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Section 5: Compact Spaces

David R. Wilkins

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5 Compact Spaces

5.1 Compact Topological Spaces

Let X be a topological space, and let A be a subset of X. A collection of subsets of X in X is said to *cover* A if and only if every point of A belongs to at least one of these subsets. In particular, an *open cover* of X is collection of open sets in X that covers X.

If $\mathcal{U}$ and $\mathcal{V}$ are open covers of some topological space X then $\mathcal{V}$ is said to be a *subcover* of $\mathcal{U}$ if and only if every open set belonging to $\mathcal{V}$ also belongs to $\mathcal{U}$.

Definition A topological space X is said to be *compact* if and only if every open cover of X possesses a finite subcover.

Lemma 5.1 *Let X be a topological space. A subset A of X is compact (with respect to the subspace topology on A) if and only if, given any collection $\mathcal{U}$ of open sets in X covering A, there exists a finite collection $V_1, V_2, \ldots, V_r$ of open sets belonging to $\mathcal{U}$ such that $A \subset V_1 \cup V_2 \cup \cdots \cup V_r$.*

Proof A subset B of A is open in A (with respect to the subspace topology on A) if and only if $B = A \cap V$ for some open set V in X. The desired result therefore follows directly from the definition of compactness. ∎

We now show that any closed bounded interval in the real line is compact. This result is known as the *Heine-Borel Theorem.* The proof of this theorem uses the *least upper bound principle* which states that, given any non-empty set S of real numbers which is bounded above, there exists a *least upper bound* (or *supremum*) $\sup S$ for the set S.

Theorem 5.2 (Heine-Borel) *Let a and b be real numbers satisfying $a < b$. Then the closed bounded interval $[a, b]$ is a compact subset of $\mathbb{R}$.*

Proof Let $\mathcal{U}$ be a collection of open sets in $\mathbb{R}$ with the property that each point of the interval $[a, b]$ belongs to at least one of these open sets. We must show that $[a, b]$ is covered by finitely many of these open sets.

Let S be the set of all $\tau \in [a, b]$ with the property that $[a, \tau]$ is covered by some finite collection of open sets belonging to $\mathcal{U}$, and let $s = \sup S$. Now $s \in W$ for some open set W belonging to $\mathcal{U}$. Moreover W is open in $\mathbb{R}$, and therefore there exists some $\delta > 0$ such that $(s - \delta, s + \delta) \subset W$. Moreover $s - \delta$ is not an upper bound for the set S, hence there exists some $\tau \in S$ satisfying $\tau > s - \delta$. It follows from the definition of S that $[a, \tau]$ is covered by some finite collection $V_1, V_2, \ldots, V_r$ of open sets belonging to $\mathcal{U}$.

Let $t \in [a, b]$ satisfy $\tau \leq t < s + \delta$. Then

$$[a, t] \subset [a, \tau] \cup (s - \delta, s + \delta) \subset V_1 \cup V_2 \cup \cdots \cup V_r \cup W,$$

and thus $t \in S$. In particular $s \in S$, and moreover $s = b$, since otherwise s would not be an upper bound of the set S. Thus $b \in S$, and therefore $[a, b]$ is covered by a finite collection of open sets belonging to $\mathcal{U}$, as required. ∎

Lemma 5.3 *Let A be a closed subset of some compact topological space X. Then A is compact.*

Proof Let $\mathcal{U}$ be any collection of open sets in X covering A. On adjoining the open set $X \setminus A$ to $\mathcal{U}$, we obtain an open cover of X. This open cover of X possesses a finite subcover, since X is compact. Moreover A is covered by the open sets in the collection $\mathcal{U}$ that belong to this finite subcover. It follows from Lemma 5.1 that A is compact, as required. ∎

Lemma 5.4 *Let $f: X \to Y$ be a continuous function between topological spaces X and Y, and let A be a compact subset of X. Then $f(A)$ is a compact subset of Y.*

Proof Let $\mathcal{V}$ be a collection of open sets in Y which covers $f(A)$. Then A is covered by the collection of all open sets of the form $f^{-1}(V)$ for some $V \in \mathcal{V}$. It follows from the compactness of A that there exists a finite collection $V_1, V_2, \ldots, V_k$ of open sets belonging to $\mathcal{V}$ such that

$$A \subset f^{-1}(V_1) \cup f^{-1}(V_2) \cup \cdots \cup f^{-1}(V_k).$$

But then $f(A) \subset V_1 \cup V_2 \cup \cdots \cup V_k$. This shows that $f(A)$ is compact. ∎

Lemma 5.5 *Let $f: X \to \mathbb{R}$ be a continuous real-valued function on a compact topological space X. Then f is bounded above and below on X.*

Proof The range $f(X)$ of the function f is covered by some finite collection $I_1, I_2, \ldots, I_k$ of open intervals of the form $(-m, m)$, where $m \in \mathbb{N}$, since $f(X)$ is compact (Lemma 5.4) and $\mathbb{R}$ is covered by the collection of all intervals of this form. It follows that $f(X) \subset (-M, M)$, where $(-M, M)$ is the largest of the intervals $I_1, I_2, \ldots, I_k$. Thus the function f is bounded above and below on X, as required. ∎

