Course 214: Michaelmas Term 2006
Complex Analysis
Section 1: Complex Numbers and Euclidean Spaces

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Contents

1 Complex Numbers and Euclidean Spaces

1.1 The Least Upper Bound Principle

A widely-used basic principle of analysis, from which many important theorems ultimately derive, is the Least Upper Bound Principle.

Let D be a subset of the set $\mathbb{R}$ of real numbers. A real number u is said to be an *upper bound* of the set D of $x \leq u$ for all $x \in D$. The set D is said to be *bounded above* if such an upper bound exists.

Definition Let D be some set of real numbers which is bounded above. A real number s is said to be the *least upper bound* (or *supremum*) of D (denoted by $\sup D$) if s is an upper bound of D and $s \leq u$ for all upper bounds u of D.

Example The real number 2 is the least upper bound of the sets $\{x \in \mathbb{R} : x \leq 2\}$ and $\{x \in \mathbb{R} : x < 2\}$. Note that the first of these sets contains its least upper bound, whereas the second set does not.

The *Least Upper Bound Principle* may be stated as follows:

> if D is any non-empty subset of $\mathbb{R}$ which is bounded above then there exists a *least upper bound* $\sup D$ for the set D.

A *lower bound* of a set D of real numbers is a real number l with the property that $l \leq x$ for all $x \in D$. A set D of real numbers is said to be *bounded below* if such a lower bound exists. If D is bounded below, then there exists a greatest lower bound (or *infimum*) $\inf D$ of the set D. Indeed $\inf D = -\sup\{x \in \mathbb{R} : -x \in D\}$.

1.2 Monotonic Sequences

An infinite sequence $a_1, a_2, a_3, \ldots$ of real numbers is said to be *strictly increasing* if $a_{n+1} > a_n$ for all n, *strictly decreasing* if $a_{n+1} < a_n$ for all n, *non-decreasing* if $a_{n+1} \geq a_n$ for all n, or *non-increasing* if $a_{n+1} \leq a_n$ for all n. A sequence satisfying any one of these conditions is said to be *monotonic*; thus a monotonic sequence is either non-decreasing or non-increasing.

Theorem 1.1 *Any non-decreasing sequence of real numbers that is bounded above is convergent. Similarly any non-increasing sequence of real numbers that is bounded below is convergent.*

Proof Let $a_1, a_2, a_3, \ldots$ be a non-decreasing sequence of real numbers that is bounded above. It follows from the Least Upper Bound Principle that there exists a least upper bound l for the set $\{a_n : n \in \mathbb{N}\}$. We claim that the sequence converges to l.

Let $\varepsilon > 0$ be given. We must show that there exists some natural number N such that $|a_n - l| < \varepsilon$ whenever $n \geq N$. Now $l - \varepsilon$ is not an upper bound for the set $\{a_n : n \in \mathbb{N}\}$ (since l is the least upper bound), and therefore there must exist some natural number N such that $a_N > l - \varepsilon$. But then $l - \varepsilon < a_n \leq l$ whenever $n \geq N$, since the sequence is non-decreasing and bounded above by l. Thus $|a_n - l| < \varepsilon$ whenever $n \geq N$. Therefore $a_n \to l$ as $n \to +\infty$, as required.

If the sequence $a_1, a_2, a_3, \ldots$ is non-increasing and bounded below then the sequence $-a_1, -a_2, -a_3, \ldots$ is non-decreasing and bounded above, and is therefore convergent. It follows that the sequence $a_1, a_2, a_3, \ldots$ is also convergent. ∎

1.3 The Complex Plane

A *complex number* is a number of the form $x + iy$, where x and y are real numbers, and $i^2 = -1$. The real numbers x and y are uniquely determined by the complex number $x + iy$, and are referred to as the *real* and *imaginary* parts of this complex number.

The algebraic operations of addition, subtraction and multiplication are defined on complex numbers according to the formulae

$$(x+yi)+(u+iv) = (x+u)+i(y+v), \quad (x+yi)-(u+iv) = (x-u)+i(y-v),$$

$$(x + yi) \times (u + iv) = (xu - yv) + i(xv + yu),$$

where x, y, u and v are real numbers.

We regard a real number x as coinciding with the complex number $x + i \times 0$. Note that the operations of addition, subtraction and multiplication of complex numbers defined as above extend the corresponding operations on the set of real numbers.

The set $\mathbb{C}$ of complex numbers, with the operations of addition and multiplication defined above, has the following properties:

(i) $z_1 + z_2 = z_2 + z_1$ for all $z_1, z_2 \in \mathbb{C}$;

(ii) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ for all $z_1, z_2, z_3 \in \mathbb{C}$;

(iii) there exists a complex number 0 with the property that $z+0 = 0+z = z$ for all complex numbers $\mathbb{C}$;

(iv) given any complex number z, there exists a complex number $-z$ such that $z + (-z) = (-z) + z = 0$;

(v) $z_1 \times z_2 = z_2 \times z_1$ for all $z_1, z_2 \in \mathbb{C}$;

(vi) $z_1 \times (z_2 \times z_3) = z_1 \times (z_2 \times z_3)$ for all $z_1, z_2, z_3 \in \mathbb{C}$;

(vii) there exists a complex number 1 with the property that $z \times 1 = 1 \times z = z$ for all complex numbers $\mathbb{C}$;

(viii) given any complex number z satisfying $z \neq 0$, there exists a complex number z^{-1} such that $z \times z^{-1} = z^{-1} \times z = 1$;

(ix) $z_1 \times (z_2 + z_3) = (z_1 \times z_2) + (z_1 \times z_3)$ and $(z_1 + z_2) \times z_3 = (z_1 \times z_3) + (z_2 \times z_3)$ for all $z_1, z_2, z_3 \in \mathbb{C}$.

To verify property (viii), we note that if z is a non-zero complex number, where $z = x + iy$ for some real numbers x and y, and if z^{-1} is given by the formula

$$z^{-1} = \frac{x}{x^2 + y^2} - i\,\frac{y}{x^2 + y^2},$$

then $z \times z^{-1} = z^{-1} \times z = 1$.

A *field* is a set X provided with binary operations $+$ and $\times$, representing addition and multiplication respectively, which satisfy properties (i)–(ix) above (where complex numbers are replaced in the statements of those properties by elements of the set X as appropriate). Thus properties (i)–(ix) may be summarized in the statement that the set of complex numbers, with the usual operations of addition and multiplication, is a field. The set $\mathbb{R}$ of real numbers, with the usual algebraic operations, is also a field.

Given complex numbers z and w, with $w \neq 0$, we define the *quotient* z/w (i.e., z divided by w) by the formula $z/w = zw^{-1}$.

The *conjugate* $\overline{z}$ of a complex number z is defined such that $\overline{x + iy} = x - iy$ for all real numbers x and y. The *modulus* $|z|$ of a complex number z is defined such that $|x + iy| = \sqrt{x^2 + y^2}$ for all real numbers x and y. Note that $|\overline{z}| = |z|$ for all complex numbers z. Also $\overline{z + w} = \overline{z} + \overline{w}$ for all complex numbers z and w. The real part $\operatorname{Re} z$ of a complex number satisfies the formula $2 \operatorname{Re} z = z + \overline{z}$. Now $|\operatorname{Re} z| \leq |z|$. It follows that $|z + \overline{z}| \leq 2|z|$ for all complex numbers z.

Straightforward calculations show that $z\overline{z} = |z|^2$ for all complex numbers z, from which it easily follows that $z^{-1} = |z|^{-2}\overline{z}$ for all non-zero complex numbers z.

Let z and w be complex numbers, and let $z = x + iy$ and $w = u + iv$, where x, y, u and v are real numbers. Then

$$\begin{aligned} |zw|^2 &= (xu - yv)^2 + (xv + yu)^2 \\ &= (x^2u^2 + y^2v^2 - 2xyuv) + (x^2v^2 + y^2u^2 + 2xyuv) \\ &= (x^2 + y^2)(u^2 + v^2) = |z|^2\,|w|^2 \end{aligned}$$

It follows that $|zw| = |z|\,|w|$ for all complex numbers z and w.

Let z and w be complex numbers. Then

$$\begin{aligned} |z + w|^2 &= (z + w)\,(\overline{z} + \overline{w}) = z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \\ &= |z|^2 + 2\,\mathrm{Re}\,z\overline{w} + |w|^2 \\ &\leq |z|^2 + 2|z\overline{w}| + |w|^2 = |z|^2 + 2|z|\,|w| + |w|^2 \\ &= (|z| + |w|)^2. \end{aligned}$$

It follows that $|z + w| \leq |z| + |w|$ for all complex numbers z and w.

We define the *distance* from a complex number z to a complex number w to be the quantity $|w - z|$. Thus if $z = x + iy$ and $w = u + iv$ then

$$|w - z| = \sqrt{(x - u)^2 + (y - v)^2}.$$

We picture the complex numbers as representing points of the Euclidean plane. A complex number $x + iy$, where x and y are real numbers, represents the point of the plane whose Cartesian coordinates (with respect to an appropriate origin) are (x, y). The fact that $|w - z|$ represents the distance between the points of the plane represented by the complex numbers z and w is an immediate consequence of Pythagoras' Theorem.

Let z_1, z_2 and z_3 be complex numbers. Then

$$|z_3 - z_1| = |(z_3 - z_2) + (z_2 - z_1)| \leq |z_3 - z_2| + |z_2 - z_1|.$$

This important inequality is known as the *Triangle Inequality.* It corresponds to the geometric statement that the length of any side of a triangle in the Euclidean plane is less than or equal to the sum of the lengths of the other two sides.

