# Course 214
# Basic Properties of Holomorphic Functions
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## Contents

# 7 Basic Properties of Holomorphic Functions

## 7.1 Taylor's Theorem for Holomorphic Functions

**Theorem 7.1** (Taylor's Theorem for Holomorphic Functions) *Let $w$ be a complex number, let $r$ be a positive real number, and let $f\colon D_{w,r} \to \mathbb{C}$ be a holomorphic function on the open disk $D_{w,r}$ of radius $r$ about $w$. Then the function $f$ may be differentiated any number of times on $D_{w,r}$, and there exist complex numbers $a_0, a_1, a_2, \ldots$ such that*

$$f(z) = \sum_{n=0}^{+\infty} a_n (z-w)^n.$$

*Moreover*

$$a_n = \frac{f^{(n)}(w)}{n!} = \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(z)}{(z-w)^{n+1}}\, dz,$$

*where $R$ is any real number satisfying $0 < R < r$ and $\gamma_R\colon [0,1] \to D_{w,r}$ is the closed path defined such that $\gamma_R(t) = w + Re^{2\pi i t}$ for all $t \in [0,1]$.*

**Proof** Choose a real number $R$ satisfying $0 < R < r$, and let $z$ be a complex number satisfying $|z - w| < R$. It follows from Corollary 6.18 that

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Now

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - w} \times \frac{1}{1 - \dfrac{z-w}{\zeta - w}} = \sum_{n=0}^{+\infty} \frac{(z-w)^n}{(\zeta - w)^{n+1}}$$

and

$$\left| \frac{(z-w)^n}{(\zeta - w)^{n+1}} \right| = \frac{1}{R} \left( \frac{|z-w|}{R} \right)^n$$

for all $\zeta \in \mathbb{C}$ satisfying $|\zeta - w| = R$. Moreover $|z - w| < R$, and therefore the infinite series $\displaystyle\sum_{n=0}^{+\infty} \left( \frac{|z-w|}{R} \right)^n$ is convergent. On applying the Weierstass $M$-Test (Proposition 2.8), we find that the infinite series

$$\sum_{n=0}^{+\infty} \frac{f(\zeta)(z-w)^n}{(\zeta - w)^{n+1}}$$

converges uniformly in $\zeta$ on the circle $\{\zeta \in \mathbb{C} : |\zeta - w| = R\}$. It follows that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\gamma_R} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_{\gamma_R}\left(\sum_{n=0}^{+\infty}\frac{(z-w)^n f(\zeta)}{(\zeta-w)^{n+1}}\right)d\zeta \\ &= \sum_{n=0}^{+\infty}\frac{(z-w)^n}{2\pi i}\int_{\gamma_R}\frac{f(\zeta)}{(\zeta-w)^{n+1}}\,d\zeta, \end{aligned}$$

provided that $|z - w| \leq R$. (The interchange of integration and summation above is justified by the uniform convergence of the infinite series of continuous functions occuring in the integrand.) The choice of $R$ satisfying $0 < R < r$ is arbitrary. Thus $f(z) = \sum_{n=0}^{+\infty} a_n(z-w)^n$ for all complex numbers $z$ satisfying $|z-w| < r$, where the coefficients of this power series are given by the formula

$$a_n = \frac{1}{2\pi i}\int_{\gamma_R}\frac{f(z)}{(z-w)^{n+1}}\,dz.$$

It then follows directly from Corollary 5.7 that the function $f$ can be differentiated any number of times on the open disk $D_{w,r}$, and $a_n = f^{(n)}(w)/n!$ for all positive integers $n$. ∎

**Corollary 7.2** (Cauchy's Inequalities) *Let $\sum_{j=0}^{+\infty} a_n z^n$ be a power series, and let $R$ be a positive real number that does not exceed the radius of convergence of the power series. Let $f(z) = \sum_{j=0}^{+\infty} a_n z^n$ for all complex numbers $z$ for which the power series converges. Suppose that $|f(z)| \leq M$ for all complex numbers $z$ satisfying $|z| = R$. Then $|a_n| \leq MR^{-n}$ and thus $|f^{(n)}(0)| \leq n!MR^{-n}$ for all non-negative integers $n$.*