Proposition 5.6 *Let $f: X \to \mathbb{R}$ be a continuous real-valued function on a compact topological space X. Then there exist points u and v of X such that $f(u) \leq f(x) \leq f(v)$ for all $x \in X$.*

Proof Let $m = \inf\{f(x) : x \in X\}$ and $M = \sup\{f(x) : x \in X\}$. There must exist $v \in X$ satisfying $f(v) = M$, for if $f(x) < M$ for all $x \in X$ then the function $x \mapsto 1/(M - f(x))$ would be a continuous real-valued function on X that was not bounded above, contradicting Lemma 5.5. Similarly there must exist $u \in X$ satisfying $f(u) = m$, since otherwise the function $x \mapsto 1/(f(x)-m)$ would be a continuous function on X that was not bounded above, again contradicting Lemma 5.5. But then $f(u) \leq f(x) \leq f(v)$ for all $x \in X$, as required. ∎

Proposition 5.7 *Let A be a compact subset of a metric space X. Then A is closed in X.*

Proof Let p be a point of X that does not belong to A, and let $f(x) = d(x,p)$, where d is the distance function on X. It follows from Proposition 5.6 that there is a point q of A such that $f(a) \geq f(q)$ for all $a \in A$, since A is compact. Now $f(q) > 0$, since $q \neq p$. Let δ satisfy $0 < \delta \leq f(q)$. Then the open ball of radius δ about the point p is contained in the complement of A, since $f(x) < f(q)$ for all points x of this open ball. It follows that the complement of A is an open set in X, and thus A itself is closed in X. ∎

Proposition 5.8 *Let X be a Hausdorff topological space, and let K be a compact subset of X. Let x be a point of $X \setminus K$. Then there exist open sets V and W in X such that $x \in V$, $K \subset W$ and $V \cap W = \emptyset$.*

Proof For each point $y \in K$ there exist open sets $V_{x,y}$ and $W_{x,y}$ such that $x \in V_{x,y}$, $y \in W_{x,y}$ and $V_{x,y} \cap W_{x,y} = \emptyset$ (since X is a Hausdorff space). But then there exists a finite set $\{y_1, y_2, \ldots, y_r\}$ of points of K such that K is contained in $W_{x,y_1} \cup W_{x,y_2} \cup \cdots \cup W_{x,y_r}$, since K is compact. Define

$$V = V_{x,y_1} \cap V_{x,y_2} \cap \cdots \cap V_{x,y_r}, \qquad W = W_{x,y_1} \cup W_{x,y_2} \cup \cdots \cup W_{x,y_r}.$$

Then V and W are open sets, $x \in V$, $K \subset W$ and $V \cap W = \emptyset$, as required. ∎

Corollary 5.9 *A compact subset of a Hausdorff topological space is closed.*

Proof Let K be a compact subset of a Hausdorff topological space X. It follows immediately from Proposition 5.8 that, for each $x \in X \setminus K$, there exists an open set V_x such that $x \in V_x$ and $V_x \cap K = \emptyset$. But then $X \setminus K$ is equal to the union of the open sets V_x as x ranges over all points of $X \setminus K$, and any set that is a union of open sets is itself an open set. We conclude that $X \setminus K$ is open, and thus K is closed. ∎

Proposition 5.10 *Let X be a Hausdorff topological space, and let K_1 and K_2 be compact subsets of X, where $K_1 \cap K_2 = \emptyset$. Then there exist open sets U_1 and U_2 such that $K_1 \subset U_1$, $K_2 \subset U_2$ and $U_1 \cap U_2 = \emptyset$.*

Proof It follows from Proposition 5.8 that, for each point x of K_1, there exist open sets V_x and W_x such that $x \in V_x$, $K_2 \subset W_x$ and $V_x \cap W_x = \emptyset$. But then there exists a finite set $\{x_1, x_2, \ldots, x_r\}$ of points of K_1 such that

$$K_1 \subset V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_r},$$

since K_1 is compact. Define

$$U_1 = V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_r}, \qquad U_2 = W_{x_1} \cap W_{x_2} \cap \cdots \cap W_{x_r}.$$

Then U_1 and U_2 are open sets, $K_1 \subset U_1$, $K_2 \subset U_2$ and $U_1 \cap U_2 = \emptyset$, as required. ∎

Lemma 5.11 *Let $f: X \to Y$ be a continuous function from a compact topological space X to a Hausdorff space Y. Then $f(K)$ is closed in Y for every closed set K in X.*

Proof If K is a closed set in X, then K is compact (Lemma 5.3), and therefore $f(K)$ is compact (Lemma 5.4). But any compact subset of a Hausdorff space is closed (Corollary 5.9). Thus $f(K)$ is closed in Y, as required. ∎

Remark If the Hausdorff space Y in Lemma 5.11 is a metric space, then Proposition 5.7 may be used in place of Corollary 5.9 in the proof of the lemma.