1.4 Euclidean Spaces

We denote by $\mathbb{R}^n$ the set consisting of all n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers. The set $\mathbb{R}^n$ represents n-dimensional *Euclidean space* (with respect to the standard Cartesian coordinate system). Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$, where

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \quad \mathbf{y} = (y_1, y_2, \ldots, y_n),$$

and let λ be a real number. We define

$$\begin{aligned}
\mathbf{x}+\mathbf{y} &= (x_1+y_1, x_2+y_2, \ldots, x_n+y_n), \\
\mathbf{x}-\mathbf{y} &= (x_1-y_1, x_2-y_2, \ldots, x_n-y_n), \\
\lambda\mathbf{x} &= (\lambda x_1, \lambda x_2, \ldots, \lambda x_n), \\
\mathbf{x}\cdot\mathbf{y} &= x_1y_1 + x_2y_2 + \cdots + x_ny_n, \\
|\mathbf{x}| &= \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.
\end{aligned}$$

The quantity $\mathbf{x}\cdot\mathbf{y}$ is the *scalar product* (or *inner product*) of $\mathbf{x}$ and $\mathbf{y}$, and the quantity $|\mathbf{x}|$ is the *Euclidean norm* of $\mathbf{x}$. Note that $|\mathbf{x}|^2 = \mathbf{x}\cdot\mathbf{x}$. The *Euclidean distance* between two points $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{R}^n$ is defined to be the Euclidean norm $|\mathbf{y}-\mathbf{x}|$ of the vector $\mathbf{y}-\mathbf{x}$.

Lemma 1.2 (Schwarz' Inequality) *Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$. Then $|\mathbf{x}\cdot\mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$.*

Proof We note that $|\lambda\mathbf{x}+\mu\mathbf{y}|^2 \geq 0$ for all real numbers λ and μ. But

$$|\lambda\mathbf{x}+\mu\mathbf{y}|^2 = (\lambda\mathbf{x}+\mu\mathbf{y}).(\lambda\mathbf{x}+\mu\mathbf{y}) = \lambda^2|\mathbf{x}|^2 + 2\lambda\mu\mathbf{x}\cdot\mathbf{y} + \mu^2|\mathbf{y}|^2.$$

Therefore $\lambda^2|\mathbf{x}|^2 + 2\lambda\mu\mathbf{x}\cdot\mathbf{y} + \mu^2|\mathbf{y}|^2 \geq 0$ for all real numbers λ and μ. In particular, suppose that $\lambda = |\mathbf{y}|^2$ and $\mu = -\mathbf{x}\cdot\mathbf{y}$. We conclude that

$$|\mathbf{y}|^4|\mathbf{x}|^2 - 2|\mathbf{y}|^2(\mathbf{x}\cdot\mathbf{y})^2 + (\mathbf{x}\cdot\mathbf{y})^2|\mathbf{y}|^2 \geq 0,$$

so that $\left(|\mathbf{x}|^2|\mathbf{y}|^2 - (\mathbf{x}\cdot\mathbf{y})^2\right)|\mathbf{y}|^2 \geq 0$. Thus if $\mathbf{y} \neq \mathbf{0}$ then $|\mathbf{y}| > 0$, and hence

$$|\mathbf{x}|^2|\mathbf{y}|^2 - (\mathbf{x}\cdot\mathbf{y})^2 \geq 0.$$

But this inequality is trivially satisfied when $\mathbf{y} = \mathbf{0}$. Thus $|\mathbf{x}\cdot\mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$, as required. ∎

It follows easily from Schwarz' Inequality that $|\mathbf{x}+\mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. For

$$\begin{aligned}
|\mathbf{x}+\mathbf{y}|^2 &= (\mathbf{x}+\mathbf{y}).(\mathbf{x}+\mathbf{y}) = |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2\mathbf{x}\cdot\mathbf{y} \\
&\leq |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2|\mathbf{x}||\mathbf{y}| = (|\mathbf{x}| + |\mathbf{y}|)^2.
\end{aligned}$$

It follows that

$$|\mathbf{z}-\mathbf{x}| \leq |\mathbf{z}-\mathbf{y}| + |\mathbf{y}-\mathbf{x}|$$

for all points $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ of $\mathbb{R}^n$. This important inequality is known as the *Triangle Inequality*. It expresses the geometric fact the the length of any triangle in a Euclidean space is less than or equal to the sum of the lengths of the other two sides.

1.5 Convergence of Sequences in Euclidean Spaces

Definition Let n be a positive integer, and let $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots$ be an infinite sequence of points in n-dimensional Euclidean space $\mathbb{R}^n$. This sequence of points is said to *converge* to some point $\mathbf{r}$ of $\mathbb{R}^n$ if, given any real number ε satisfying $\varepsilon > 0$, there exists some positive integer N such that $|\mathbf{p}_j - \mathbf{r}| < \varepsilon$ whenever $j \geq N$.

Lemma 1.3 *Let $\mathbf{p}$ be a point of $\mathbb{R}^n$, where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. Then a sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ converges to $\mathbf{p}$ if and only if the ith components of the elements of this sequence converge to p_i for $i = 1, 2, \ldots, n$.*

Proof Let x_{ji} and p_i denote the ith components of $\mathbf{x}_j$ and $\mathbf{p}$. Then $|x_{ji} - p_i| \leq |\mathbf{x}_j - \mathbf{p}|$ for all j. It follows directly from the definition of convergence that if $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$ then $x_{ji} \to p_i$ as $j \to +\infty$.

Conversely suppose that, for each i, $x_{ji} \to p_i$ as $j \to +\infty$. Let $\varepsilon > 0$ be given. Then there exist natural numbers $N_1, N_2, \ldots, N_n$ such that $|x_{ji} - p_i| < \varepsilon/\sqrt{n}$ whenever $j \geq N_i$. Let N be the maximum of $N_1, N_2, \ldots, N_n$. If $j \geq N$ then

$$|\mathbf{x}_j - \mathbf{p}|^2 = \sum_{i=1}^{n} (x_{ji} - p_i)^2 < n(\varepsilon/\sqrt{n})^2 = \varepsilon^2,$$

so that $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$. ∎

1.6 The Bolzano-Weierstrass Theorem

We say that an infinite sequence $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots$ of points in $\mathbb{R}^n$ is *bounded* if there exists some positive real number R such that $|\mathbf{p}_j| \leq R$ for all positive integers j.

Let $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots$ be an infinite sequence of points in $\mathbb{R}^n$. A *subsequence* of this sequence is a sequence that is of the form $\mathbf{p}_{m_1}, \mathbf{p}_{m_2}, \mathbf{p}_{m_3}, \ldots$, where $m_1, m_2, m_3, \ldots$ are natural numbers satisfying $m_1 < m_2 < m_3 < \cdots$. Thus, for example, $\mathbf{p}_2, \mathbf{p}_4, \mathbf{p}_6, \ldots$ and $\mathbf{p}_1, \mathbf{p}_4, \mathbf{p}_9, \ldots$ are subsequences of the given sequence.

The following theorem may be regarded as the Bolzano-Weierstrass Theorem in n dimensions.

Theorem 1.4 *Every bounded sequence of points in $\mathbb{R}^n$ has a convergent subsequence.*

Proof Let $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots$ be a bounded sequence of points in $\mathbb{R}^n$.

Given any point $\mathbf{b}$ of $\mathbb{R}^n$, where $\mathbf{b} = (b_1, b_2, \ldots, b_n)$, and given any positive real number L, let us denote by $H(\mathbf{b}, L)$ the hypercube in $\mathbb{R}^n$ consisting of those points $(x_1, x_2, \ldots, x_n)$ of $\mathbb{R}^n$ whose coordinates satisfy the inequalities $b_i \leq x_i \leq b_i + L$ for $i = 1, 2, \ldots, n$. This hypercube may be decomposed as the union of 2^n hypercubes whose sides are of length $\frac{1}{2}L$: these smaller hypercubes are the sets of the form $H(\mathbf{c}, \frac{1}{2}L)$, where $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ and, for each i, either $c_i = b_i$ or else $c_i = b_i + \frac{1}{2}L$. Now if the larger hypercube $L(\mathbf{b}, L)$ contains infinitely many members of the sequence $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots$ then at least one of the smaller hypercubes into which it is decomposed must also contain infinitely many members of this sequence.

Let $\mathbf{v}_1$ be a point of $\mathbb{R}^n$ and let M be a positive real number chosen such that the hypercube $H(\mathbf{v}_1, M)$ contains all the members of the sequence $(\mathbf{p}_j : j \in \mathbb{N})$. (The boundedness of this sequence ensures that such a hypercube exists.) Then there exists in infinite sequence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ of points of $\mathbb{R}^n$ such that, for each integer j satisfying $j > 1$, the hypercube H_j defined by $H_j = H(\mathbf{v}_j, M/2^{j-1})$ contains infinitely many members of the sequence $(\mathbf{p}_j : j \in \mathbb{N})$, has sides of length $M/2^{j-1}$, and is contained in the hypercube H_{j-1}. Let $\mathbf{v}_j = (v_j^{(1)}, v_j^{(2)}, \ldots, v_j^{(n)})$ for each positive integer j. Then, for each integer i between 1 and n, the sequence $v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, \ldots$ is a bounded non-decreasing sequence of real numbers. Such a sequence is guaranteed to converge to some real number r_i (Theorem 1.1). Let $\mathbf{r} = (r_1, r_2, \ldots, r_n)$. Then the sequence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ converges to the point $\mathbf{r}$. We claim that the sequence $(\mathbf{p}_j : j \in \mathbb{N})$ has a subsequence that converges to this point $\mathbf{r}$.