**Proof** It follows from Lemma 4.2 that

$$|a_n| = \frac{1}{2\pi}\left|\int_{\gamma_R}\frac{f(z)}{(z-w)^{n+1}}\,dz\right| \leq \frac{1}{2\pi} \times \frac{M}{R^{n+1}} \times 2\pi R = \frac{M}{R^n},$$

where $\gamma_R\colon [0,1] \to \mathbb{C}$ denotes the closed path of length $2\pi R$ defined such that $\gamma_R(t) = Re^{2\pi i t}$ for all $t \in [0,1]$. Therefore $|f^{(n)}(0)| = n!|a_n| \leq n!MR^{-n}$, as required. ∎

## 7.2 Liouville's Theorem

**Theorem 7.3** (Liouville's Theorem) *Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function defined over the entire complex plane. Suppose that there exists some non-negative real number $M$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Then the function $f$ is constant on $\mathbb{C}$.*

**Proof** It follows from Theorem 7.1 that there exists an infinite sequence $a_0, a_1, a_2, \ldots$ of complex numbers such that $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ for all $z \in \mathbb{C}$. Cauchy's Inequalities then ensure that $|a_n| \leq MR^{-n}$ for all non-negative integers $n$ and for all positive real numbers $R$ (see Corollary 7.2). This requires that $a_n = 0$ when $n > 0$. Thus $f$ is constant on $\mathbb{C}$, as required. ∎

## 7.3 Laurent's Theorem

**Theorem 7.4** (Laurent's Theorem) *Let $r$ be a positive real number, and let $f$ be a holomorphic function on $D_{0,r}$, where $D_{0,r} = \{z \in \mathbb{C} : 0 < |z| < r\}$. Then there exist complex numbers $a_n$ for all integers $n$ such that*

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n + \sum_{n=1}^{+\infty} a_{-n} z^{-n}$$

*for all complex numbers $z$ satisfying $0 < |z| < r$. Moreover*

$$a_n = \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(z)}{z^{n+1}}\, dz,$$

*for all integers $n$, where $R$ is any real number satisfying $0 < R < r$ and $\gamma_R\colon [0,1] \to D_{0,r}$ is the closed path defined such that $\gamma_R(t) = Re^{2\pi i t}$ for all $t \in [0,1]$.*

**Proof** Choose real numbers $R_1$ and $R_2$ such that $0 < R_1 < R_2 < r$, and, for each real number $R$ satisfying $0 < R < r$, let $\gamma_R\colon [0,1] \to \mathbb{C}$ be the closed path defined such that $\gamma_R(t) = Re^{2\pi i t}$ for all $t \in [0,1]$. A straightforward application of Theorem 6.16 shows that follows from Corollary 6.18 that

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_{R_2}} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int_{\gamma_{R_1}} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

for all $z \in \mathbb{C}$ satisfying $R_1 < |z| < R_2$. But

$$\frac{1}{\zeta - z} = \sum_{n=0}^{+\infty} \frac{z^n}{\zeta^{n+1}}$$

when $|z| < R_2$ and $|\zeta| = R_2$, and moreover the infinite series on the right-hand side of this equality converges uniformly in $\zeta$, for values of $\zeta$ that lie on the circle $|\zeta| = R_2$. Also

$$\frac{1}{\zeta - z} = -\sum_{n=1}^{+\infty} \frac{\zeta^{n-1}}{z^n}$$

when $|z| > R_1$ and $|\zeta| = R_1$, and the infinite series on the right-hand side of this equality converges uniformly in $\zeta$, for values of $\zeta$ that lie on the circle $|\zeta| = R_1$. It follows that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\gamma_{R_2}} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_{\gamma_{R_1}} \frac{f(\zeta)}{\zeta - z}\,d\zeta \\ &= \sum_{n=0}^{+\infty} \frac{z^n}{2\pi i}\int_{\gamma_{R_2}} \frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta + \sum_{n=1}^{+\infty} \frac{z^{-n}}{2\pi i}\int_{\gamma_{R_1}} f(\zeta)\zeta^{n-1}\,d\zeta \\ &= \sum_{n=0}^{+\infty} a_n z^n + \sum_{n=1}^{+\infty} a_{-n} z^{-n}, \end{aligned}$$

