$$

for all subsets A of X. Also

$$\lambda(B) = \lambda(B \cap F) + \lambda(B \setminus F) = \lambda(B \cap F) + \lambda(B \cap F^c).$$

for all subsets B of X. Therefore

$$\begin{aligned} \lambda(A \cap E) &= \lambda(A \cap E \cap F) + \lambda((A \cap E \cap F^c), \\ \lambda(A \cap E^c) &= \lambda(A \cap E^c \cap F) + \lambda((A \cap E^c \cap F^c), \end{aligned}$$

and thus

$$\begin{aligned} \lambda(A) &= \lambda(A \cap E) + \lambda(A \cap E^c) \\ &= \lambda(A \cap E \cap F) + \lambda(A \cap E \cap F^c) + \lambda(A \cap E^c \cap F) \\ &\quad + \lambda(A \cap E^c \cap F^c) \end{aligned}$$

for all subsets A of X. Let A be a subset of X, and let $B = A \cap (E \cup F)$. Then

$$A \cap E \cap F \subset B, \quad A \cap E \cap F^c \subset B, \quad A \cap E^c \cap F \subset B,$$

$$A \cap E^c \cap F^c \subset X \setminus B,$$

and therefore

$$B \cap E \cap F = A \cap E \cap F, \quad B \cap E \cap F^c = A \cap E \cap F^c,$$

$$B \cap E^c \cap F = A \cap E^c \cap F, \quad B \cap E^c \cap F^c = \emptyset.$$

It follows that

$$\begin{aligned} \lambda(A \cap (E \cup F)) &= \lambda(B) \\ &= \lambda(B \cap E \cap F) + \lambda(B \cap E \cap F^c) + \lambda(B \cap E^c \cap F) \\ &\quad + \lambda(B \cap E^c \cap F^c) \\ &= \lambda(A \cap E \cap F) + \lambda(A \cap E \cap F^c) + \lambda(A \cap E^c \cap F). \end{aligned}$$

Also $A \cap E^c \cap F^c = A \cap (E \cup F)^c$. We conclude therefore that

$$\begin{aligned} \lambda(A) &= \lambda(A \cap E \cap F) + \lambda(A \cap E \cap F^c) + \lambda(A \cap E^c \cap F) \\ &\quad + \lambda(A \cap E^c \cap F^c) \\ &= \lambda(A \cap (E \cup F)) + \lambda(A \cap (E \cup F)^c) \end{aligned}$$

for all subsets A of X. This shows that if E and F are λ-measurable subsets of X, then so is $E \cup F$.

Let E and F be λ-measurable subsets of X. Then $X \setminus E$ and $X \setminus F$ are λ-measurable sets, and therefore $(X \setminus E) \cup (X \setminus F)$ is a λ-measurable set. But $(X \setminus E) \cup (X \setminus F) = X \setminus (E \cap F)$. Thus the complement $X \setminus (E \cap F)$ of $E \cap F$ is a λ-measurable set, and therefore $E \cap F$ is itself a λ-measurable set. Thus the intersection of any two λ-measurable subsets of X is a λ-measurable set. It follows from this that the intersection of any finite collection of λ-measurable subsets of X is itself λ-measurable.

Let E and F be λ-measurable subsets of X. Then $E \setminus F = E \cap (X \setminus F)$, and E and $X \setminus F$ are both λ-measurable sets. It follows that the difference $E \setminus F$ of any two λ-measurable subsets E and F of X is itself λ-measurable. This completes the proof. ∎

It follows from the above proposition that any finite union or intersection of measurable sets is measurable.

Proposition 7.11 *Let λ be an outer measure on a set X. Then the union of any countable collection of λ-measurable subsets of X is λ-measurable.*

Proof The union of any two λ-measurable sets is λ-measurable (Proposition 7.10). It follows from this that the union of any finite collection of λ-measurable sets is λ-measurable.