Theorem 5.12 *A continuous bijection $f: X \to Y$ from a compact topological space X to a Hausdorff space Y is a homeomorphism.*

Proof Let $g: Y \to X$ be the inverse of the bijection $f: X \to Y$. If U is open in X then $X \setminus U$ is closed in X, and hence $f(X \setminus U)$ is closed in Y, by Lemma 5.11. But $f(X \setminus U) = g^{-1}(X \setminus U) = Y \setminus g^{-1}(U)$. It follows that $g^{-1}(U)$ is open in Y for every open set U in X. Therefore $g: Y \to X$ is continuous, and thus $f: X \to Y$ is a homeomorphism. ∎

We recall that a function $f: X \to Y$ from a topological space X to a topological space Y is said to be an *identification map* if it is surjective and satisfies the following condition: a subset U of Y is open in Y if and only if $f^{-1}(U)$ is open in X.

Proposition 5.13 *A continuous surjection $f\colon X \to Y$ from a compact topological space X to a Hausdorff space Y is an identification map.*

Proof Let U be a subset of Y. We claim that $Y \setminus U = f(K)$, where $K = X \setminus f^{-1}(U)$. Clearly $f(K) \subset Y \setminus U$. Also, given any $y \in Y \setminus U$, there exists $x \in X$ satisfying $y = f(x)$, since $f\colon X \to Y$ is surjective. Moreover $x \in K$, since $f(x) \notin U$. Thus $Y \setminus U \subset f(K)$, and hence $Y \setminus U = f(K)$, as claimed.

We must show that the set U is open in Y if and only if $f^{-1}(U)$ is open in X. First suppose that $f^{-1}(U)$ is open in X. Then K is closed in X, and hence $f(K)$ is closed in Y, by Lemma 5.11. It follows that U is open in Y. Conversely if U is open in Y then $f^{-1}(Y)$ is open in X, since $f\colon X \to Y$ is continuous. Thus the surjection $f\colon X \to Y$ is an identification map. ∎

Example Let S^1 be the unit circle in $\mathbb{R}^2$, defined by $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, and let $q\colon [0, 1] \to S^1$ be defined by $q(t) = (\cos 2\pi t, \sin 2\pi t)$ for all $t \in [0, 1]$. It has been shown that the map q is an identification map. This also follows directly from the fact that $q\colon [0, 1] \to S^1$ is a continuous surjection from the compact space $[0, 1]$ to the Hausdorff space S^1.

We shall show that a finite Cartesian product of compact spaces is compact. To prove this, we apply the following result, known as the *Tube Lemma*.

Lemma 5.14 *Let X and Y be topological spaces, let K be a compact subset of Y, and U be an open set in $X \times Y$. Let $V = \{x \in X : \{x\} \times K \subset U\}$. Then V is an open set in X.*

Proof Let $x \in V$. For each $y \in K$ there exist open subsets D_y and E_y of X and Y respectively such that $(x, y) \in D_y \times E_y$ and $D_y \times E_y \subset U$. Now there exists a finite set $\{y_1, y_2, \ldots, y_k\}$ of points of K such that $K \subset E_{y_1} \cup E_{y_2} \cup \cdots \cup E_{y_k}$, since K is compact. Set $N_x = D_{y_1} \cap D_{y_2} \cap \cdots \cap D_{y_k}$. Then N_x is an open set in X. Moreover

$$N_x \times K \subset \bigcup_{i=1}^{k} (N_x \times E_{y_i}) \subset \bigcup_{i=1}^{k} (D_{y_i} \times E_{y_i}) \subset U,$$

so that $N_x \subset V$. It follows that V is the union of the open sets N_x for all $x \in V$. Thus V is itself an open set in X, as required. ∎

Theorem 5.15 *A Cartesian product of a finite number of compact spaces is itself compact.*

Proof It suffices to prove that the product of two compact topological spaces X and Y is compact, since the general result then follows easily by induction on the number of compact spaces in the product.

Let $\mathcal{U}$ be an open cover of $X \times Y$. We must show that this open cover possesses a finite subcover.

Let x be a point of X. The set $\{x\} \times Y$ is a compact subset of $X \times Y$, since it is the image of the compact space Y under the continuous map from Y to $X \times Y$ which sends $y \in Y$ to (x, y), and the image of any compact set under a continuous map is itself compact (Lemma 5.4). Therefore there exists a finite collection $U_1, U_2, \ldots, U_r$ of open sets belonging to the open cover $\mathcal{U}$ such that $\{x\} \times Y$ is contained in $U_1 \cup U_2 \cup \cdots \cup U_r$. Let V_x denote the set of all points x' of X for which $\{x'\} \times Y$ is contained in $U_1 \cup U_2 \cup \cdots \cup U_r$. Then $x \in V_x$, and Lemma 5.14 ensures that V_x is an open set in X. Note that $V_x \times Y$ is covered by finitely many of the open sets belonging to the open cover $\mathcal{U}$.

