Course 221: Michaelmas Term 2007.

Assignment I.

To be handed in by Wednesday, December 5, 2007.
Please include both name and student number on any work handed in.

1. For each of the following subsets of $\mathbb{R}^3$, determine whether or not that subset is an open set in $\mathbb{R}^3$, and also determine whether or not the subset is a closed set in $\mathbb{R}^3$:

   (a) $\{(x, y, z) \in \mathbb{R}^3 : (x-1)^2 + (y-2)^2 + z^2 < 9 \text{ and } z > 1\}$;

   (b) $\{(x, y, z) \in \mathbb{R}^3 : (x-1)^2 + (y-2)^2 + z^2 < 9 \text{ and } z \geq 1\}$;

   (c) $\{(x, y, z) \in \mathbb{R}^3 : (x-1)^2 + (y-2)^2 + z^2 \geq 9 \text{ and } z \geq 1\}$.

2. Let X be a metric space with distance function d, and let $d^*(x, y) = q(d(x, y))$ for all $x, y \in X$, where q is a real-valued function defined on the set $[0, +\infty)$ of non-negative real numbers which satisfies the following properties:

   (i) $q(u) \geq 0$ for all non-negative real numbers u;

   (ii) $q(u) \leq q(v)$ for all non-negative real numbers u and v satisfying $u \leq v$;

   (iii) $q(u + v) \leq q(u) + q(v)$ for all non-negative real numbers u and v;

   (iv) $q(u) = 0$ if and only if $u = 0$.

   Prove that the function d^* satisfies the axioms required of a distance function on a metric space, so that X can be considered as a metric space with distance function d^*.

3. Let the set X, the function q and the distance functions d and d^* be as in the previous question (so that $d^*(x, y) = q(d(x, y))$ for all $x, y \in X$, where q is a function satisfying properties (i), (ii), (iii) and (iv) above). Let V be a subset of X. Prove that if V is an open set in X when X is considered as a metric space with distance function d, then V is also an open set in X when X is considered as a metric space with distance function d^*.