Let $E_1, E_2, E_3, \ldots$ be an infinite sequence of pairwise disjoint λ-measurable subsets of X. We shall prove that the union of these sets is λ-measurable. Let A be a subset of X. Now $\bigcup_{k=1}^{m} E_k$ is a λ-measurable set for each positive integer m, because any finite union of λ-measurable sets is λ-measurable, and therefore

$$\lambda(A) = \lambda\left(A \cap \bigcup_{k=1}^{m} E_k\right) + \lambda\left(A \setminus \bigcup_{k=1}^{m} E_k\right)$$

for all positive integers m. Moreover it follows from Lemma 7.9 that

$$\lambda\left(A \cap \bigcup_{k=1}^{m} E_k\right) = \sum_{k=1}^{m} \lambda(A \cap E_k).$$

Also

$$A \setminus \bigcup_{k=1}^{+\infty} E_k \subset A \setminus \bigcup_{k=1}^{m} E_k,$$

and therefore

$$\lambda\left(A \setminus \bigcup_{k=1}^{m} E_k\right) \geq \lambda\left(A \setminus \bigcup_{k=1}^{+\infty} E_k\right).$$

It follows that

$$\lambda(A) \geq \sum_{k=1}^{m} \lambda(A \cap E_k) + \lambda\left(A \setminus \bigcup_{k=1}^{+\infty} E_k\right),$$

and therefore

$$\begin{aligned} \lambda(A) &\geq \lim_{m\to+\infty} \sum_{k=1}^{m} \lambda(A \cap E_k) + \lambda\left(A \setminus \bigcup_{k=1}^{+\infty} E_k\right) \\ &= \sum_{k=1}^{+\infty} \lambda(A \cap E_k) + \lambda\left(A \setminus \bigcup_{k=1}^{+\infty} E_k\right). \end{aligned}$$

However it follows from the definition of outer measures that

$$\lambda\left(A \cap \bigcup_{k=1}^{+\infty} E_k\right) = \lambda\left(\bigcup_{k=1}^{+\infty} (A \cap E_k)\right) \leq \sum_{k=1}^{+\infty} \lambda(A \cap E_k).$$

Therefore

$$\lambda(A) \geq \lambda\left(A \cap \bigcup_{k=1}^{+\infty} E_k\right) + \lambda\left(A \setminus \bigcup_{k=1}^{+\infty} E_k\right).$$

But the set A is the union of the sets $A \cap \bigcup_{k=1}^{+\infty} E_k$ and $A \setminus \bigcup_{k=1}^{+\infty} E_k$, and therefore

$$\lambda(A) \leq \lambda\left(A \cap \bigcup_{k=1}^{+\infty} E_k\right) + \lambda\left(A \setminus \bigcup_{k=1}^{+\infty} E_k\right).$$

We conclude therefore that

$$\lambda(A) = \lambda\left(A \cap \bigcup_{k=1}^{+\infty} E_k\right) + \lambda\left(A \setminus \bigcup_{k=1}^{+\infty} E_k\right)$$

for all subsets A of X. We conclude from this that the union of any pairwise disjoint sequence of λ-measurable subsets of X. is itself λ-measurable.

Now let $E_1, E_2, E_3, \ldots$ be a countable sequence of (not necessarily pairwise disjoint) λ-measurable sets. Then $\bigcup_{k=1}^{+\infty} E_k = \bigcup_{k=1}^{+\infty} F_k$, where $F_1 = E_1$, and $F_k = E_k \setminus \bigcup_{j=1}^{k-1} E_j$ for all integers k satisfying $k > 1$. Now we have proved that any finite union of λ-measurable sets is λ-measurable, and any difference of λ-measurable sets is λ-measurable. It follows that the sets $F_1, F_2, F_3, \ldots$ are all λ-measurable. These sets are also pairwise disjoint. We conclude that the union of the sets $F_1, F_2, F_3, \ldots$ is λ-measurable, and therefore the union of the sets $E_1, E_2, E_3, \ldots$ is λ-measurable.