Now $\{V_x : x \in X\}$ is an open cover of the space X. It follows from the compactness of X that there exists a finite set $\{x_1, x_2, \ldots, x_r\}$ of points of X such that $X = V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_r}$. Now $X \times Y$ is the union of the sets $V_{x_j} \times Y$ for $j = 1, 2, \ldots, r$, and each of these sets can be covered by a finite collection of open sets belonging to the open cover $\mathcal{U}$. On combining these finite collections, we obtain a finite collection of open sets belonging to $\mathcal{U}$ which covers $X \times Y$. This shows that $X \times Y$ is compact. ∎

Theorem 5.16 *Let K be a subset of $\mathbb{R}^n$. Then K is compact if and only if K is both closed and bounded.*

Proof Suppose that K is compact. Then K is closed, since $\mathbb{R}^n$ is Hausdorff, and a compact subset of a Hausdorff space is closed (by Corollary 5.9). For each natural number m, let B_m be the open ball of radius m about the origin, given by $B_m = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < m\}$. Then $\{B_m : m \in \mathbb{N}\}$ is an open cover of $\mathbb{R}^n$. It follows from the compactness of K that there exist natural numbers $m_1, m_2, \ldots, m_k$ such that $K \subset B_{m_1} \cup B_{m_2} \cup \cdots \cup B_{m_k}$. But then $K \subset B_M$, where M is the maximum of $m_1, m_2, \ldots, m_k$, and thus K is bounded.

Conversely suppose that K is both closed and bounded. Then there exists some real number L such that K is contained within the closed cube C given by

$$C = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : -L \leq x_j \leq L \text{ for } j = 1, 2, \ldots, n\}.$$

Now the closed interval $[-L, L]$ is compact by the Heine-Borel Theorem (Theorem 5.2), and C is the Cartesian product of n copies of the compact set $[-L, L]$. It follows from Theorem 5.15 that C is compact. But K is a closed subset of C, and a closed subset of a compact topological space is itself compact, by Lemma 5.3. Thus K is compact, as required. ∎

5.2 Compact Metric Spaces

We recall that a metric or topological space is said to be *compact* if every open cover of the space has a finite subcover. We shall obtain some equivalent characterizations of compactness for *metric spaces* (Theorem 5.22); these characterizations do not generalize to arbitrary topological spaces.

Proposition 5.17 *Every sequence of points in a compact metric space has a convergent subsequence.*

Proof Let X be a compact metric space, and let $x_1, x_2, x_3, \ldots$ be a sequence of points of X. We must show that this sequence has a convergent subsequence. Let F_n denote the closure of $\{x_n, x_{n+1}, x_{n+2}, \ldots\}$. We claim that the intersection of the sets $F_1, F_2, F_3, \ldots$ is non-empty. For suppose that this intersection were the empty set. Then X would be the union of the sets $V_1, V_2, V_3, \ldots$, where $V_n = X \setminus F_n$ for all n. But $V_1 \subset V_2 \subset V_3 \subset \cdots$, and each set V_n is open. It would therefore follow from the compactness of X that X would be covered by finitely many of the sets $V_1, V_2, V_3, \ldots$, and therefore $X = V_n$ for some sufficiently large n. But this is impossible, since F_n is non-empty for all natural numbers n. Thus the intersection of the sets $F_1, F_2, F_3, \ldots$ is non-empty, as claimed, and therefore there exists a point p of X which belongs to F_n for all natural numbers n.

We now obtain, by induction on n, a subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ which satisfies $d(x_{n_j}, p) < 1/j$ for all natural numbers j. Now p belongs to the closure F_1 of the set $\{x_1, x_2, x_3, \ldots\}$. Therefore there exists some natural number n_1 such that $d(x_{n_1}, p) < 1$. Suppose that x_{n_j} has been chosen so that $d(x_{n_j}, p) < 1/j$. The point p belongs to the closure F_{n_j+1} of the set $\{x_n : n > n_j\}$. Therefore there exists some natural number n_{j+1} such that $n_{j+1} > n_j$ and $d(x_{n_{j+1}}, p) < 1/(j+1)$. The subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ constructed in this manner converges to the point p, as required. ∎

We shall also prove the converse of Proposition 5.17: if X is a metric space, and if every sequence of points of X has a convergent subsequence, then X is compact (see Theorem 5.22 below).

Let X be a metric space with distance function d. A *Cauchy sequence* in X is a sequence $x_1, x_2, x_3, \ldots$ of points of X with the property that, given any $\varepsilon > 0$, there exists some natural number N such that $d(x_j, x_k) < \varepsilon$ for all j and k satisfying $j \geq N$ and $k \geq N$.

A metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges to some point of X.

Proposition 5.18 *Let X be a metric space with the property that every sequence of points of X has a convergent subsequence. Then X is complete.*

Proof Let $x_1, x_2, x_3, \ldots$ be a Cauchy sequence in X. This sequence then has a subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ which converges to some point p of X. We claim that the given Cauchy sequence also converges to p.

