Course 214
Applications of Cauchy's Residue Theorem
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Contents

8 Applications of Cauchy's Residue Theorem

Lemma 8.1 *Let R be a positive real number, and let f be a continuous complex-valued function defined everywhere on the semicircle S_R, where*

$$S_R = \{z \in \mathbb{C} : |z| = R \text{ and } \operatorname{Im}[z] \geq 0\}.$$

Suppose that there exists a non-negative real number $M(R)$ such that $|f(z)| \leq M(R)$ for all $z \in S_R$. Then

$$\left| \int_{\sigma_R} f(z) e^{isz} \, dz \right| \leq \frac{\pi M(R)}{s}$$

for all $s > 0$, where $\sigma_R \colon [0, \pi] \to \mathbb{C}$ is the path with $[\sigma_R] = S_R$ defined such that $\sigma_R(\theta) = Re^{i\theta}$ for all $\theta \in [0, \pi]$.

Proof It follows from the definition of the path integral that

$$\begin{aligned} \int_{\sigma_R} f(z) e^{isz} \, dz &= \int_0^\pi f(\sigma_R(\theta)) e^{is\sigma_R(\theta)} \sigma_R'(\theta) \, d\theta \\ &= \int_0^\pi f(Re^{i\theta}) e^{iRs\cos\theta - Rs\sin\theta} \left(iRe^{i\theta} \right) \, d\theta. \end{aligned}$$

Therefore

$$\begin{aligned} \left| \int_{\sigma_R} f(z) e^{isz} \, dz \right| &\leq R \int_0^\pi |f(Re^{i\theta})| |e^{iRs\cos\theta - Rs\sin\theta}| \, d\theta \\ &\leq RM(R) \int_0^\pi e^{-Rs\sin\theta} \, d\theta. \end{aligned}$$

Now $\sin\theta \geq 2\theta/\pi$ when $0 \leq \theta \leq \pi/2$, and therefore

$$\int_0^{\frac{\pi}{2}} e^{-Rs\sin\theta} \, d\theta \leq \int_0^{\frac{\pi}{2}} e^{\frac{-2Rs\theta}{\pi}} \, d\theta \leq \frac{\pi}{2Rs}.$$

Also

$$\int_{\frac{\pi}{2}}^{\pi} e^{-Rs\sin\theta} \, d\theta = \int_0^{\frac{\pi}{2}} e^{-Rs\sin\theta} \, d\theta.$$

(This follows on making the substitution that replaces θ by $\pi - \theta$.) Therefore

$$\int_0^\pi e^{-Rs\sin\theta} \, d\theta \leq \frac{\pi}{Rs}.$$

It follows that

$$\left| \int_{\sigma_R} f(z) e^{isz} \, dz \right| \leq \frac{\pi M(R)}{s},$$

as required. ∎

Example We shall apply Cauchy's Residue Theorem (Theorem 6.16) and Lemma 8.1 in order to evaluate

$$\int_{-\infty}^{\infty} \frac{e^{isx}}{x^2 + a^2}\, dx$$

when $s > 0$.

Let a be a positive real number, let R be a real number satisfying $R > a$, and let $\sigma_R\colon [0, \pi] \to \mathbb{C}$ be the path that sends $\theta \in [0, \pi]$ to $Re^{i\theta}$. (Thus $\sigma_R(\theta)$ traverses a semicircle of radius R in the upper half of the complex plane from R to $-R$ as θ increases from 0 to π. Now it follows from the Triangle Inequality that $|z^2| \le |z^2 + a^2| + |a^2|$, and thus $|z^2 + a^2| \ge |z|^2 - a^2$ for all complex numbers z, and therefore

$$\left| \frac{1}{z^2 + a^2} \right| \le \frac{1}{R^2 - a^2}$$

for all complex numbers z satisfying $|z| \ge R$. It now follows from Lemma 8.1 that

$$\left| \int_{\sigma_R} \frac{e^{isz}}{z^2 + a^2}\, dz \right| \le \frac{\pi}{s(R^2 - a^2)}$$

for all real numbers s and R satisfying $s > 0$ and $R > a$. Therefore

$$\lim_{R \to +\infty} \int_{\sigma_R} \frac{e^{isz}}{z^2 + a^2}\, dz = 0.$$

Now the function f has poles at ia and $-ia$. Moreover

$$\lim_{z \to ia} \left((z - ia) \frac{e^{isz}}{z^2 + a^2} \right) = \lim_{z \to ia} \frac{e^{isz}}{z + ia} = \frac{e^{-sa}}{2ia},$$

and therefore the meromorphic function that sends z to $\dfrac{e^{isz}}{z^2 + a^2}$ as a simple pole at ia with residue $e^{-sa}/2ia$. Thus if we apply Cauchy's Residue Theorem (Theorem 6.16) in order to evaluate the path integral of this function around the boundary of the set