We have now shown that the union of any finite collection of λ-measurable sets is λ-measurable, and the union of any infinite sequence of λ-measurable sets is λ-measurable. We conclude that the union of any countable collection of λ-measurable sets is λ-measurable, as required. ∎

Corollary 7.12 *Let λ be an outer measure on a set X. Then the intersection of any countable collection of λ-measurable subsets of X is λ-measurable.*

Proof Let $\mathcal{C}$ be a countable collection of λ-measurable subsets of X. Then $X \setminus \bigcap_{E\in\mathcal{C}} E = \bigcup_{E\in\mathcal{C}} (X \setminus E)$ (i.e., the complement of the intersection of the sets in the collection is the union of the complements of those sets.) Now $X \setminus E$ is λ-measurable for every $E \in \mathcal{C}$. Therefore the complement $X \setminus \bigcap_{E\in\mathcal{C}} E$ of $\bigcap_{E\in\mathcal{C}} E$ is a union of λ-measurable sets, and is thus itself λ-measurable. It follows that intersection $\bigcap_{E\in\mathcal{C}} E$ of the sets in the collection is λ-measurable, as required. ∎

Proposition 7.13 *Let λ be an outer measure on a set X, let A be a subset of X, and let $\mathcal{C}$ be a countable collection of pairwise disjoint λ-measurable sets. Then*

$$\lambda\left(A \cap \bigcup_{E \in \mathcal{C}} E\right) = \sum_{E \in \mathcal{C}} \lambda(A \cap E).$$

Proof It follows from Lemma 7.9 that the required identity holds for any finite collection of pairwise disjoint λ-measurable sets.

Let $E_1, E_2, E_3, \ldots$ be an infinite sequence of pairwise disjoint λ-measurable subsets of X. Then

$$\sum_{k=1}^{m} \lambda(A \cap E_k) = \lambda\left(A \cap \bigcup_{k=1}^{m} E_k\right) \leq \lambda\left(A \cap \bigcup_{k=1}^{+\infty} E_k\right)$$

for all positive integers m. It follows that

$$\sum_{k=1}^{+\infty} \lambda(A \cap E_k) = \lim_{m \to +\infty} \sum_{k=1}^{m} \lambda(A \cap E_k) \leq \lambda\left(A \cap \bigcup_{k=1}^{+\infty} E_k\right).$$

But the definition of outer measures ensures that

$$\lambda\left(A \cap \bigcup_{k=1}^{+\infty} E_k\right) = \lambda\left(\bigcup_{k=1}^{+\infty} (A \cap E_k)\right) \leq \sum_{k=1}^{+\infty} \lambda(A \cap E_k)$$

We conclude therefore that $\lambda\left(A \cap \bigcup\limits_{k=1}^{+\infty} E_k\right) = \sum\limits_{k=1}^{+\infty} \lambda(A \cap E_k)$ for any infinite sequence $E_1, E_2, E_3, \ldots$ of pairwise disjoint λ-measurable subsets of X.

Thus the required identity holds for any countable collection of pairwise disjoint λ-measurable subsets of X, as required. ∎

7.4 Measure Spaces

Definition Let X be a set. A collection $\mathcal{A}$ of subsets of X is said to a σ-*algebra* (or *sigma-algebra*) of subsets of X if it has the following properties:

(i) the empty set $\emptyset$ is a member of $\mathcal{A}$;

(ii) the complement $X \setminus E$ of any member E of $\mathcal{A}$ is itself a member of $\mathcal{A}$;

(iii) the union of any countable collection of members of $\mathcal{A}$ is itself a member of $\mathcal{A}$.

Lemma 7.14 *Let X be a set, and let $\mathcal{A}$ be a σ-algebra of subsets of X. Then the intersection of any countable collection of members of the σ-algebra $\mathcal{A}$ is itself a member of $\mathcal{A}$.*

Proof Let $\mathcal{C}$ be a countable collection of sets belonging to $\mathcal{A}$. Then $X \setminus E \in \mathcal{A}$ for all $E \in \mathcal{C}$, and therefore $\bigcup_{E \in \mathcal{C}} (X \setminus E) \in \mathcal{A}$. But $\bigcup_{E \in \mathcal{C}} (X \setminus E) = X \setminus \bigcap_{E \in \mathcal{C}} E$. It follows that the complement of the intersection $\bigcap_{E \in \mathcal{C}} E$ of the sets in the collection $\mathcal{C}$ is itself a member of $\mathcal{A}$, and therefore the intersection $\bigcap_{E \in \mathcal{C}} E$ of those sets is a member of the σ-algebra $\mathcal{A}$, as required. ∎

