

Course 221: Michaelmas Term 2006.

Assignment I.

To be handed in by Monday 22 January, 2007.

Please include both name and student number on any work handed in.

1. Let X be a set. A collection β of subsets of X is said to be a *basis* for a topology on X if, given any two subsets B_1 and B_2 of X belonging to the collection β , and given any point p of $B_1 \cap B_2$, there exists a subset B of X belonging to β such that $p \in B$ and $B \subset B_1 \cap B_2$.

In this question, we show that any such basis β determines a well-defined topology on the set X .

Throughout this question, let X be a set, and let β be a basis for a topology on X .

- (a) Let B_1, B_2, \dots, B_k be subsets of X belonging to β , and let p be a point of $B_1 \cap B_2 \cap \dots \cap B_k$. Prove that there exists a subset B of X belonging to β such that $p \in B$ and $B \subset B_1 \cap B_2 \cap \dots \cap B_k$.

[Note that the number of sets B_1, B_2, \dots, B_k involved is finite. As a result, it is possible to make use of the Principle of Mathematical Induction in the proof of (a).]

Let τ denote the collection consisting of all subsets V of X with the property that, given any point p of V , there exists some subset B belonging to the basis β such that $p \in B$ and $B \subset V$. We regard the empty set as belonging to the collection τ , and it is obvious that X belongs to the collection τ .

- (b) Prove that, given any collection of subsets of X belonging to τ , the union of those subsets belongs to τ .

- (c) Prove that, given any *finite* collection of subsets of X belonging to τ , the intersection of those subsets belongs to τ .

The results of (b) and (c) ensure that τ is a topology on X whose open sets are those subsets of X which belong to the collection τ . This topology τ is referred to as the topology generated by the basis β . Note that $\beta \subset \tau$.

- (d) Let τ be a topology on the set X generated by some basis β , so that open sets in X are the subsets of X that belong to the topology τ . Prove that a subset of X is an open set if and only if it is a union of subsets of X that belong to the basis β .
2. Throughout this question, let X be a metric space with distance function d . Given any point p of X , and given any positive real number δ , we denote by $B_X(p, \delta)$ the open ball in X of radius δ about the point p .
- (a) Explain why the collection of open balls in X is a basis for a topology on X .

Note that it follows directly from the definition of an open set in a metric space that the topology on X generated by the collection of open balls in X is that generated by the distance function d .

A subset S of a metric space X is said to be *dense* in X if $\overline{S} = X$, where \overline{S} denotes the closure of S . It is easily seen that a subset S of X is dense in X if and only if, given any point x of X , and given any positive real number δ , there exists some point s of S satisfying $d(s, x) < \delta$.

- (b) Let S be a dense subset of the metric space X , let p be a point of X , and let δ be a positive real number. Prove that there exists an element s of S and a positive *rational number* θ such that $p \in B_X(s, \theta)$ and $B_X(s, \theta) \subset B_X(p, \delta)$.

[You will almost certainly need to make use of the Triangle Inequality in proving the above result. In discovering the proof of the result, you may find it helpful to work out the proof first in the special case when $X = \mathbb{R}^2$, the distance function on \mathbb{R}^2 is the Euclidean distance function, and the dense subset S of \mathbb{R}^2 is the subset \mathbb{Q}^2 of \mathbb{R}^2 consisting of all points in \mathbb{R}^2 whose Cartesian coordinates are rational numbers. Having seen how to prove the result in this special case, using geometric intuition as appropriate, you can then generalize the argument in order to prove the general result applicable to any dense subset in any metric space.]

- (c) A metric space is said to be *separable* if it contains a countable dense subset. Prove that, given any separable metric space X , there exists a countable collection β of subsets of X which is a basis for the topology on X that is generated by the distance function on X .