$$\{z \in \mathbb{C} : |z| \le R \text{ and } \operatorname{Im}[z] \ge 0\},$$

we find that

$$\lim_{R \to +\infty} \int_{-R}^{R} \frac{e^{isx}\, dx}{x^2 + a^2}\, dx + \lim_{R \to +\infty} \int_{\sigma_R} \frac{e^{isz}}{z^2 + a^2} = 2\pi i \times \frac{e^{-sa}}{2ia} = \frac{\pi e^{-sa}}{a}$$

when $s > 0$. If we then take the limit of the left hand side of this identity as $R \to +\infty$, we find that

$$\int_{-\infty}^{\infty} \frac{e^{isx}}{x^2 + a^2}\, dx = \frac{\pi e^{-sa}}{a}.$$

when $s > 0$. This formula does not hold when $s \leq 0$. And indeed, if we take the complex conjugate of the above identity we find that

$$\int_{-\infty}^{\infty} \frac{e^{-isx}}{x^2 + a^2}\, dx = \frac{\pi e^{-sa}}{a}.$$

when $s > 0$. It follows that

$$\int_{-\infty}^{\infty} \frac{e^{isx}}{x^2 + a^2}\, dx = \frac{\pi e^{-|s|a}}{a}.$$

when $|s| \neq 0$. This identity also holds when $s = 0$, though this does not follow from the above calculations.

Example We can also evaluate the above integral by applying Cauchy's Residue Theorem to the path integral taken around the boundary of a rectangle in the complex plane with vertices at $-R$, R, $R + iM$ and $-R + iM$, where R and M are large positive real numbers.

Let a be a real number, and let R and M be real numbers satisfying $R > a$ and $M > a$. The inequality

$$\left| \frac{1}{z^2 + a^2} \right| \leq \frac{1}{R^2 - a^2}$$

is satisfied for all complex numbers z for which $|z| > a$. It follows from this that

$$\begin{aligned} \left| \int_{[R,R+iM]} \frac{e^{isz}}{z^2 + a^2}\, dz \right| &\leq \frac{1}{R^2 - a^2} \int_0^M |e^{is(R-iy)}|\, dy = \frac{1}{R^2 - a^2} \int_0^M e^{-sy}\, dy \\ &\leq \frac{M}{s(R^2 - a^2)}. \end{aligned}$$

Similarly

$$\left| \int_{[-R,-R+iM]} \frac{e^{isz}}{z^2 + a^2}\, dz \right| \leq \frac{M}{s(R^2 - a^2)},$$

and

$$\left| \int_{[-R+iM,R+iM]} \frac{e^{isz}}{z^2 + a^2}\, dz \right| \leq \frac{2Me^{-sM}}{M^2 - a^2}.$$

If we take, for example, $M = R$ in these inequalities, and let R tend to $+\infty$, we find that

$$\lim_{R\to+\infty} \int_{[R,R+iR]} \frac{e^{isz}}{z^2+a^2}\,dz = 0,$$

$$\lim_{R\to+\infty} \int_{[-R,-R+iR]} \frac{e^{isz}}{z^2+a^2}\,dz = 0,$$

$$\lim_{R\to+\infty} \int_{[-R+iR,R+iR]} \frac{e^{isz}}{z^2+a^2}\,dz = 0.$$

Also, Cauchy's Residue Theorem ensures that

$$\begin{aligned}\frac{\pi e^{-sa}}{a} &= \int_{[-R,R]} \frac{e^{isz}}{z^2+a^2}\,dz + \int_{[R,R+iR]} \frac{e^{isz}}{z^2+a^2}\,dz \\ &\quad - \int_{[-R+iR,R+iR]} \frac{e^{isz}}{z^2+a^2}\,dz - \int_{[-R,-R+iR]} \frac{e^{isz}}{z^2+a^2}\,dz\end{aligned}$$

when $s > 0$. It follows that

$$\int_{-\infty}^{\infty} \frac{e^{-isx}}{x^2+a^2}\,dx = \lim_{R\to+\infty} \int_{[-R,R]} \frac{e^{isz}}{z^2+a^2}\,dz = \frac{\pi e^{-sa}}{a}$$

when $s > 0$.

Example Let α be a real number satisfying $0 < \alpha < 1$. We evaluate the integral

$$\int_0^{\infty} \frac{x^\alpha}{x(x+1)}\,dx$$

through an application of Cauchy's Residue Theorem. Let

$$D = \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\},$$

so that D is the open set obtained on removing the negative real axis from the complex plane, let $\log\colon D \to \mathbb{C}$ denote the principal branch of the logarithm that sends $re^{i\theta}$ to $\log r + i\theta$ for all real numbers r and θ satisfying $r > 0$ and $-\pi < \theta < \pi$, and let $z^\alpha = \exp(\alpha \log z)$ for all $z \in D$. Then the function f that sends $z \in D\setminus\{1\}$ to $\dfrac{z^\alpha}{z(z-1)}$ is a meromorphic function on D. The only pole of this function that lies within the open set D is a simple pole at $z = 1$ with residue 1. Let R and η be real numbers satisfying $R > 1$ and $0 < \eta < 1$, and let $\theta_\eta \in [\frac{3}{4}\pi, \pi]$ be determined such that $-R + i\eta = \sqrt{R^2+\eta^2}\,\exp(i\theta_\eta)$. It follows from Cauchy's Residue Theorem (Theorem 6.16) that