Let X be a set, and let $\mathcal{C}$ be a collection of subsets of X. The collection of all subsets of X is a σ-algebra. Also the intersection of any collection of σ-algebras of subsets of X is itself a σ-algebra. Let $\mathcal{A}$ be the intersection of all σ-algebras $\mathcal{B}$ of subsets of X that have the property that $\mathcal{C} \subset \mathcal{B}$. Then $\mathcal{A}$ is a σ-algebra, and $\mathcal{C} \subset \mathcal{A}$. Moreover if $\mathcal{B}$ is a σ-algebra of subsets of X, and if $\mathcal{C} \subset \mathcal{B}$ then $\mathcal{A} \subset \mathcal{B}$. The σ-algebra $\mathcal{A}$ may therefore be regarded as the smallest σ-algebra of subsets of X for which $\mathcal{C} \subset \mathcal{A}$. We shall refer to this σ-algebra $\mathcal{A}$ as the σ-algebra of subsets of X *generated* by $\mathcal{C}$. We see therefore that any collection of subsets of a set X generates a σ-algebra of subsets of X which is the smallest σ-algebra of subsets of X that contains the given collection of subsets.

Definition Let X be a set, and let $\mathcal{A}$ be a σ-algebra of subsets of X. A *measure* on $\mathcal{A}$ is a function $\mu: \mathcal{A} \to [0, +\infty]$, taking values in the set $[0, +\infty]$ of non-negative extended real numbers, which has the property that

$$\mu\left(\bigcup_{E \in \mathcal{C}} E\right) = \sum_{E \in \mathcal{C}} \mu(E)$$

for any countable collection $\mathcal{C}$ of pairwise disjoint members of the σ-algebra $\mathcal{A}$.

Definition A *measure space* $(X, \mathcal{A}, \mu)$ consists of a set X, a σ-algebra $\mathcal{A}$ of subsets of X, and a measure $\mu: \mathcal{A} \to [0, +\infty]$ defined on this σ-algebra $\mathcal{A}$. A subset E of a measure space $(X, \mathcal{A}, \mu)$ is said to be *measurable* (or *μ-measurable*) if it belongs to the σ-algebra $\mathcal{A}$.

Theorem 7.15 *Let λ be an outer measure on a set X. Then the collection $\mathcal{A}_\lambda$ of all λ-measurable subsets of X is a σ-algebra. The members of this σ-algebra are those subsets E of X with the property that $\lambda(A) = \lambda(A \cap E) + \lambda(A \setminus E)$ for any subset of A. Moreover the restriction of the outer measure λ to the λ-measurable sets defines a measure μ on the σ-algebra $\mathcal{A}_\lambda$. Thus $(X, \mathcal{A}, \mu)$ is a measure space.*

Proof Immediate from Propositions 7.10, 7.11 and 7.13. ∎

Definition A *measure space* $(X, \mathcal{A}, \mu)$ is said to be *complete* if, given any measurable subset E of X satisfying $\mu(E) = 0$, and given any subset F of E, the subset F is also measurable. The measure μ on $\mathcal{A}$ is then said to be *complete.*

Lemma 7.16 *Let λ be an outer measure on a set X, let $\mathcal{A}$ be the σ-algebra consisting of the λ-measurable subsets of X, and let μ be the measure on $\mathcal{A}$ obtained by restricting the outer measure λ to the members of $\mathcal{A}$. Then $(X, \mathcal{A}, \mu)$ is a complete measure space.*

Proof Let E be a measurable set in X satisfying $\mu(E) = 0$, let F be a subset of E, and let A be a subset of X. Then $A \cap F \subset A \cap E$ and $A \setminus E \subset A \setminus F \subset A$, and therefore $0 \leq \lambda(A \cap F) \leq \lambda(A \cap E)$ and $\lambda(A \setminus E) \leq \lambda(A \setminus F) \leq \lambda(A)$. Now it follows from the definition of measurable sets in X that $\lambda(A) = \lambda(A \cap E) + \lambda(A \setminus E)$. Moreover $0 \leq \lambda(A \cap E) \leq \lambda(E) = \mu(E) = 0$. It follows that $\lambda(A \cap E) = 0$ and $\lambda(A \setminus E) = \lambda(A)$. The inequalities above then ensure that $\lambda(A \cap F) = 0$ and $\lambda(A \setminus F) = \lambda(A)$. But then $\lambda(A) = \lambda(A \cap F) + \lambda(A \setminus F)$, and thus F is λ-measurable, as required. ∎