We claim that, for every real number ε satisfying $\varepsilon > 0$, there are infinitely many members $\mathbf{p}_j$ of the sequence $(\mathbf{p}_j : j \in \mathbb{N})$ satisfying the inequality $|\mathbf{p}_j - \mathbf{r}| < \varepsilon$. Now, given $\varepsilon > 0$, there exists some integer j such that $M\sqrt{n}/2^{j-1} < \frac{1}{2}\varepsilon$ and $|\mathbf{v}_j - \mathbf{r}| < \frac{1}{2}\varepsilon$. Now if $\mathbf{x}$ is a point of the hypercube H_j then

$$|\mathbf{x} - \mathbf{v}_j| \leq \frac{M\sqrt{n}}{2^{j-1}} < \frac{1}{2}\varepsilon$$

and therefore

$$|\mathbf{x} - \mathbf{r}| \leq |\mathbf{x} - \mathbf{v}_j| + |\mathbf{v}_j - \mathbf{r}| < \varepsilon.$$

But the hypercube H_j contains infinitely many members of the sequence $(\mathbf{p}_j : j \in \mathbb{N})$. Thus infinitely many members $\mathbf{p}_j$ of this sequence satisfy the inequality $|\mathbf{p}_j - \mathbf{r}| < \varepsilon$. It follows that there exists a subsequence $\mathbf{p}_{m_1}, \mathbf{p}_{m_2}, \mathbf{p}_{m_3}, \ldots.$ of the given sequence such that $m_{j+1} > m_j$ and $|\mathbf{p}_{m_j} - \mathbf{r}| < 1/j$ for each positive integer j. This subsequence converges to the point $\mathbf{r}$. ∎

Each complex number determines a point of the Euclidean plane $\mathbb{R}^2$ whose Cartesian coordinates are the real and imaginary parts of the complex number. Accordingly a sequence of complex numbers corresponds to a sequence of points in $\mathbb{R}^2$, and converges if and only if the corresponding sequence of points in $\mathbb{R}^2$ converges. The following theorem is thus a special case of Theorem 1.4.

Theorem 1.5 *Every bounded sequence of complex numbers has a convergent subsequence.*

1.7 Cauchy Sequences

Definition A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in a Euclidean space is said to be a *Cauchy sequence* if, given any $\varepsilon > 0$, there exists some natural number N such that $|\mathbf{x}_j - \mathbf{x}_k| < \varepsilon$ for all integers j and k satisfying $j \geq N$ and $k \geq N$.

Lemma 1.6 *Every convergent sequence in a Euclidean space is a Cauchy sequence.*

Proof Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points in a Euclidean space $\mathbb{R}^n$ which converges to some point $\mathbf{p}$ of $\mathbb{R}^n$. Given any $\varepsilon > 0$, there exists some natural number N such that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon/2$ whenever $j \geq N$. But then it follows from the Triangle Inequality that

$$|\mathbf{x}_j - \mathbf{x}_k| \leq |\mathbf{x}_j - \mathbf{p}| + |\mathbf{p} - \mathbf{x}_k| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $j \geq N$ and $k \geq N$. ∎

Theorem 1.7 *Every Cauchy sequence in $\mathbb{R}^n$ converges to some point of $\mathbb{R}^n$.*

Proof Let $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots$ be a Cauchy sequence in $\mathbb{R}^n$. Then, given any real number ε satisfying $\varepsilon > 0$, there exists some natural number N such that $|\mathbf{p}_j - \mathbf{p}_k| < \varepsilon$ whenever $j \geq N$ and $k \geq N$. In particular, there exists some natural number L such that $|\mathbf{p}_j - \mathbf{p}_k| < 1$ whenever $j \geq L$ and $k \geq L$. Let R be the maximum of the numbers $|\mathbf{p}_1|, |\mathbf{p}_2|, \ldots, |\mathbf{p}_{L-1}|$ and $|\mathbf{p}_L| + 1$. Then $|\mathbf{p}_j| \leq R$ whenever $j < L$. Moreover if $j \geq L$ then

$$|\mathbf{p}_j| \leq |\mathbf{p}_L| + |\mathbf{p}_j - \mathbf{p}_L| < |\mathbf{p}_L| + 1 \leq R.$$

Thus $|\mathbf{p}_j| \leq R$ for all positive integers j. We conclude that the Cauchy sequence $(\mathbf{p}_j : j \in N)$ is bounded.

It now follows from the Bolzano-Weierstrass theorem in n dimensions (Theorem 1.4) that the Cauchy sequence $(\mathbf{p}_j : j \in N)$ has a convergent subsequence $(\mathbf{p}_{k_j} : j \in N)$, where $k_1, k_2, k_3, \ldots$ are positive integers satisfying $k_1 < k_2 < k_3 < \cdots$. Let the point $\mathbf{q}$ of $\mathbb{R}^n$ be the limit of this subsequence. Then, given any positive number ε satisfying $\varepsilon > 0$, there exists some positive integer M such that $|\mathbf{p}_{k_m} - \mathbf{q}| < \frac{1}{2}\varepsilon$ whenever $m \geq M$. Also there exists some positive integer N such that $|\mathbf{p}_j - \mathbf{p}_k| < \frac{1}{2}\varepsilon$ whenever $j \geq N$ and $k \geq N$. Choose m large enough to ensure that $m \geq M$ and $k_m \geq N$. If $j \geq N$ then

$$|\mathbf{p}_j - \mathbf{q}| \leq |\mathbf{p}_j - \mathbf{p}_{k_m}| + |\mathbf{p}_{k_m} - \mathbf{q}| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

It follows that the Cauchy sequence $(\mathbf{p}_j : j \in N)$ converges to the point $\mathbf{q}$. Thus every Cauchy sequence is convergent, as required. ∎

Theorem 1.8 (Cauchy's Criterion for Convergence) *A sequence of complex numbers is convergent if and only if it is a Cauchy sequence.*

Proof Every convergent sequence of complex numbers is a Cauchy sequence (Lemma 1.6). Now every complex number corresponds to a point of the Euclidean plane $\mathbb{R}^2$. Moreover a sequence of complex numbers is a Cauchy sequence if and only if the corresponding sequence of points of $\mathbb{R}^2$ is a Cauchy sequence. The result is therefore follows from Theorem 1.7. ∎

1.8 Continuity

Definition Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. A function $f: X \to Y$ from X to Y is said to be *continuous* at a point $\mathbf{p}$ of X if and only if the following criterion is satisfied:—

> given any real number ε satisfying $\varepsilon > 0$ there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all points $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta$.

The function $f: X \to Y$ is said to be continuous on X if and only if it is continuous at every point $\mathbf{p}$ of X.

Lemma 1.9 *The functions $s: \mathbb{R}^2 \to \mathbb{R}$ and $p: \mathbb{R}^2 \to \mathbb{R}$ defined by $s(x, y) = x + y$ and $p(x, y) = xy$ are continuous.*

Proof Let $(u, v) \in \mathbb{R}^2$. We first show that $s: \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v). Let $\varepsilon > 0$ be given. Let $\delta = \frac{1}{2}\varepsilon$. If (x, y) is any point of $\mathbb{R}^2$ whose distance from (u, v) is less than δ then $|x - u| < \delta$ and $|y - v| < \delta$, and hence

$$|s(x, y) - s(u, v)| = |x + y - u - v| \leq |x - u| + |y - v| < 2\delta = \varepsilon.$$

This shows that $s: \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v).

Next we show that $p: \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v). Now

$$p(x, y) - p(u, v) = xy - uv = (x - u)(y - v) + u(y - v) + (x - u)v$$

for all points (x, y) of $\mathbb{R}^2$. Thus if the distance from (x, y) to (u, v) is less than δ then $|x - u| < \delta$ and $|y - v| < \delta$, and hence $|p(x, y) - p(u, v)| < \delta^2 + (|u| + |v|)\delta$. Let $\varepsilon > 0$ is given. If $\delta > 0$ is chosen to be the minimum of 1 and $\varepsilon/(1 + |u| + |v|)$ then $\delta^2 + (|u| + |v|)\delta < (1 + |u| + |v|)\delta < \varepsilon$, and thus $|p(x, y) - p(u, v)| < \varepsilon$ for all points (x, y) of $\mathbb{R}^2$ whose distance from (u, v) is less than δ. This shows that $p: \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v). ∎

Lemma 1.10 *Let X, Y and Z be subsets of $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^k$ respectively, let $f: X \to Y$ be a function mapping X into Y, and let $g: Y \to Z$ be a function mapping Y into Z. Let $\mathbf{p}$ be a point of X. Suppose that f is continuous at $\mathbf{p}$ and g is continuous at $f(\mathbf{p})$. Then the composition function $g \circ f: X \to Z$ is continuous at $\mathbf{p}$.*

Proof Let $\varepsilon > 0$ be given. Then there exists some $\eta > 0$ such that $|g(\mathbf{y}) - g(f(\mathbf{p}))| < \varepsilon$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y} - f(\mathbf{p})| < \eta$. But then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \eta$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that $|g(f(\mathbf{x})) - g(f(\mathbf{p}))| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, and thus $g \circ f$ is continuous at $\mathbf{p}$, as required. ∎

Lemma 1.10 guarantees that a composition of continuous functions between subsets of Euclidean spaces is continuous.

