UNIVERSITY OF DUBLIN

TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

Trinity Term 2007

COURSE 214

SAMPLE PAPER

Credit will be given for the best 4 questions answered.

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

1. (a) Let X be a subset of some Euclidean space, and let $f_1, f_2, f_3, \ldots$ be an infinite sequence of functions mapping X into some Euclidean space $\mathbb{R}^n$. Define what is meant by saying that the sequence $f_1, f_2, f_3, \ldots$ converges *uniformly* to some function $f\colon X \to \mathbb{R}^n$.

   (b) Let X be a subset of some Euclidean space, and let $f_1, f_2, f_3, \ldots$ be an infinite sequence of continuous functions mapping X into some Euclidean space $\mathbb{R}^n$ which converges uniformly to some function $f\colon X \to \mathbb{R}^n$. Prove that the function f is also continuous.

   (c) Let X be a subset of some Euclidean space, and let $f\colon X \to \mathbb{R}^n$ be a function mapping X into some Euclidean space $\mathbb{R}^n$. What is meant by saying that this function is *uniformly continuous*?

   (d) Let X be a subset of some Euclidean space, and let $f\colon X \to \mathbb{R}^n$ be a continuous function mapping X into some Euclidean space $\mathbb{R}^n$. Suppose that the set X is both closed and bounded. Prove that the function f is then uniformly continuous.

   [You may use without proof the result that any bounded sequence of points in a Euclidean space has a convergent subsequence. You may also use without proof the result that if a sequence of points in some closed set converges to some point $\mathbf{p}$, then this point $\mathbf{p}$ belongs to the closed set.]

2. (a) Give the definition of the *winding number* $n(\gamma, w)$ of a closed path $\gamma\colon [a,b] \to \mathbb{C}$ about some point w of the complex plane that does not lie on γ.

   (b) State and prove the *Fundamental Theorem of Algebra*.

   [You may use without proof the result that if $\gamma_s\colon [a,b] \to \mathbb{C}$ is a closed path for each real number s in some interval $[c,d]$, then the value of the winding number $n(\gamma_s, w)$ of γ_s about some complex number w is independent of the value of s, provided that $\gamma_s(t)$ is a continuous function of s and t, and provided also that none of the paths γ_s passes through w.]

3. Let $f: D \to \mathbb{C}$ be a holomorphic function defined over an open set D in $\mathbb{C}$, and let T be a closed triangle contained in D. Prove that

$$\int_{\partial T} f(z)\,dz = 0$$

where $\int_{\partial T} f(z)\,dz$ denotes the path integral of f taken round the boundary of the triangle T in the anti-clockwise direction.

4. Let w be a complex number, let r be a positive real number, and let f be a holomorphic function on $\{z \in \mathbb{C} : 0 < |z| < r\}$. Laurent's Theorem asserts that there exist complex numbers a_n for all integers n such that

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n + \sum_{n=1}^{+\infty} a_{-n} z^{-n}$$

for all complex numbers z satisfying $0 < |z| < r$, and that moreover

$$a_n == \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(z)}{z^{n+1}}\,dz$$

for all integers n, where $0 < R < r$, and $\gamma_R: [0,1] \to \mathbb{C}$ is the closed path defined such that $\gamma_R(t) = Re^{2\pi i t}$ for all $t \in [0,1]$. Prove this theorem.

5. Use the method of contour integration to evaluate

$$\int_{-\infty}^{+\infty} \frac{e^{isx}}{x^2+4}\,dx$$

and

$$\int_{-\infty}^{+\infty} \frac{e^{isx}}{x^4+5x^2+4}\,dx$$

when s is a real number satisfying $s > 0$.

[Briefly justify your answers. You may use, without proof, the result that if R is a positive real number, if f is a continuous complex-valued function defined everywhere on the semicircle S_R, where

$$S_R = \{z \in \mathbb{C} : |z| = R \text{ and } \mathrm{Im}[z] \geq 0\},$$

and if there exists a non-negative real number $M(R)$ such that $|f(z)| \leq M(R)$ for all $z \in S_R$ then

$$\left| \int_{\sigma_R} f(z) e^{isz}\,dz \right| \leq \frac{\pi M(R)}{s}$$

for all $s > 0$, where $\sigma_R\colon [0,\pi] \to \mathbb{C}$ is the path with $[\sigma_R] = S_R$ defined such that $\sigma_R(\theta) = Re^{i\theta}$ for all $\theta \in [0,\pi]$.]

6. (a) What is an *elliptic function*?

(b) What is a *fundamental region* for an elliptic function?

(c) Let f be an elliptic function, and let X be a fundamental region for f. Prove that the sum of the residues of f at those poles of f located in the fundamental region X is zero.