7.5 Lebesgue Measure on Euclidean Spaces

We are now in a position to give the definition of *Lebesgue measure* on n-dimensional Euclidean space $\mathbb{R}^n$. We have already defined an outer measure μ^* on $\mathbb{R}^n$ known as *Lebesgue outer measure.* We defined a *brick* in $\mathbb{R}^n$ to be a subset of $\mathbb{R}^n$ that is a Cartesian product of n bounded intervals. The product of the lengths of those intervals is the *content* of the brick. Then, given any subset E of $\mathbb{R}^n$, we defined the *Lebesgue outer measure* $\mu^*(E)$ of the set E to be the infimum of the quantities $\sum_{B \in \mathcal{C}} m(B)$, where the infimum is taken over all countable collections of bricks in $\mathbb{R}^n$ that cover the set E, and where $m(B)$ denotes the content of a brick B in such a collection. Thus

$$\sum_{B \in \mathcal{C}} m(B) \geq \mu^*(E)$$

for every countable collection $\mathcal{C}$ of bricks in $\mathbb{R}^n$ that covers E; and, moreover, given any positive real number ε, there exists a countable collection $\mathcal{C}$ of bricks in $\mathbb{R}^n$ covering E for which

$$\mu^*(E) \leq \sum_{B \in \mathcal{C}} m(B) \leq \mu^*(E) + \varepsilon.$$

These properties characterize the Lebesgue outer measure $\mu^*(E)$ of the set E.

We say that a subset E of $\mathbb{R}^n$ is *Lebesgue-measurable* if and only if it is μ^*-measurable, where μ^* denotes Lebesgue outer measure on $\mathbb{R}^n$. Thus a subset E of $\mathbb{R}^n$ is Lebesgue-measurable if and only if $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E)$ for all subsets A of $\mathbb{R}^n$. The collection $\mathcal{L}_n$ of all Lebesgue-measurable sets is a σ-algebra of subsets of $\mathbb{R}^n$, and therefore the difference of any two Lebesgue-measurable subsets of $\mathbb{R}^n$ is Lebesgue-measurable, and any countable union or intersection of Lebesgue-measurable sets is Lebesgue-measurable. The *Lebesgue measure* $\mu(E)$ of a Lebesgue-measurable subset E of $\mathbb{R}^n$ is defined to be the Lebesgue outer measure $\mu^*(E)$ of that set. Thus Lebesgue measure μ is the restriction of Lebesgue outer measure μ^* to the σ-algebra $\mathcal{L}_n$ of Lebesgue-measurable subsets of $\mathbb{R}^n$.

It follows from Lemma 7.16 that Lebesgue measure is a complete measure on $\mathbb{R}^n$.

Remark The Lebesgue measure $\mu(E)$ of a subset E of $\mathbb{R}^2$ may be regarded as the area of that set. It is not possible to assign an area to every subset of $\mathbb{R}^2$ in such a way that the areas assigned to such subsets have all the properties that one would expect from a well-defined notion of area. One might at first sight expect that Lebesgue outer measure would provide a natural definition of area, applicable to all subsets of the plane, that would have the properties that one would expect of a well-defined notion of area. One would expect in particular that the area of a disjoint union of two subsets of the plane would be the sum of the areas of those sets. However one it is possible to construct examples of disjoint subsets E and F in the plane which interpenetrate one another to such an extent as to ensure that $\mu^*(E \cap F) < \mu^*(E) + \mu^*(F)$, where μ^* denotes Lebesgue outer measure on $\mathbb{R}^2$. The σ-algebra $\mathcal{L}_2$ consisting of the Lebesgue-measurable subsets of the plane $\mathbb{R}^2$ is in fact that largest collection of subsets of the plane for which the sets in the collection have a well-defined area; the Lebesgue measure of a Lebesgue-measurable subset of the plane can be regarded as the area of that set. Similarly the σ-algebra $\mathcal{L}_3$ of Lebesgue-measurable subsets of three-dimensional Euclidean space $\mathbb{R}^3$ is the largest collection of subsets of $\mathbb{R}^3$ for which the sets in the collection have a well-defined volume.