1.9 Convergent Sequences and Continuous Functions

Lemma 1.11 *Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f: X \to Y$ be a continuous function from X to Y. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of X which converges to some point $\mathbf{p}$ of X. Then the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$.*

Proof Let $\varepsilon > 0$ be given. Then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, since the function f is continuous at $\mathbf{p}$. Also there exists some natural number N such that $|\mathbf{x}_j - \mathbf{p}| < \delta$ whenever $j \geq N$, since the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to $\mathbf{p}$. Thus if $j \geq N$ then $|f(\mathbf{x}_j) - f(\mathbf{p})| < \varepsilon$. Thus the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$, as required. ∎

Proposition 1.12 *Let $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ be convergent infinite sequences of complex numbers. Then the sum, difference and product of these sequences are convergent, and*

$$\lim_{j \to +\infty} (a_j + b_j) = \lim_{j \to +\infty} a_j + \lim_{j \to +\infty} b_j,$$

$$
\begin{aligned}
\lim_{j\to+\infty}(a_j - b_j) &= \lim_{j\to+\infty} a_j - \lim_{j\to+\infty} b_j, \\
\lim_{j\to+\infty}(a_j b_j) &= \left(\lim_{j\to+\infty} a_j\right)\left(\lim_{j\to+\infty} b_j\right).
\end{aligned}
$$

If in addition $b_j \neq 0$ for all n and $\lim_{j\to+\infty} b_j \neq 0$, then the quotient of the sequences (a_j) and (b_j) is convergent, and

$$
\lim_{j\to+\infty} \frac{a_j}{b_j} = \frac{\lim\limits_{j\to+\infty} a_j}{\lim\limits_{j\to+\infty} b_j}.
$$

Proof Throughout this proof let $l = \lim_{j\to+\infty} a_j$ and $m = \lim_{j\to+\infty} b_j$.

First of all, consider the case when a_j and b_j are real numbers for all positive integers j. Then $a_j + b_j = s(a_j, b_j)$ and $a_j b_j = p(a_j, b_j)$, where $s\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $p\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are the functions given by $s(x,y) = x+y$ and $p(x,y) = xy$ for all real numbers x and y. Now the sequence $((a_j, b_j) : j \in \mathbb{N})$ is a sequence of points in $\mathbb{R}^2$ which converges to the point (l,m), since its components are sequences of real numbers converging to the limits l and m (Lemma 1.3). Also the functions s and p are continuous (Lemma 1.9). It now follows from Lemma 1.11 that

$$
\lim_{j\to+\infty}(a_j + b_j) = \lim_{j\to+\infty} s(a_j, b_j) = s\left(\lim_{j\to+\infty}(a_j, b_j)\right) = s(l,m) = l + m,
$$

and

$$
\lim_{j\to+\infty}(a_j b_j) = \lim_{j\to+\infty} p(a_j, b_j) = p\left(\lim_{j\to+\infty}(a_j, b_j)\right) = p(l,m) = lm.
$$

Also the sequence $(-b_j : j \in \mathbb{N})$ converges to $-m$, and therefore $\lim_{j\to+\infty}(a_j - b_j) = l - m$. Now the reciprocal function $r\colon \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is continuous on $\mathbb{R}\setminus\{0\}$, where $r(x) = 1/x$ for all non-zero real numbers x. It follows from Lemma 1.11 that if $b_j \neq 0$ for all positive integers j then $1/b_j$ converges to $1/m$ as $j \to +\infty$. But then $\lim_{j\to+\infty}(a_j/b_j) = l/m$. This completes the proof of Proposition 1.12 in the case when $(a_j : j \in \mathbb{N})$ and $(b_j : j \in \mathbb{N})$ are convergent sequences of real numbers.

The result for convergent sequences of complex numbers now follows easily on considering the real and imaginary parts of the sequences involved and using the result that a sequence of complex numbers converges to a complex number $u + iv$, where u and v are real numbers, if and only if the real parts of those complex numbers converge to u and the imaginary parts converge to v. ∎

1.10 Components of Continuous Functions

Let $f\colon X \to \mathbb{R}^n$ be a function mapping a mapping a set X into n-dimensional Euclidean space $\mathbb{R}^n$. Then

$$f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$$

for all $x \in X$, where $f_1, f_2, \ldots, f_n$ are functions from X to $\mathbb{R}$, referred to as the *components* of the function f.

Proposition 1.13 *Let X be a subset of some Euclidean space, and let $\mathbf{p}$ be a point of X. A function $f\colon X \to \mathbb{R}^n$ mapping X into the Euclidean space $\mathbb{R}^n$ is continuous at $\mathbf{p}$ if and only if its components are continuous at $\mathbf{p}$.*

Proof Note that the ith component f_i of f is given by $f_i = p_i \circ f$, where $p_i\colon \mathbb{R}^n \to \mathbb{R}$ is the continuous function which maps $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ onto its ith coordinate y_i. It therefore follows immediately from Lemma 1.10 that if f is continuous the point $\mathbf{p}$, then so are the components of f.

Conversely suppose that the components of f are continuous at $\mathbf{p} \in X$. Let $\varepsilon > 0$ be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots, \delta_n$ such that $|f_i(\mathbf{x}) - f_i(\mathbf{p})| < \varepsilon/\sqrt{n}$ for $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_i$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_n$. If $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then

$$|f(\mathbf{x}) - f(\mathbf{p})|^2 = \sum_{i=1}^{n} |f_i(\mathbf{x}) - f_i(\mathbf{p})|^2 < \varepsilon^2,$$

and hence $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$. Thus the function f is continuous at $\mathbf{p}$, as required. ∎

Corollary 1.14 *Let $f\colon X \to \mathbb{C}$ be a complex-valued function defined on a subset X of some Euclidean space. Then the function f is continuous if and only if the real and imaginary parts of f are continuous real-valued functions on X.*

Proposition 1.15 *Let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be real-valued functions defined on some subset X of a Euclidean space, and let $\mathbf{p}$ be a point of X. Suppose that the functions f and g are continuous at the point $\mathbf{p}$. Then so are the functions $f + g$, $f - g$ and $f \cdot g$. If in addition $g(x) \neq 0$ for all $x \in X$ then the quotient function f/g is continuous at $\mathbf{p}$.*

Proof Note that $f + g = s \circ h$ and $f \cdot g = p \circ h$, where $h\colon X \to \mathbb{R}^2$, $s\colon \mathbb{R}^2 \to \mathbb{R}$ and $p\colon \mathbb{R}^2 \to \mathbb{R}$ are given by $h(x) = (f(x), g(x))$, $s(u, v) = u + v$ and $p(u, v) = uv$ for all $x \in X$ and $u, v \in \mathbb{R}$. If the functions f and g are continuous at $\mathbf{p}$ then so is the function h (Proposition 1.13). The functions s and p are continuous on $\mathbb{R}^2$. It therefore follows from Lemma 1.10 that the composition functions $s \circ h$ and $p \circ h$ are continuous at $\mathbf{p}$. Thus the functions $f + g$ and $f \cdot g$ are continuous at $\mathbf{p}$. Now $f - g = f + (-g)$, and both f and $-g$ are continuous. Therefore $f - g$ is continuous.

Now suppose that $g(x) \neq 0$ for all $x \in X$. Note that $1/g = r \circ g$, where $r: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is the reciprocal function, defined by $r(t) = 1/t$. Now the reciprocal function r is continuous. It now follows on applying Lemma 1.10 that the function $1/g$ is continuous at $\mathbf{p}$. The function f/g, being the product of the functions f and $1/g$ is therefore continuous at $\mathbf{p}$. ∎

Proposition 1.16 *Let $f: X \to \mathbb{C}$ and $g: X \to \mathbb{C}$ be complex-valued functions defined on some subset X of a Euclidean space, and let $\mathbf{p}$ be a point of X. Suppose that the functions f and g are continuous at the point $\mathbf{p}$. Then so are the functions $f + g$, $f - g$ and $f \cdot g$. If in addition $g(x) \neq 0$ for all $x \in X$ then the quotient function f/g is continuous at $\mathbf{p}$.*

Proof Let $f(x) = u_1(x) + iv_1(x)$ and $g(x) = u_2 + iv_2(x)$ for all $x \in X$, where u_1, u_2, v_1 and v_2 are real valued functions on X. It follows from Corollary 1.14 that the functions u_1, u_2, v_1 and v_2 are all continuous at $\mathbf{p}$. Now

$$f + g = (u_1 + u_2) + i(v_1 + v_2), \quad f - g = (u_1 - u_2) + i(v_1 - v_2),$$

$$fg = (u_1 v_1 - u_2 v_2) + i(u_1 v_2 + u_2 v_1).$$

Moreover

$$1/g = (u_2 - iv_2)/(u_2^2 + v_2^2),$$

provided that the function g is non-zero at all points of its domain. It therefore follows from repeated applications of Proposition 1.15 that the real and imaginary parts of the functions $f + g$, $f - g$ and fg are continuous at $\mathbf{p}$. The same is true of the function $1/g$, provided that the function g is non-zero throughout its domain. It then follows from Corollary 1.14 that the functions $f + g$, $f - g$ and fg are themselves continuous at $\mathbf{p}$. Also f/g is continuous at $\mathbf{p}$, provided that the function g is non-zero throughout its domain. ∎

1.11 Limits of Functions

Let X be a subset of some Euclidean space $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of $\mathbb{R}^n$. We say that the point $\mathbf{p}$ is a *limit point* of X if, given any real number δ satisfying $\delta > 0$, there exists a point $\mathbf{x}$ of X satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta$. It follows easily from this that a point $\mathbf{p}$ of $\mathbb{R}^n$ is a limit point of X if and only if there exists a sequence of points of $X \setminus \{\mathbf{p}\}$ which converges to the point $\mathbf{p}$.