Let $\varepsilon > 0$ be given. Then there exists some natural number N such that $d(x_m, x_n) < \frac{1}{2}\varepsilon$ whenever $m \geq N$ and $n \geq N$, since $x_1, x_2, x_3, \ldots$ is a Cauchy sequence. Moreover n_j can be chosen large enough to ensure that $n_j \geq N$ and $d(x_{n_j}, p) < \frac{1}{2}\varepsilon$. If $n \geq N$ then

$$d(x_n, p) \leq d(x_n, x_{n_j}) + d(x_{n_j}, p) < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

This shows that the Cauchy sequence $x_1, x_2, x_3, \ldots$ converges to the point p. Thus X is complete, as required. ∎

Definition Let X be a metric space with distance function d. A subset A of X is said to be *bounded* if there exists a non-negative real number K such that $d(x, y) \leq K$ for all $x, y \in A$. The smallest real number K with this property is referred to as the *diameter* of A, and is denoted by $\operatorname{diam} A$. (Note that $\operatorname{diam} A$ is the supremum of the values of $d(x, y)$ as x and y range over all points of A.)

Let X be a metric space with distance function d, and let A be a subset of X. The *closure* $\overline{A}$ of A is the intersection of all closed sets in X that contain the set A: it can be regarded as the smallest closed set in X containing A. Let x be a point of the closure $\overline{A}$ of A. Given any $\varepsilon > 0$, there exists some point x' of A such that $d(x, x') < \varepsilon$. (Indeed the open ball in X of radius ε about the point x must intersect the set A, since otherwise the complement of this open ball would be a closed set in X containing the set A but not including the point x, which is not possible if x belongs to the closure of A.)

Lemma 5.19 *Let X be a metric space, and let A be a subset of X. Then* $\operatorname{diam} A = \operatorname{diam} \overline{A}$, *where $\overline{A}$ is the closure of A.*

Proof Clearly $\operatorname{diam} A \leq \operatorname{diam} \overline{A}$. Let x and y be points of $\overline{A}$. Then, given any $\varepsilon > 0$, there exist points x' and y' of A satisfying $d(x, x') < \varepsilon$ and $d(y, y') < \varepsilon$. It follows from the Triangle Inequality that

$$d(x, y) \leq d(x, x') + d(x', y') + d(y', y) < \operatorname{diam} A + 2\varepsilon.$$

Thus $d(x, y) < \operatorname{diam} A + 2\varepsilon$ for all $\varepsilon > 0$, and hence $d(x, y) \leq \operatorname{diam} A$. This shows that $\operatorname{diam} \overline{A} \leq \operatorname{diam} A$, as required. ∎

Definition A metric space X is said to be *totally bounded* if, given any $\varepsilon > 0$, the set X can be expressed as a finite union of subsets of X, each of which has diameter less than ε.

A subset A of a totally bounded metric space X is itself totally bounded. For if X is the union of the subsets $B_1, B_2, \ldots, B_k$, where $\operatorname{diam} B_n < \varepsilon$ for $n = 1, 2, \ldots, k$, then A is the union of $A \cap B_n$ for $n = 1, 2, \ldots, k$, and $\operatorname{diam} A \cap B_n < \varepsilon$.

Proposition 5.20 *Let X be a metric space. Suppose that every sequence of points of X has a convergent subsequence. Then X is totally bounded.*

Proof Suppose that X were not totally bounded. Then there would exist some $\varepsilon > 0$ with the property that no finite collection of subsets of X of diameter less than 3ε covers the set X. There would then exist an infinite sequence $x_1, x_2, x_3, \ldots$ of points of X with the property that $d(x_m, x_n) \geq \varepsilon$ whenever $m \neq n$. Indeed suppose that points $x_1, x_2, \ldots, x_{k-1}$ of X have already been chosen satisfying $d(x_m, x_n) \geq \varepsilon$ whenever $m < k$, $n < k$ and $m \neq n$. The diameter of each open ball $B_X(x_m, \varepsilon)$ is less than or equal to 2ε. Therefore X could not be covered by the sets $B_X(x_m, \varepsilon)$ for $m < k$, and thus there would exist a point x_k of X which does not belong to $B(x_m, \varepsilon)$ for any $m < k$. Then $d(x_m, x_k) \geq \varepsilon$ for all $m < k$. In this way we can successively choose points $x_1, x_2, x_3, \ldots$ to form an infinite sequence with the required property. However such an infinite sequence would have no convergent subsequence, which is impossible. This shows that X must be totally bounded, as required. ∎

Proposition 5.21 *Every complete totally bounded metric space is compact.*

Proof Let X be some totally bounded metric space. Suppose that there exists an open cover $\mathcal{V}$ of X which has no finite subcover. We shall prove the existence of a Cauchy sequence $x_1, x_2, x_3, \ldots$ in X which cannot converge to any point of X. (Thus if X is not compact, then X cannot be complete.)

Let $\varepsilon > 0$ be given. Then X can be covered by finitely many closed sets whose diameter is less than ε, since X is totally bounded and every subset of X has the same diameter as its closure (Lemma 5.19). At least one of these closed sets cannot be covered by a finite collection of open sets belonging to $\mathcal{V}$ (since if every one of these closed sets could be covered by a such a finite collection of open sets, then we could combine these collections to obtain a finite subcover of $\mathcal{V}$). We conclude that, given any $\varepsilon > 0$, there exists a closed subset of X of diameter less than ε which cannot be covered by any finite collection of open sets belonging to $\mathcal{V}$.