Proposition 7.17 *Every open set in $\mathbb{R}^n$ is Lebesgue-measurable.*

Proof Let $\mathcal{W}$ be the collection of all open bricks in $\mathbb{R}^n$ that are Cartesian products of intervals whose endpoints are rational numbers. Now the set $\mathcal{I}$ of all open intervals in $\mathbb{R}^n$ whose endpoints are rational numbers is a countable set, as the function that sends such an interval to its endpoints defines an injective function from $\mathcal{I}$ to the countable set $\mathbb{Q} \times \mathbb{Q}$. Moreover there is a bijection from the countable set $\mathcal{I}^n$ to $\mathcal{W}$ that sends each ordered n-tuple $(I_1, I_2, \ldots, I_n)$ of open intervals to the open brick $I_1 \times I_2 \times \cdots \times I_n$. It follows that the collection $\mathcal{W}$ is countable.

Let V be an open set in $\mathbb{R}^n$, and let $\mathbf{v}$ be a point of V. Then there exists some positive real number δ such that $B(\mathbf{v}, \delta) \subset V$, where $B(\mathbf{v}, \delta) \subset V$ denotes the open ball of radius δ centred on $\mathbf{v}$. Moreover there exist open bricks W belonging to $\mathcal{W}$ for which $\mathbf{v} \in W$ and $W \subset B(\mathbf{v}, \delta)$. It follows that the open set V is the union of the countable collection

$$\{W \in \mathcal{W} : W \subset V\}$$

of open bricks. Now each open brick is a Lebesgue-measurable set, and any countable union of Lebesgue-measurable sets is itself a Lebesgue-measurable set. Therefore the open set V is a Lebesgue-measurable set, as required. ∎

Corollary 7.18 *Every closed set in $\mathbb{R}^n$ is Lebesgue-measurable.*

Proof This follows immediately from Proposition 7.17, since the complement of any Lebesgue-measurable set is itself Lebesgue measurable set. ∎

Definition A subset of $\mathbb{R}^n$ is said to be a *Borel set* if it belongs to the σ-algebra generated by the collection of open sets in $\mathbb{R}^n$.

All open sets and closed sets in $\mathbb{R}^n$ are Borel sets. The collection of all Borel sets is a σ-algebra in $\mathbb{R}^n$ and is the smallest such σ-algebra containing all open subsets of $\mathbb{R}^n$.

Definition A measure defined on a σ-algebra $\mathcal{A}$ of subsets of $\mathbb{R}^n$ is said to be a *Borel measure* if the σ-algebra $\mathcal{A}$ contains all the open sets in $\mathbb{R}^n$.

Corollary 7.19 *Lebesgue measure on $\mathbb{R}^n$ is a Borel measure, and thus every Borel set in $\mathbb{R}^n$ is Lebesgue-measurable.*

Remark The definitions of Borel sets and Borel measures generalize in the obvious fashion to arbitrary topological spaces. The collection of Borel sets in a topological space X is the σ-algebra generated by the open subsets of X. A measure defined on a σ-ring of subsets of X is said to be a Borel measure if every Borel set is measurable.

7.6 Basic Properties of Measures

Let $(X, \mathcal{A}, \mu)$ be a measure space. Then the measure μ is defined on the σ-algebra $\mathcal{A}$ of measurable subsets of X, and takes values in the set $[0, +\infty]$, where $[0, +\infty] = [0, +\infty) \cup \{+\infty\}$. Thus $\mu(E)$ is defined for each measurable subset E of X, and is either a non-negative real number, or else has the value $+\infty$. The measure μ is by definition *countably additive*, so that

$$\mu\left(\bigcup_{E \in \mathcal{C}} E\right) = \sum_{E \in \mathcal{C}} \mu(E)$$

for every countable collection $\mathcal{C}$ of pairwise disjoint measurable subsets of X. In particular μ is *finitely additive*, so that if $E_1, E_2, \ldots, E_r$ are measurable subsets of X that are pairwise disjoint, then

$$\mu(E_1 \cup E_2 \cup \cdots \cup E_r) = \mu(E_1) + \mu(E_2) + \cdots + \mu(E_r).$$

Also

$$\mu\left(\bigcup_{j=1}^{+\infty} E_j\right) = \sum_{j=1}^{+\infty} \mu(E_j)$$

for any infinite sequence $E_1, E_2, E_3, \ldots$ of pairwise disjoint measurable subsets of X.