Definition Let X be a subset of a Euclidean space $\mathbb{R}^m$, let $f: X \to \mathbb{R}^n$ mapping X into a Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of X, and let $\mathbf{q}$ be a point of $\mathbb{R}^n$. We say that $\mathbf{q}$ is the *limit* of $f(\mathbf{x})$ as $\mathbf{x}$ tends to $\mathbf{p}$ in X if, given any real number ε satisfying $\varepsilon > 0$, there exists some real number δ satisfying $\delta > 0$ such that $|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$ for all points $\mathbf{x}$ of X satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta$. If the point $\mathbf{q}$ is the limit of $f(\mathbf{x})$ as $\mathbf{x}$ tends to $\mathbf{p}$ in X, then we denote this fact by writing: $\mathbf{q} = \lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x})$.

Lemma 1.17 *Let X and Y be subsets of Euclidean spaces, let $f: X \to Y$ be a function from X to Y, and let $\mathbf{p}$ be a point of X that is also limit point of X. Then the function f is continuous at $\mathbf{p}$ if and only if $\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = f(\mathbf{p})$.*

Proof The result follows immediately on comparing the definitions of convergence and of limits of functions. ∎

Let X be a subset of some Euclidean space. A point $\mathbf{p}$ of X is said to be an *isolated point* of X if it is not a limit point of X. A point $\mathbf{p}$ of X is an isolated point of X if and only if there exists some real number δ satisfying $\delta > 0$ such that the only point of X whose distance from $\mathbf{p}$ is less than δ is the point $\mathbf{p}$ itself. It follows directly from the definition of continuity that any function between subsets of Euclidean space is continuous at all the isolated points of its domain.

Lemma 1.18 *Let X, Y and Z be subsets of Euclidean spaces, let $\mathbf{p}$ be a limit point of X, and let $f: X \to Y$ and $g: Y \to Z$ be functions. Suppose that $\lim_{x \to p} f(x) = \mathbf{q}$. Suppose also that the function g is defined and is continuous at $\mathbf{q}$. Then $\lim_{x \to p} g(f(\mathbf{x})) = g(\mathbf{q})$.*

Proof The function g is continuous at $\mathbf{q}$. Therefore there exists some $\eta > 0$ such that $|g(\mathbf{y}) - g(\mathbf{q})| < \varepsilon$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y} - \mathbf{q}| < \eta$. But then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - \mathbf{q}| < \eta$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta$. Thus $|g(f(\mathbf{x})) - g(\mathbf{q})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta$, showing that $\lim_{\mathbf{x} \to \mathbf{p}} g(f(\mathbf{x})) = g(\mathbf{q})$, as required. ∎

Let X be a subset of some Euclidean space, let $f: X \to \mathbb{R}^n$ be a function mapping X into n-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of the set X, and let $\mathbf{q}$ be a point in $\mathbb{R}^n$. Let $\tilde{f}: X \cup \{\mathbf{p}\} \to \mathbb{R}^n$ be the function on $X \cup \{\mathbf{p}\}$ defined such that

$$\tilde{f}(\mathbf{x}) = \begin{cases} f(\mathbf{x}) & \text{if } \mathbf{x} \in X \setminus \{\mathbf{p}\}; \\ \mathbf{q} & \text{if } \mathbf{x} = \mathbf{p}. \end{cases}$$

Then $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$ if and only if the function $\tilde{f}$ is continuous at $\mathbf{p}$. This enables one to deduce basic results concerning limits of functions from the corresponding results concerning continuity of functions.

The following result is thus a conseqence of Proposition 1.13.

Proposition 1.19 *Let X be a subset of some Euclidean space, let $f\colon X \to \mathbb{R}^n$ be a function mapping X into n-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of the set X, and let $\mathbf{q}$ be a point in $\mathbb{R}^n$. Let the real-valued functions $f_1, f_2, \ldots, f_n$ be the components of f, so that*

$$f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))$$

for all $\mathbf{x} \in X$, and let $\mathbf{q} = (q_1, q_2, \ldots, q_n)$. Then $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$ if and only if $\lim_{\mathbf{x}\to\mathbf{p}} f_i(\mathbf{x}) = q_i$ for $i = 1, 2, \ldots, n$.

The following result is a consequence of Proposition 1.16.

Proposition 1.20 *Let X be a subset of some Euclidean space, let $\mathbf{p}$ be a limit point of X, and let $f\colon X \to \mathbb{R}^n$ and $g\colon X \to \mathbb{R}^n$ be functions on X taking values in some Euclidean space $\mathbb{R}^n$. Suppose that the limits $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})$ and $\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})$ exist. Then*

$$\begin{aligned}
\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) + g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) + \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}), \\
\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) - g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) - \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}), \\
\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) g(\mathbf{x})) &= \left(\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})\right) \left(\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})\right).
\end{aligned}$$

If moreover $\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}) \neq 0$ and the function g is non-zero throughout its domain X then

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})}{\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})}.$$

1.12 The Intermediate Value Theorem

Proposition 1.21 *Let $f\colon [a, b] \to \mathbb{Z}$ continuous integer-valued function defined on a closed interval $[a, b]$. Then the function f is constant.*

Proof Let

$$S = \{x \in [a,b] : f \text{ is constant on the interval } [a,x]\},$$

and let $s = \sup S$. Now $s \in [a,b]$, and therefore the function f is continuous at s. Therefore there exists some real number δ satisfying $\delta > 0$ such that $|f(x) - f(s)| < \frac{1}{2}$ for all $x \in [a,b]$ satisfying $|x - s| < \delta$. But the function f is integer-valued. It follows that $f(x) = f(s)$ for all $x \in [a,b]$ satisfying $|x - s| < \delta$. Now $s - \delta$ is not an upper bound for the set S. Therefore there exists some element x_0 of S satisfying $s - \delta < x_0 \leq s$. But then $f(s) = f(x_0) = f(a)$, and therefore the function f is constant on the interval $[a,x]$ for all $x \in [a,b]$ satisfying $s \leq x < s + \delta$. Thus $x \in [a,b] \cap [s, s+\delta) \subset S$. In particular $s \in S$. Now S cannot contain any elements x of $[a,b]$ satisfying $x > s$. Therefore $[a,b] \cap [s, s+\delta) = \{s\}$, and therefore $s = b$. This shows that $b \in S$, and thus the function f is constant on the interval $[a,b]$, as required. ∎

Theorem 1.22 (The Intermediate Value Theorem) *Let a and b be real numbers satisfying $a < b$, and let $f\colon [a,b] \to \mathbb{R}$ be a continuous function defined on the interval $[a,b]$. Let c be a real number which lies between $f(a)$ and $f(b)$ (so that either $f(a) \leq c \leq f(b)$ or else $f(a) \geq c \geq f(b)$.) Then there exists some $s \in [a,b]$ for which $f(s) = c$.*

Proof Let c be a real number which lies between $f(a)$ and $f(b)$, and let $g_c\colon \mathbb{R} \setminus \{c\} \to \mathbb{Z}$ be the continuous integer-valued function on $\mathbb{R} \setminus \{c\}$ defined such that $g_c(x) = 0$ whenever $x < c$ and $g_c(x) = 1$ if $x > c$. Suppose that c were not in the range of the function f. Then the composition function $g_c \circ f\colon [a,b] \to \mathbb{R}$ would be a continuous integer-valued function defined throughout the interval $[a,b]$. This function would not be constant, since $g_c(f(a)) \neq g_c(f(b))$. But every continuous integer-valued function on the interval $[a,b]$ is constant (Proposition 1.21). It follows that every real number c lying between $f(a)$ and $f(b)$ must belong to the range of the function f, as required. ∎

Corollary 1.23 *Let $f\colon [a,b] \to [c,d]$ be a strictly increasing continuous function mapping an interval $[a,b]$ into an interval $[c,d]$, where a, b, c and d are real numbers satisfying $a < b$ and $c < d$. Suppose that $f(a) = c$ and $f(b) = d$. Then the function f has a continuous inverse $f^{-1}\colon [c,d] \to [a,b]$.*

Proof Let x_1 and x_2 be distinct real numbers belonging to the interval $[a,b]$ then either $x_1 < x_2$, in which case $f(x_1) < f(x_2)$ or $x_1 > x_2$, in which case $f(x_1) > f(x_2)$. Thus $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. It follows that the function f is injective. The Intermediate Value Theorem (Theorem 1.22) ensures that f is surjective. It follows that the function f has a well-defined inverse $f^{-1}\colon [c,d] \to [a,b]$. It only remains to show that this inverse function is continuous.

Let y be a real number satisfying $c < y < d$, and let x be the unique real number such that $a < x < b$ and $f(x) = y$. Let $\varepsilon > 0$ be given. We can then choose $x_1, x_2 \in [a,b]$ such that $x - \varepsilon < x_1 < x < x_2 < x + \varepsilon$. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then $y_1 < y < y_2$. Choose $\delta > 0$ such that $\delta < y - y_1$ and $\delta < y_2 - y$. If $v \in [c,d]$ satisfies $|v - y| < \delta$ then $y_1 < v < y_2$ and therefore $x_1 < f^{-1}(v) < x_2$. But then $|f^{-1}(v) - f^{-1}(y)| < \varepsilon$. We conclude that the function $f^{-1}\colon [c,d] \to [a,b]$ is continuous at all points in the interior of the interval $[a,b]$. A similar argument shows that it is continuous at the endpoints of this interval. Thus the function f has a continuous inverse, as required. ∎

1.13 Uniform Convergence

Definition Let X be a subset of some Euclidean spaces, and let $f_1, f_2, f_3, \ldots$ be a sequence of functions mapping X into some Euclidean space $\mathbb{R}^n$. The sequence (f_j) is said to converge *uniformly* to a function $f\colon X \to \mathbb{R}^n$ on X as $j \to +\infty$ if, given any real number ε satisfying $\varepsilon > 0$, there exists some positive integer N such that $|f_j(\mathbf{x}) - f(\mathbf{x})| < \varepsilon$ for all $\mathbf{x} \in X$ and for all integers j satisfying $j \geq N$ (where the value of N is independent of $\mathbf{x}$).