We claim that there exists a sequence $F_1, F_2, F_3, \ldots$ of closed sets in X satisfying $F_1 \supset F_2 \supset F_3 \supset \cdots$ such that each closed set F_n has the following properties: $\operatorname{diam} F_n < 1/2^n$, and no finite collection of open sets belonging to $\mathcal{V}$ covers F_n. For if F_n is a closed set with these properties then F_n is itself totally bounded, and thus the above remarks (applied with F_n in place of X) guarantee the existence of a closed subset F_{n+1} of F_n with the required properties. Thus the existence of the required sequence of closed sets follows by induction on n.

Choose $x_n \in F_n$ for each natural number n. Then $d(x_m, x_n) < 1/2^n$ for any $m > n$, since x_m and x_n belong to F_n and $\operatorname{diam} F_n < 1/2^n$. Therefore the sequence $x_1, x_2, x_3, \ldots$ is a Cauchy sequence. Suppose that this Cauchy sequence were to converge to some point p of X. Then $p \in F_n$ for each natural number n, since F_n is closed and $x_m \in F_n$ for all $m \geq n$. (If a sequence of points belonging to a closed subset of a metric or topological space is convergent then the limit of that sequence belongs to the closed set.) Moreover $p \in V$ for some open set V belonging to $\mathcal{V}$, since $\mathcal{V}$ is an open cover of X. But then there would exist $\delta > 0$ such that $B_X(p, \delta) \subset V$, where $B_X(p, \delta)$ denotes the open ball of radius δ in X centred on p. Thus if n were large enough to ensure that $1/2^n < \delta$, then $p \in F_n$ and $\operatorname{diam} F_n < \delta$, and hence $F_n \subset B_X(p, \delta) \subset V$, contradicting the fact that no finite collection of open sets belonging to $\mathcal{V}$ covers the set F_n. This contradiction shows that the Cauchy sequence $x_1, x_2, x_3, \ldots$ is not convergent.

We have thus shown that if X is a totally bounded metric space which is not compact then X is not complete. Thus every complete totally bounded metric space must be compact, as required. ∎

Theorem 5.22 *Let X be a metric space with distance function d. The following are equivalent:—*

(i) *X is compact,*

(ii) *every sequence of points of X has a convergent subsequence,*

(iii) *X is complete and totally bounded,*

Proof Propositions 5.17, 5.18 5.20 and 5.21 show that (i) implies (ii), (ii) implies (iii), and (iii) implies (i). It follows that (i), (ii) and (iii) are all equivalent to one another. ∎

Remark A subset K of $\mathbb{R}^n$ is complete if and only if it is closed in $\mathbb{R}^n$. Also it is easy to see that K is totally bounded if and only if K is a bounded subset of $\mathbb{R}^n$. Thus Theorem 5.22 is a generalization of the theorem which states that a subset K of $\mathbb{R}^n$ is compact if and only if it is both closed and bounded (Theorem 5.16).

5.3 The Lebesgue Lemma and Uniform Continuity

Lemma 5.23 (Lebesgue Lemma) *Let (X, d) be a compact metric space. Let $\mathcal{U}$ be an open cover of X. Then there exists a positive real number δ such that every subset of X whose diameter is less than δ is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$.*

Proof Every point of X is contained in at least one of the open sets belonging to the open cover $\mathcal{U}$. It follows from this that, for each point x of X, there exists some $\delta_x > 0$ such that the open ball $B(x, 2\delta_x)$ of radius $2\delta_x$ about the point x is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$. But then the collection consisting of the open balls $B(x, \delta_x)$ of radius δ_x about the points x of X forms an open cover of the compact space X. Therefore there exists a finite set $x_1, x_2, \ldots, x_r$ of points of X such that

$$B(x_1, \delta_1) \cup B(x_2, \delta_2) \cup \cdots \cup B(x_r, \delta_r) = X,$$

where $\delta_i = \delta_{x_i}$ for $i = 1, 2, \ldots, r$. Let $\delta > 0$ be given by

$$\delta = \text{minimum}(\delta_1, \delta_2, \ldots, \delta_r).$$

Suppose that A is a subset of X whose diameter is less than δ. Let u be a point of A. Then u belongs to $B(x_i, \delta_i)$ for some integer i between 1 and r. But then it follows that $A \subset B(x_i, 2\delta_i)$, since, for each point v of A,

$$d(v, x_i) \leq d(v, u) + d(u, x_i) < \delta + \delta_i \leq 2\delta_i.$$

But $B(x_i, 2\delta_i)$ is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$. Thus A is contained wholly within one of the open sets belonging to $\mathcal{U}$, as required. ∎

Let $\mathcal{U}$ be an open cover of a compact metric space X. A *Lebesgue number* for the open cover $\mathcal{U}$ is a positive real number δ such that every subset of X whose diameter is less than δ is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$. The Lebesgue Lemma thus states that there exists a Lebesgue number for every open cover of a compact metric space.