Let E and F be measurable subsets of X. Then $E = (E \cap F) \cup (E \setminus F)$, and the sets $E \cap F$ and $E \setminus F$ are measurable and disjoint. It therefore follows from the finite additivity of the measure μ that $\mu(E) = \mu(E \cap F) + \mu(E \setminus F)$. Also $E \cup F$ is the disjoint union of E and $F \setminus E$. and therefore

$$\mu(E \cup F) = \mu(E) + \mu(F \setminus E) = \mu(E \cap F) + \mu(E \setminus F) + \mu(F \setminus E).$$

It follows that

$$\begin{aligned} \mu(E \cup F) + \mu(E \cap F) &= (\mu(E \cap F) + \mu(E \setminus F)) + (\mu(E \cap F) + \mu(F \setminus E)) \\ &= \mu(E) + \mu(F). \end{aligned}$$

Now let E and F be measurable subsets of X that satisfy $F \subset E$. Then $\mu(E) = \mu(F) + \mu(E \setminus F)$, and $\mu(E \setminus F) \geq 0$. It follows that $\mu(F) \leq \mu(E)$. Moreover $\mu(E \setminus F) = \mu(E) - \mu(F)$, provided that $\mu(E) < +\infty$.

Lemma 7.20 *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $E_1, E_2, E_3, \ldots$ be an infinite sequence of measurable subsets of X. Suppose that $E_j \subset E_{j+1}$ for all positive integers j. Then*

$$\mu\left(\bigcup_{j=1}^{+\infty} E_j\right) = \lim_{j \to +\infty} \mu(E_j).$$

Proof Let $E = \bigcup_{j=1}^{+\infty} E_j$, let $F_1 = E_1$, and let $F_j = E_j \setminus \bigcup_{k=1}^{j-1} E_k$ for all integers j satisfying $j > 1$. Then the sets $F_1, F_2, F_3, \ldots$ are pairwise disjoint, the set E_j is the disjoint union of the sets F_k for which $1 \leq k \leq j$, and the set E is the disjoint union of all of the sets F_k. It therefore follows from the countable (and finite) additivity of the measure μ that

$$\mu(E) = \sum_{k=1}^{+\infty} \mu(F_k), \quad \mu(E_j) = \sum_{k=1}^{j} \mu(F_k).$$

But then

$$\mu(E) = \sum_{k=1}^{+\infty} \mu(F_k) = \lim_{j \to +\infty} \sum_{k=1}^{j} \mu(F_k) = \lim_{j \to +\infty} \mu(E_j),$$

as required. ∎

Lemma 7.21 *Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $E_1, E_2, E_3, \ldots$ be an infinite sequence of measurable subsets of X. Suppose that $E_{j+1} \subset E_j$ for all positive integers j, and that $\mu(E_1) < +\infty$. Then*

$$\mu\left(\bigcap_{j=1}^{+\infty} E_j\right) = \lim_{j \to +\infty} \mu(E_j).$$

Proof Let $G_j = E_1 \setminus E_j$ for all positive integers j, let $E = \bigcap_{j=1}^{+\infty} E_j$, and let let $G = \bigcup_{j=1}^{+\infty} G_j$. It then follows from Lemma 7.20 that $\mu(G) = \lim_{j \to +\infty} \mu(G_j)$. Now $E_j = E_1 \setminus G_j$ for all positive integers j, and $\mu(E_1) < \infty$. It follows that $\mu(E_j) = \mu(E_1) - \mu(G_j)$ for all positive integers j. Also $E = E_1 \setminus G$. Therefore

$$\mu(E) = \mu(E_1) - \mu(G) = \mu(E_1) - \lim_{j \to +\infty} \mu(G_j) = \lim_{j \to +\infty} \mu(E_j),$$

as required. ∎