$$\begin{aligned}\int_{[-R+i\eta,-\eta+i\eta]} f(z)\,dz &- \int_{\alpha_\eta} f(z)\,dz \\ &- \int_{[-R-i\eta,-\eta-i\eta]} f(z)\,dz + \int_{\beta_{R,\eta}} f(z)\,dz \\ = 2\pi i,&\end{aligned}$$

where $\alpha_\eta\colon [-\frac{3}{4}\pi, \frac{3}{4}\pi] \to \mathbb{C}$ is the path from $-\eta - i\eta$ to $-\eta + i\eta$ that sends $t \in [-\frac{3}{4}\pi, \frac{3}{4}\pi]$ to $\sqrt{2}\,\eta e^{it}$, and $\beta_{R,\eta}\colon [-\theta_\eta, \theta_\eta] \to \mathbb{C}$ is the path from $-R - i\eta$ to $-R + i\eta$ that sends $t \in [-\theta_\eta, \theta_\eta]$ to $\sqrt{R^2+\eta^2}e^{it}$. [Thus $\alpha_\eta(t)$ traverses a three-quarters of a circle of radius $\sqrt{2}\,\eta$ about zero in the anti-clockwise direction as t increases from $-\frac{3}{4}\pi$ to $\frac{3}{4}\pi$, and $\beta_{R,\eta}(t)$ traverses most of a circle of radius $\sqrt{R^2+\eta^2}$ about zero as t increases from $-\theta_\eta$ to θ_η.] Now the inequality $\alpha > 0$ ensures that

$$\lim_{\eta\to 0} \int_{\alpha_\eta} f(z)\,dz = 0.$$

Also

$$\lim_{\eta\to 0} \int_{\beta_{R,\eta}} f(z)\,dz = \int_{\sigma_R} f(z)\,dz,$$

where $\sigma_R\colon [-\pi, \pi] \to \mathbb{C}$ is the path that sends $t \in [-\pi, \pi]$ to Re^{it}. It follows that

$$\begin{aligned}2\pi i - \int_{\sigma_R} f(z)\,dz &= \lim_{\eta\to 0^+} \left(\int_{[-R+i\eta,-\eta+i\eta]} f(z)\,dz - \int_{[-R-i\eta,-\eta-i\eta]} f(z)\,dz\right) \\ &= \lim_{\eta\to 0^+} \left(\int_{-R}^{0} f(x+i\eta)\,dx - \int_{-R}^{0} f(x-i\eta)\,dx\right) \\ &= \int_{-R}^{0} \lim_{\eta\to 0^+} \left(f(x+i\eta) - f(x-i\eta)\right)\,dx \\ &= \int_{0}^{R} \lim_{\eta\to 0^+} \left(f(-x+i\eta) - f(-x-i\eta)\right)\,dx\end{aligned}$$

Now

$$\begin{aligned}\lim_{\eta\to 0^+} f(-x+i\eta) &= \lim_{\eta\to 0^+} \frac{(-x+i\eta)^\alpha}{(-x+i\eta)(-x-1-i\eta)} \\ &= \frac{\displaystyle\lim_{\eta\to 0^+} \exp(\alpha\log(-x+i\eta))}{x(x+1)}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\exp(\alpha(\log x + i\pi))}{x(x+1)} = \frac{\exp(\alpha \log x)\exp(i\pi\alpha))}{x(x+1)} \\
&= e^{i\pi\alpha}\frac{x^\alpha}{x(x+1)}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\lim_{\eta\to 0^+} f(-x - i\eta) &= \frac{\lim_{\eta\to 0^+} \exp(\alpha\log(-x - i\eta))}{x(x+1)} \\
&= \frac{\exp(\alpha(\log x - i\pi))}{x(x+1)} = \frac{\exp(\alpha \log x)\exp(-i\pi\alpha))}{x(x+1)} \\
&= e^{-i\pi\alpha}\frac{x^\alpha}{x(x+1)}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\lim_{\eta\to 0^+} \left(f(-x + i\eta) - f(-x - i\eta)\right) &= \left(e^{i\pi\alpha} - e^{-i\pi\alpha}\right)\frac{x^\alpha}{x(x+1)} \\
&= \frac{2ix^\alpha \sin\pi\alpha}{x(x+1)}
\end{aligned}
$$

Therefore

$$
2i\sin(\pi\alpha)\int_0^R \frac{x^\alpha}{x(x+1)}\,dz = 2\pi i - \int_{\sigma_R} f(z)\,dz.
$$

Now the inequality $\alpha < 1$ ensures that

$$
\lim_{R\to+\infty}\int_{\sigma_R} f(z)\,dz = 0.
$$

It follows that

$$
\int_0^{+\infty} \frac{x^\alpha}{x(x+1)}\,dz = \lim_{R\to+\infty}\int_0^R \frac{x^\alpha}{x(x+1)}\,dz = \frac{\pi}{\sin\pi\alpha}
$$

when $0 < \alpha < 1$.