Let X and Y be metric spaces with distance functions d_X and d_Y respectively, and let $f: X \to Y$ be a function from X to Y. The function f is said to be *uniformly continuous* on X if and only if, given $\varepsilon > 0$, there exists some $\delta > 0$ such that $d_Y(f(x), f(x')) < \varepsilon$ for all points x and x' of X satisfying $d_X(x, x') < \delta$. (The value of δ should be independent of both x and x'.)

Theorem 5.24 *Let X and Y be metric spaces. Suppose that X is compact. Then every continuous function from X to Y is uniformly continuous.*

Proof Let d_X and d_Y denote the distance functions for the metric spaces X and Y respectively. Let $f\colon X \to Y$ be a continuous function from X to Y. We must show that f is uniformly continuous.

Let $\varepsilon > 0$ be given. For each $y \in Y$, define

$$V_y = \{x \in X : d_Y(f(x), y) < \tfrac{1}{2}\varepsilon\}.$$

Note that $V_y = f^{-1}\left(B_Y(y, \frac{1}{2}\varepsilon)\right)$, where $B_Y(y, \frac{1}{2}\varepsilon)$ denotes the open ball of radius $\frac{1}{2}\varepsilon$ about y in Y. Now the open ball $B_Y(y, \frac{1}{2}\varepsilon)$ is an open set in Y, and f is continuous. Therefore V_y is open in X for all $y \in Y$. Note that $x \in V_{f(x)}$ for all $x \in X$.

Now $\{V_y : y \in Y\}$ is an open cover of the compact metric space X. It follows from the Lebesgue Lemma (Lemma 5.23) that there exists some $\delta > 0$ such that every subset of X whose diameter is less than δ is a subset of some set V_y. Let x and x' be points of X satisfying $d_X(x, x') < \delta$. The diameter of the set $\{x, x'\}$ is $d_X(x, x')$, which is less than δ. Therefore there exists some $y \in Y$ such that $x \in V_y$ and $x' \in V_y$. But then $d_Y(f(x), y) < \frac{1}{2}\varepsilon$ and $d_Y(f(x'), y) < \frac{1}{2}\varepsilon$, and hence

$$d_Y(f(x), f(x')) \leq d_Y(f(x), y) + d_Y(y, f(x')) < \varepsilon.$$

This shows that $f\colon X \to Y$ is uniformly continuous, as required. ∎

Let K be a closed bounded subset of $\mathbb{R}^n$. It follows from Theorem 5.16 and Theorem 5.24 that any continuous function $f\colon K \to \mathbb{R}^k$ is uniformly continuous.

5.4 The Equivalence of Norms on a Finite-Dimensional Vector Space

Let $\|.\|$ and $\|.\|_*$ be norms on a real or complex vector space X. The norms $\|.\|$ and $\|.\|_*$ are said to be *equivalent* if and only if there exist constants c and C, where $0 < c \leq C$, such that

$$c\|x\| \leq \|x\|_* \leq C\|x\|$$

for all $x \in X$.

Lemma 5.25 *Two norms $\|.\|$ and $\|.\|_*$ on a real or complex vector space X are equivalent if and only if they induce the same topology on X.*

Proof Suppose that the norms $\|.\|$ and $\|.\|_*$ induce the same topology on X. Then there exists some $\delta > 0$ such that

$$\{x \in X : \|x\| < \delta\} \subset \{x \in X : \|x\|_* < 1\},$$

since the set $\{x \in X : \|x\|_* < 1\}$ is open with respect to the topology on X induced by both $\|.\|_*$ and $\|.\|$. Let C be any positive real number satisfying $C\delta > 1$. Then

$$\left\| \frac{1}{C\|x\|} x \right\| = \frac{1}{C} < \delta,$$

and hence

$$\|x\|_* = C\|x\| \left\| \frac{1}{C\|x\|} x \right\|_* < C\|x\|.$$

for all non-zero elements x of X, and thus $\|x\|_* \leq C\|x\|$ for all $x \in X$. On interchanging the roles of the two norms, we deduce also that there exists a positive real number c such that $\|x\| \leq (1/c)\|x\|_*$ for all $x \in X$. But then $c\|x\| \leq \|x\|_* \leq C\|x\|$ for all $x \in X$. We conclude that the norms $\|.\|$ and $\|.\|_*$ are equivalent.