Theorem 1.24 *Let $f_1, f_2, f_3, \ldots$ be a sequence of continuous functions mapping some subset X of a Euclidean space into $\mathbb{R}^n$. Suppose that this sequence converges uniformly on X to some function $f\colon X \to \mathbb{R}^n$. Then this limit function f is continuous.*

Proof Let $\mathbf{p}$ be an element of X, and let $\varepsilon > 0$ be given. If j is chosen sufficiently large then $|f(\mathbf{x}) - f_j(\mathbf{x})| < \frac{1}{3}\varepsilon$ for all $\mathbf{x} \in X$, since $f_j \to f$ uniformly on X as $j \to +\infty$. It then follows from the continuity of f_j that there exists some $\delta > 0$ such that $|f_j(\mathbf{x}) - f_j(\mathbf{p})| < \frac{1}{3}\varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. But then

$$\begin{aligned} |f(\mathbf{x}) - f(\mathbf{p})| &\leq |f(\mathbf{x}) - f_j(\mathbf{x})| + |f_j(\mathbf{x}) - f_j(\mathbf{p})| + |f_j(\mathbf{p}) - f(\mathbf{p})| \\ &< \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon \end{aligned}$$

whenever $|\mathbf{x} - \mathbf{p}| < \delta$. Thus the function f is continuous at $\mathbf{p}$, as required. ∎

1.14 Open Sets in Euclidean Spaces

Let X be a subset of $\mathbb{R}^n$. Given a point $\mathbf{p}$ of X and a non-negative real number r, the *open ball* $B_X(\mathbf{p}, r)$ in X of *radius* r about $\mathbf{p}$ is defined to be the subset of X given by

$$B_X(\mathbf{p}, r) \equiv \{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < r\}.$$

(Thus $B_X(\mathbf{p}, r)$ is the set consisting of all points of X that lie within a sphere of radius r centred on the point $\mathbf{p}$.)

Definition Let X be a subset of $\mathbb{R}^n$. A subset V of X is said to be *open* in X if and only if, given any point $\mathbf{p}$ of V, there exists some $\delta > 0$ such that $B_X(\mathbf{p}, \delta) \subset V$.

By convention, we regard the empty set $\emptyset$ as being an open subset of X. (The criterion given above is satisfied vacuously in the case when V is the empty set.)

In particular, a subset V of $\mathbb{R}^n$ is said to be an *open set* (in $\mathbb{R}^n$) if and only if, given any point $\mathbf{p}$ of V, there exists some $\delta > 0$ such that $B(\mathbf{p}, \delta) \subset V$, where $B(\mathbf{p}, r) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < r\}$.

Example Let $H = \{(x, y, z) \in \mathbb{R}^3 : z > c\}$, where c is some real number. Then H is an open set in $\mathbb{R}^3$. Indeed let $\mathbf{p}$ be a point of H. Then $\mathbf{p} = (u, v, w)$, where $w > c$. Let $\delta = w - c$. If the distance from a point (x, y, z) to the point (u, v, w) is less than δ then $|z - w| < \delta$, and hence $z > c$, so that $(x, y, z) \in H$. Thus $B(\mathbf{p}, \delta) \subset H$, and therefore H is an open set.

The previous example can be generalized. Given any integer i between 1 and n, and given any real number c_i, the sets

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i > c_i\}, \qquad \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i < c_i\}$$

are open sets in $\mathbb{R}^n$.

Example Let U be an open set in $\mathbb{R}^n$. Then for any subset X of $\mathbb{R}^n$, the intersection $U \cap X$ is open in X. (This follows directly from the definitions.) Thus for example, let S^2 be the unit sphere in $\mathbb{R}^3$, given by

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

and let N be the subset of S^2 given by

$$N = \{(x, y, z) \in \mathbb{R}^n : x^2 + y^2 + z^2 = 1 \text{ and } z > 0\}.$$

Then N is open in S^2, since $N = H \cap S^2$, where H is the open set in $\mathbb{R}^3$ given by

$$H = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}.$$

Note that N is not itself an open set in $\mathbb{R}^3$. Indeed the point $(0, 0, 1)$ belongs to N, but, for any $\delta > 0$, the open ball (in $\mathbb{R}^3$ of radius δ about $(0, 0, 1)$ contains points (x, y, z) for which $x^2 + y^2 + z^2 \neq 1$. Thus the open ball of radius δ about the point $(0, 0, 1)$ is not a subset of N.

Lemma 1.25 *Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any positive real number r, the open ball $B_X(\mathbf{p}, r)$ in X of radius r about $\mathbf{p}$ is open in X.*

Proof Let $\mathbf{x}$ be an element of $B_X(\mathbf{p}, r)$. We must show that there exists some $\delta > 0$ such that $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{p}, r)$. Let $\delta = r - |\mathbf{x} - \mathbf{p}|$. Then $\delta > 0$, since $|\mathbf{x} - \mathbf{p}| < r$. Moreover if $\mathbf{y} \in B_X(\mathbf{x}, \delta)$ then

$$|\mathbf{y} - \mathbf{p}| \leq |\mathbf{y} - \mathbf{x}| + |\mathbf{x} - \mathbf{p}| < \delta + |\mathbf{x} - \mathbf{p}| = r,$$

by the Triangle Inequality, and hence $\mathbf{y} \in B_X(\mathbf{p}, r)$. Thus $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{p}, r)$. This shows that $B_X(\mathbf{p}, r)$ is an open set, as required. ∎

Lemma 1.26 *Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any non-negative real number r, the set $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| > r\}$ is an open set in X.*

Proof Let $\mathbf{x}$ be a point of X satisfying $|\mathbf{x} - \mathbf{p}| > r$, and let $\mathbf{y}$ be any point of X satisfying $|\mathbf{y} - \mathbf{x}| < \delta$, where $\delta = |\mathbf{x} - \mathbf{p}| - r$. Then

$$|\mathbf{x} - \mathbf{p}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{p}|,$$

by the Triangle Inequality, and therefore

$$|\mathbf{y} - \mathbf{p}| \geq |\mathbf{x} - \mathbf{p}| - |\mathbf{y} - \mathbf{x}| > |\mathbf{x} - \mathbf{p}| - \delta = r.$$

Thus $B_X(\mathbf{x}, \delta)$ is contained in the given set. The result follows. ∎

Proposition 1.27 *Let X be a subset of $\mathbb{R}^n$. The collection of open sets in X has the following properties:—*

(i) *the empty set $\emptyset$ and the whole set X are both open in X;*

(ii) *the union of any collection of open sets in X is itself open in X;*

(iii) *the intersection of any* finite *collection of open sets in* X *is itself open in* X.

Proof The empty set $\emptyset$ is an open set by convention. Moreover the definition of an open set is satisfied trivially by the whole set X. This proves (i).

Let $\mathcal{A}$ be any collection of open sets in X, and let U denote the union of all the open sets belonging to $\mathcal{A}$. We must show that U is itself open in X. Let $\mathbf{x} \in U$. Then $\mathbf{x} \in V$ for some set V belonging to the collection $\mathcal{A}$. It follows that there exists some $\delta > 0$ such that $B_X(\mathbf{x}, \delta) \subset V$. But $V \subset U$, and thus $B_X(\mathbf{x}, \delta) \subset U$. This shows that U is open in X. This proves (ii).

Finally let $V_1, V_2, V_3, \ldots, V_k$ be a *finite* collection of subsets of X that are open in X, and let V denote the intersection $V_1 \cap V_2 \cap \cdots \cap V_k$ of these sets. Let $\mathbf{x} \in V$. Now $\mathbf{x} \in V_j$ for $j = 1, 2, \ldots, k$, and therefore there exist strictly positive real numbers $\delta_1, \delta_2, \ldots, \delta_k$ such that $B_X(\mathbf{x}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. (This is where we need the fact that we are dealing with a finite collection of sets.) Now $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{x}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$, and thus $B_X(\mathbf{x}, \delta) \subset V$. Thus the intersection V of the sets $V_1, V_2, \ldots, V_k$ is itself open in X. This proves (iii). ∎

Example The set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 4 \text{ and } z > 1\}$ is an open set in $\mathbb{R}^3$, since it is the intersection of the open ball of radius 2 about the origin with the open set $\{(x, y, z) \in \mathbb{R}^3 : z > 1\}$.

Example The set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 4 \text{ or } z > 1\}$ is an open set in $\mathbb{R}^3$, since it is the union of the open ball of radius 2 about the origin with the open set $\{(x, y, z) \in \mathbb{R}^3 : z > 1\}$.

Example The set

$$\{(x, y, z) \in \mathbb{R}^3 : (x - n)^2 + y^2 + z^2 < \tfrac{1}{4} \text{ for some } n \in \mathbb{Z}\}$$

is an open set in $\mathbb{R}^3$, since it is the union of the open balls of radius $\frac{1}{2}$ about the points $(n, 0, 0)$ for all integers n.

Example For each natural number k, let

$$V_k = \{(x, y, z) \in \mathbb{R}^3 : k^2(x^2 + y^2 + z^2) < 1\}.$$

Now each set V_k is an open ball of radius $1/k$ about the origin, and is therefore an open set in $\mathbb{R}^3$. However the intersection of the sets V_k for all natural numbers k is the set $\{(0, 0, 0)\}$, and thus the intersection of the sets V_k for all natural numbers k is not itself an open set in $\mathbb{R}^3$. This example demonstrates that infinite intersections of open sets need not be open.

Lemma 1.28 *A sequence* $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ *of points in* $\mathbb{R}^n$ *converges to a point* $\mathbf{p}$ *if and only if, given any open set* U *which contains* $\mathbf{p}$*, there exists some natural number* N *such that* $\mathbf{x}_j \in U$ *for all* j *satisfying* $j \geq N$.

Proof Suppose that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ has the property that, given any open set U which contains $\mathbf{p}$, there exists some natural number N such that $\mathbf{x}_j \in U$ whenever $j \geq N$. Let $\varepsilon > 0$ be given. The open ball $B(\mathbf{p}, \varepsilon)$ of radius ε about $\mathbf{p}$ is an open set by Lemma 1.25. Therefore there exists some natural number N such that $\mathbf{x}_j \in B(\mathbf{p}, \varepsilon)$ whenever $j \geq N$. Thus $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$. This shows that the sequence converges to $\mathbf{p}$.

Conversely, suppose that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to $\mathbf{p}$. Let U be an open set which contains $\mathbf{p}$. Then there exists some $\varepsilon > 0$ such that the open ball $B(\mathbf{p}, \varepsilon)$ of radius ε about $\mathbf{p}$ is a subset of U. Thus there exists some $\varepsilon > 0$ such that U contains all points $\mathbf{x}$ of X that satisfy $|\mathbf{x} - \mathbf{p}| < \varepsilon$. But there exists some natural number N with the property that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$, since the sequence converges to $\mathbf{p}$. Therefore $\mathbf{x}_j \in U$ whenever $j \geq N$, as required. ∎

1.15 Interiors

Definition Let X be a subset of some Euclidean space, and let A be a subset of X. The *interior* of A in X is the subset of A consisting of those points $\mathbf{p}$ of A for which there exists some positive real number δ such that $B_X(\mathbf{p}, \delta) \subset A$. (Here $B_X(\mathbf{p}, \delta)$ denotes the open ball in X of radius δ centred on $\mathbf{p}$.)

A straightforward application of Lemma 1.25 shows that if X is a subset of some Euclidean space, and if A is a subset of X then the interior of A is open in X.

1.16 Closed Sets in Euclidean Spaces

Let X be a subset of $\mathbb{R}^n$. A subset F of X is said to be *closed* in X if and only if its complement $X \setminus F$ in X is open in X. (Recall that $X \setminus F = \{\mathbf{x} \in X : \mathbf{x} \notin F\}$.)

Example The sets $\{(x, y, z) \in \mathbb{R}^3 : z \geq c\}$, $\{(x, y, z) \in \mathbb{R}^3 : z \leq c\}$, and $\{(x, y, z) \in \mathbb{R}^3 : z = c\}$ are closed sets in $\mathbb{R}^3$ for each real number c, since the complements of these sets are open in $\mathbb{R}^3$.

Example Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{x}_0$ be a point of X. Then the sets $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{x}_0| \leq r\}$ and $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{x}_0| \geq r\}$ are closed for each non-negative real number r. In particular, the set $\{\mathbf{x}_0\}$ consisting of the single point $\mathbf{x}_0$ is a closed set in X. (These results follow immediately using Lemma 1.25 and Lemma 1.26 and the definition of closed sets.)

Let $\mathcal{A}$ be some collection of subsets of a set X. Then

$$X \setminus \bigcup_{S \in \mathcal{A}} S = \bigcap_{S \in \mathcal{A}} (X \setminus S), \qquad X \setminus \bigcap_{S \in \mathcal{A}} S = \bigcup_{S \in \mathcal{A}} (X \setminus S)$$

(i.e., the complement of the union of some collection of subsets of X is the intersection of the complements of those sets, and the complement of the intersection of some collection of subsets of X is the union of the complements of those sets). The following result therefore follows directly from Proposition 1.27.

Proposition 1.29 *Let X be a subset of $\mathbb{R}^n$. The collection of closed sets in X has the following properties:—*

(i) *the empty set $\emptyset$ and the whole set X are both closed in X;*

(ii) *the intersection of any collection of closed sets in X is itself closed in X;*

(iii) *the union of any* finite *collection of closed sets in X is itself closed in X.*

Lemma 1.30 *Let X be a subset of $\mathbb{R}^n$, and let F be a subset of X which is closed in X. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of F which converges to a point $\mathbf{p}$ of X. Then $\mathbf{p} \in F$.*

Proof The complement $X \setminus F$ of F in X is open, since F is closed. Suppose that $\mathbf{p}$ were a point belonging to $X \setminus F$. It would then follow from Lemma 1.28 that $\mathbf{x}_j \in X \setminus F$ for all values of j greater than some positive integer N, contradicting the fact that $\mathbf{x}_j \in F$ for all j. This contradiction shows that $\mathbf{p}$ must belong to F, as required. ∎

Lemma 1.31 *Let F be a closed bounded set in $\mathbb{R}^n$, and let U be an open set in $\mathbb{R}^n$. Suppose that $F \subset U$. Then there exists positive real number δ such that $|\mathbf{x} - \mathbf{y}| \geq \delta > 0$ for all $\mathbf{x} \in F$ and $\mathbf{y} \in \mathbb{R}^n \setminus U$.*

Proof Suppose that such a positive real number δ did not exist. Then there would exist an infinite sequence $(\mathbf{x}_j : j \in \mathbb{N})$ of points of F and a correspondinding infinite sequence $(\mathbf{y}_j : j \in \mathbb{N})$ of points of $\mathbb{R}^n \setminus U$ such that $|\mathbf{x}_j - \mathbf{y}_j| < 1/j$ for all positive integers j. The sequence $(\mathbf{x}_j : j \in \mathbb{N})$ would be a bounded sequence of points of $\mathbb{R}^n$, and would therefore have a convergent subsequence $(\mathbf{x}_{m_j} : j \in \mathbb{N})$ (Theorem 1.4). Let $\mathbf{p} = \lim_{j \to +\infty} \mathbf{x}_{m_j}$. Then $\mathbf{p} = \lim_{j \to +\infty} \mathbf{y}_{m_j}$, because $\lim_{j \to +\infty} (\mathbf{x}_{m_j} - \mathbf{y}_{m_j}) = \mathbf{0}$. But then $\mathbf{p} \in F$ and $\mathbf{p} \in \mathbb{R}^n \setminus U$, because the sets F and $\mathbb{R}^n \setminus U$ are closed (Lemma 1.30). But this is impossible, as $F \subset U$. It follows that there must exist some positive real number δ with the required properties. ∎

1.17 Closures

Definition Let X be a subset of some Euclidean space $\mathbb{R}^n$, and let A be a subset of X. The *closure* of A in X is the subset of X consisting of all points $\mathbf{x}$ of X with the property that, given any real number δ satisfying $\delta > 0$, there exists some point $\mathbf{a}$ of A such that $|\mathbf{x} - \mathbf{a}| < \delta$. We denote the closure of A in X by $\overline{A}$.

Let X be a subset of some Euclidean space, and let A be a subset of X. Note that a point $\mathbf{x}$ of X belongs to the closure of A in X if and only if $B_X(\mathbf{x}, \delta) \cap A$ is a non-empty set for all positive real numbers δ, where $B_X(\mathbf{x}, \delta)$ denotes the open ball in X of radius δ centred on $\mathbf{x}$ consisting of all points of X whose distance from $\mathbf{x}$ is less than δ.

Lemma 1.32 *Let X be a subset of some Euclidean space, let A be a subset of X, and let $\mathbf{p}$ be a point of the closure $\overline{A}$ of A in X. Then there exists an infinite sequence of points in A which converges to $\mathbf{p}$.*

Proof For each positive integer j let $\mathbf{x}_j$ be a point of A satisfying $|\mathbf{p} - \mathbf{x}_j| < 1/j$. Then $\lim_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$. ∎

Proposition 1.33 *Let X be a subset of some Euclidean space $\mathbb{R}^n$, and let A be a subset of X. Then the closure $\overline{A}$ of A in X is closed in X. Moreover if F is a subset of X which is closed in X, and if $A \subset F$ then $\overline{A} \subset F$.*

Proof Let $\mathbf{p}$ be a point belonging to the complement $X \setminus \overline{A}$ of $\overline{A}$ in X. Then there exists some real number δ such that $B_X(\mathbf{p}, 2\delta) \cap A = \emptyset$. Let $\mathbf{x}$ be a point of $\overline{A}$. Then there exists some point $\mathbf{a}$ of A such that $|\mathbf{x} - \mathbf{a}| < \delta$. Then

$$2\delta \leq |\mathbf{p} - \mathbf{a}| \leq |\mathbf{p} - \mathbf{x}| + |\mathbf{x} - \mathbf{a}| < |\mathbf{p} - \mathbf{x}| + \delta,$$

and therefore $|\mathbf{p} - \mathbf{x}| > \delta$. This shows that $B_X(\mathbf{p}, \delta) \cap \overline{A} = \emptyset$. We deduce that the complement of $\overline{A}$ is open in X, and therefore $\overline{A}$ is closed in X.

Now let F be a subset of X which is closed in X. Suppose that $A \subset F$. Let $\mathbf{p}$ be a point belonging to the complement $X \setminus F$ of F in X. Then there exists some real number δ satisfying $\delta > 0$ for which $B_X(\mathbf{p}, \delta) \cap F = \emptyset$. But then $B_X(\mathbf{p}, \delta) \cap A = \emptyset$ and therefore $\mathbf{p} \notin \overline{A}$. Thus $X \setminus F \subset X \setminus \overline{A}$, and therefore $\overline{A} \subset F$, as required. ∎

1.18 Continuous Functions and Open and Closed Sets

Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and let $f\colon X \to Y$ be a function from X to Y. We recall that the function f is continuous at a point $\mathbf{p}$ of X if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that $|f(\mathbf{u}) - f(\mathbf{p})| < \varepsilon$ for all points $\mathbf{u}$ of X satisfying $|\mathbf{u} - \mathbf{p}| < \delta$. Thus the function $f\colon X \to Y$ is continuous at $\mathbf{p}$ if and only if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that the function f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$ (where $B_X(\mathbf{p}, \delta)$ and $B_Y(f(\mathbf{p}), \varepsilon)$ denote the open balls in X and Y of radius δ and ε about $\mathbf{p}$ and $f(\mathbf{p})$ respectively).