Conversely suppose that the norms $\|.\|$ and $\|.\|_*$ are equivalent. Then there exist constants c and C, where $0 < c \leq C$, such that $c\|x\| \leq \|x\|_* \leq C\|x\|$ for all $x \in X$. Let U be a subset of X that is open with respect to the topology on X induced by the norm $\|.\|_*$, and let $u \in U$. Then there exists some $\delta > 0$ such that

$$\{x \in X : \|x - u\|_* < C\delta\} \subset U.$$

But then

$$\{x \in X : \|x - u\| < \delta\} \subset \{x \in X : \|x - u\|_* < C\delta\} \subset U,$$

showing that U is open with respect to the topology induced by the norm $\|.\|$. Similarly any subset of X that is open with respect to the topology induced by the norm $\|.\|$ must also be open with respect to the topology induced by $\|.\|_*$. Thus equivalent norms induce the same topology on X. ∎

It follows immediately from Lemma 5.25 that if $\|.\|$, $\|.\|_*$ and $\|.\|_\sharp$ are norms on a real (or complex) vector space X, if the norms $\|.\|$ and $\|.\|_*$ are equivalent, and if the norms $\|.\|_*$ and $\|.\|_\sharp$ are equivalent, then the norms $\|.\|$ and $\|.\|_\sharp$ are also equivalent. This fact can easily be verified directly from the definition of equivalence of norms.

We recall that the usual topology on $\mathbb{R}^n$ is that generated by the Euclidean norm on $\mathbb{R}^n$.

Lemma 5.26 *Let $\|.\|$ be a norm on $\mathbb{R}^n$. Then the function $\mathbf{x} \mapsto \|\mathbf{x}\|$ is continuous with respect to the usual topology on on $\mathbb{R}^n$.*

Proof Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ denote the basis of $\mathbb{R}^n$ given by

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \cdots, \quad \mathbf{e}_n = (0, 0, 0, \ldots, 1).$$

Let $\mathbf{x}$ and $\mathbf{y}$ be points of $\mathbb{R}^n$, given by

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \qquad \mathbf{y} = (y_1, y_2, \ldots, y_n).$$

Using Schwarz' Inequality, we see that

$$\begin{aligned} \|\mathbf{x} - \mathbf{y}\| &= \left\| \sum_{j=1}^{n} (x_j - y_j)\mathbf{e}_j \right\| \leq \sum_{j=1}^{n} |x_j - y_j| \, \|\mathbf{e}_j\| \\ &\leq \left(\sum_{j=1}^{n} (x_j - y_j)^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} \|\mathbf{e}_j\|^2 \right)^{\frac{1}{2}} = C\|\mathbf{x} - \mathbf{y}\|_2, \end{aligned}$$

where

$$C^2 = \|\mathbf{e}_1\|^2 + \|\mathbf{e}_2\|^2 + \cdots + \|\mathbf{e}_n\|^2$$

and $\|\mathbf{x} - \mathbf{y}\|_2$ denotes the Euclidean norm of $\mathbf{x} - \mathbf{y}$, defined by

$$\|\mathbf{x} - \mathbf{y}\|_2 = \left(\sum_{j=1}^{n} (x_j - y_j)^2 \right)^{\frac{1}{2}}.$$

Also $|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq \|\mathbf{x} - \mathbf{y}\|$, since

$$\|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\|, \qquad \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{x}\|.$$

We conclude therefore that

$$|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq C\|\mathbf{x} - \mathbf{y}\|_2,$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and thus the function $\mathbf{x} \mapsto \|\mathbf{x}\|$ is continuous on $\mathbb{R}^n$ (with respect to the usual topology on $\mathbb{R}^n$). ∎

Theorem 5.27 *Any two norms on $\mathbb{R}^n$ are equivalent, and induce the usual topology on $\mathbb{R}^n$.*

Proof Let $\|.\|$ be any norm on $\mathbb{R}^n$. We show that $\|.\|$ is equivalent to the Euclidean norm $\|.\|_2$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, defined by

$$S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 = 1\},$$

and let $f\colon S^{n-1} \to \mathbb{R}$ be the real-valued function on S^{n-1} defined such that $f(\mathbf{x}) = \|\mathbf{x}\|$ for all $\mathbf{x} \in S^{n-1}$. Now the function f is a continuous function on S^{n-1} (Lemma 5.26). Also the function f is non-zero at each point of S^{n-1}, and therefore the function sending $\mathbf{x} \in S^{n-1}$ to $1/f(\mathbf{x})$ is continuous. Now any continuous real-valued function on a closed bounded subset of $\mathbb{R}^n$ is bounded on that set (Proposition). It follows that there exist positive real numbers C and D such that $f(\mathbf{x}) \leq C$ and $1/f(\mathbf{x}) \leq D$ for all $\mathbf{x} \in S^{n-1}$. Let $c = D^{-1}$. Then $c \leq \|\mathbf{x}\| \leq C$ for all $\mathbf{x} \in S^{n-1}$.

Now

$$\frac{\|\mathbf{x}\|}{\|\mathbf{x}\|_2} = f\left(\|\mathbf{x}\|_2^{-1}\mathbf{x}\right)$$

for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. (This is an immediate consequence of the fact that $\|\lambda\mathbf{x}\| = |\lambda|\,\|\mathbf{x}\|$ for all $\mathbf{x} \in IR^n$ and $\lambda \in \mathbb{R}$.) It follows that $c\|\mathbf{x}\|_2 \leq \|\mathbf{x}\| \leq C\|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. These inequalities also hold when $\mathbf{x} = \mathbf{0}$. The result follows. ∎