Given any function $f\colon X \to Y$, we denote by $f^{-1}(V)$ the *preimage* of a subset V of Y under the map f, defined by $f^{-1}(V) = \{\mathbf{x} \in X : f(\mathbf{x}) \in V\}$.

Proposition 1.34 *Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and let $f\colon X \to Y$ be a function from X to Y. The function f is continuous if and only if $f^{-1}(V)$ is open in X for every open subset V of Y.*

Proof Suppose that $f\colon X \to Y$ is continuous. Let V be an open set in Y. We must show that $f^{-1}(V)$ is open in X. Let $\mathbf{p} \in f^{-1}(V)$. Then $f(\mathbf{p}) \in V$. But V is open, hence there exists some $\varepsilon > 0$ with the property that $B_Y(f(\mathbf{p}), \varepsilon) \subset V$. But f is continuous at $\mathbf{p}$. Therefore there exists some $\delta > 0$ such that f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$ (see the remarks above). Thus $f(\mathbf{x}) \in V$ for all $\mathbf{x} \in B_X(\mathbf{p}, \delta)$, showing that $B_X(\mathbf{p}, \delta) \subset f^{-1}(V)$. This shows that $f^{-1}(V)$ is open in X for every open set V in Y.

Conversely suppose that $f\colon X \to Y$ is a function with the property that $f^{-1}(V)$ is open in X for every open set V in Y. Let $\mathbf{p} \in X$. We must show that f is continuous at $\mathbf{p}$. Let $\varepsilon > 0$ be given. Then $B_Y(f(\mathbf{p}), \varepsilon)$ is an open set in Y, by Lemma 1.25, hence $f^{-1}\left(B_Y(f(\mathbf{p}), \varepsilon)\right)$ is an open set in X which contains $\mathbf{p}$. It follows that there exists some $\delta > 0$ such that $B_X(\mathbf{p}, \delta) \subset f^{-1}\left(B_Y(f(\mathbf{p}), \varepsilon)\right)$. Thus, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$. We conclude that f is continuous at $\mathbf{p}$, as required. ∎

Let X be a subset of $\mathbb{R}^n$, let $f\colon X \to \mathbb{R}$ be continuous, and let c be some real number. Then the sets $\{\mathbf{x} \in X : f(\mathbf{x}) > c\}$ and $\{\mathbf{x} \in X : f(\mathbf{x}) < c\}$ are open in X, and, given real numbers a and b satisfying $a < b$, the set $\{\mathbf{x} \in X : a < f(\mathbf{x}) < b\}$ is open in X.

1.19 Continuous Functions on Closed Bounded Sets

We shall prove that continuous functions between subsets of Euclidean spaces map closed bounded sets to closed bounded sets.

Lemma 1.35 *Let X be a closed bounded subset of some Euclidean space $\mathbb{R}^m$, and let $f\colon X \to \mathbb{R}^n$ be a continuous function mapping X into some Euclidean space $\mathbb{R}^n$. Then there exists some non-negative real number M such that $|f(\mathbf{x})| \leq M$ for all $\mathbf{x} \in X$.*

Proof Let $f\colon X \to \mathbb{R}^n$ be a continuous function defined on some closed set X in $\mathbb{R}^m$. Suppose that the function f is not bounded on X. We shall prove that the domain X of f must then be unbounded.

Now if the function f is not bounded on X then there must exist a sequence $(\mathbf{x}_j : j \in \mathbb{N})$ of points of X such that $|f(\mathbf{x}_j)| > j$ for all positive integers j. If $(\mathbf{x}_{m_j} : j \in \mathbb{N})$ were a subsequence of $(\mathbf{x}_j : j \in \mathbb{N})$ converging to some point $\mathbf{p}$ of $\mathbb{R}^m$, then $\mathbf{p}$ would belong to X, since X is closed (Lemma 1.30). Then $\lim\limits_{j \to +\infty} f(\mathbf{x}_{m_j}) = f(\mathbf{p})$, and therefore $|f(\mathbf{x}_{m_j})| \leq |f(\mathbf{p})| + 1$ for all sufficiently large positive integers j. But this is impossible because $|f(\mathbf{x}_j)| > j$ for all positive integers j. Thus the sequence $(\mathbf{x}_j : j \in \mathbb{N})$ cannot have any convergent subsequences. Now every bounded sequence of points in $\mathbb{R}^m$ has a convergent subsequence (Theorem 1.4). It follows that $(\mathbf{x}_j : j \in \mathbb{N})$ is not a bounded sequence of points in $\mathbb{R}^m$, and therefore X is an unbounded set.

It follows from this that if the domain X of the continuous function $f\colon X \to \mathbb{R}^n$ is both closed and bounded then the function f must be bounded on X, as required. ∎

Theorem 1.36 *Let X be a closed bounded set in some Euclidean space, and let $f\colon X \to \mathbb{R}^n$ be a continuous function mapping X into some Euclidean space $\mathbb{R}^n$. Then the function f maps X onto a closed bounded set $f(X)$ in $\mathbb{R}^n$.*

Proof It follows from Lemma 1.35 that the set $f(X)$ must be bounded.

Let $\mathbf{q}$ be a point belonging to the closure $\overline{f(A)}$ of $f(A)$. Then there exists a sequence $(\mathbf{x}_j : j \in \mathbb{N})$ of points of X such that $\lim\limits_{j \to +\infty} f(\mathbf{x}_j) = \mathbf{q}$ (Lemma 1.32). Because the set X is both closed and bounded, this sequence is a bounded sequence in Euclidean space, and therefore has a convergent subsequence $(\mathbf{x}_{m_j} : j \in \mathbb{N})$ (Theorem 1.4). Let $\mathbf{p} = \lim\limits_{j \to +\infty} \mathbf{x}_{m_j}$. Then $\mathbf{p} \in X$, because X is closed (Lemma 1.30). But then $\mathbf{q} = \lim\limits_{j \to +\infty} f(\mathbf{x}_{m_j}) = f(\mathbf{p})$, and therefore $\mathbf{q} \in f(A)$. Thus every point of the closure of $f(A)$ belongs to $f(A)$ itself, and therefore $f(A)$ is closed, as required. ∎

1.20 Uniform Continuity

Definition Let X and Y be subsets of Euclidean spaces. A function $f\colon X \to Y$ from X to Y is said to be to be *uniformly continuous* if, given any $\varepsilon > 0$, there exists some $\delta > 0$ (which does not depend on either $\mathbf{x}'$ or $\mathbf{x}$) such that $|f(\mathbf{x}') - f(\mathbf{x})| < \varepsilon$ for all points $\mathbf{x}'$ and $\mathbf{x}$ of X satisfying $|\mathbf{x}' - \mathbf{x}| < \delta$.

Theorem 1.37 *Let X be a subset of $\mathbb{R}^m$ that is both closed and bounded. Then any continuous function $f\colon X \to \mathbb{R}^n$ is uniformly continuous.*

Proof Let $\varepsilon > 0$ be given. Suppose that there did not exist any $\delta > 0$ such that $|f(\mathbf{x}') - f(\mathbf{x})| < \varepsilon$ for all points $\mathbf{x}', \mathbf{x} \in X$ satisfying $|\mathbf{x}' - \mathbf{x}| < \delta$. Then, for each natural number j, there would exist points $\mathbf{u}_j$ and $\mathbf{v}_j$ in X such that $|\mathbf{u}_j - \mathbf{v}_j| < 1/j$ and $|f(\mathbf{u}_j) - f(\mathbf{v}_j)| \geq \varepsilon$. But the sequence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$ would be bounded, since X is bounded, and thus would possess a subsequence $\mathbf{u}_{j_1}, \mathbf{u}_{j_2}, \mathbf{u}_{j_3}, \ldots$ converging to some point $\mathbf{p}$ (Theorem 1.4). Moreover $\mathbf{p} \in X$, since X is closed. The sequence $\mathbf{v}_{j_1}, \mathbf{v}_{j_2}, \mathbf{v}_{j_3}, \ldots$ would also converge to $\mathbf{p}$, since $\lim\limits_{k \to +\infty} |\mathbf{v}_{j_k} - \mathbf{u}_{j_k}| = 0$. But then the sequences $f(\mathbf{u}_{j_1}), f(\mathbf{u}_{j_2}), f(\mathbf{u}_{j_3}), \ldots$ and $f(\mathbf{v}_{j_1}), f(\mathbf{v}_{j_2}), f(\mathbf{v}_{j_3}), \ldots$ would converge to $f(\mathbf{p})$, since f is continuous (Lemma 1.11), and thus $\lim\limits_{k \to +\infty} |f(\mathbf{u}_{j_k}) - f(\mathbf{v}_{j_k})| = 0$. But this is impossible, since $\mathbf{u}_j$ and $\mathbf{v}_j$ have been chosen so that $|f(\mathbf{u}_j) - f(\mathbf{v}_j)| \geq \varepsilon$ for all j. We conclude therefore that there must exist some $\delta > 0$ such that $|f(\mathbf{x}') - f(\mathbf{x})| < \varepsilon$ for all points $\mathbf{x}', \mathbf{x} \in X$ satisfying $|\mathbf{x}' - \mathbf{x}| < \delta$, as required. ∎