when $R_1 < |z| < R_2$, where

$$a_n = \frac{1}{2\pi i}\int_{\gamma_{R_2}} \frac{f(z)}{z^{n+1}}\,dz$$

when $n \leq 0$, and

$$a_n = \frac{1}{2\pi i}\int_{\gamma_{R_1}} \frac{f(z)}{z^{n+1}}\,dz$$

when $n < 0$. A straightforward application of Corollary 6.12 shows that

$$a_n = \frac{1}{2\pi i}\int_{\gamma_R} \frac{f(z)}{z^{n+1}}\,dz,$$

for all integer $n$, where $R$ is any real number satisfying $0 < R < r$. The result follows. ∎

## 7.4 Morera's Theorem

**Theorem 7.5** (Morera's Theorem) *Let $f\colon D \to \mathbb{C}$ be a continuous function defined over an open set $D$ in $\mathbb{C}$. Suppose that*

$$\int_{\partial T} f(z)\,dz = 0$$

*for all closed triangles $T$ contained in $D$. Then $f$ is holomorphic on $D$.*

**Proof** Let $D_1$ be an open disk with $D_1 \subset D$. It follows from Proposition 6.5 that there exists a holomorphic function $F\colon D_1 \to \mathbb{R}$ such that $f(z) = F'(z)$ for all $z \in D_1$. But it follows from Theorem 7.1 and Corollary 5.7 that the derivative of a holomorphic function is itself a holomorphic function. Therefore the function $f$ is holomorphic on the open disk $D_1$. It follows that the derivative of $f$ exists at every point of $D$, and thus $f$ is holomorphic on $D$, as required. ∎

## 7.5 Meromorphic Functions

**Definition** Let $f$ be a complex-valued function defined over some subset of the complex plane, and let $w$ be a complex number. The function $f$ is said to be *meromorphic* at $w$ if there exists an integer $m$, a positive real number $r$, and a holomorphic function $g$ on the open disk $D_{w,r}$ of radius $r$ about $w$ such that $f(z) = (z-w)^m g(z)$ for all $z \in D_{w,r}$. The function $f$ is said to be meromorphic on some open set $D$ if it is meromorphic at each element of $D$.

Holomorphic functions are meromorphic.

Let $w$ be a complex number, and let $f$ be a complex-valued function that is meromorphic at $w$, but is not identically zero over any open set containing $w$. Then there exists an integer $m_0$, a positive real number $r$, and a holomorphic function $g_0$ on the open disk $D_{w,r}$ of radius $r$ about $w$ such that $f(z) = (z-w)^{m_0} g_0(z)$ for all $z \in D_{w,r}$. Now it follows from Theorem 7.1 (Taylor's Theorem) that there exists a sequence $a_1, a_2, a_3, \ldots$ of complex numbers such that the power series $\sum_{n=0}^{+\infty} a_n (z-w)^n$ converges to $g_0(z)$ for all $z \in D_{w,r}$. Let $k$ be the smallest non-negative integer for which $a_k \neq 0$. Then $g_0(z) = (z-w)^k g(z)$ for all $z \in D_{w,r}$, where $g(z) = \sum_{n=k}^{+\infty} a_n (z-w)^{n-k}$ Let $m = m_0 + k$. Then $f(z) = (z-w)^m g(z)$ where $g$ is a holomorphic function on $D_{w,r}$ and $g(w) \neq 0$. The value of $m$ is uniquely determined by $f$ and $w$. If $m > 0$ we say that the function $f$ has a *zero* of order $m$ at $w$. If $m < 0$, we say that $f$ has a *pole* of order $-m$ at $w$. A pole is said to be a *simple pole* if it is of order 1.

The following result is a direct consequence of Theorem 7.1 (Taylor's Theorem) and the definition of a meromorphic function.

**Lemma 7.6** *Let $w$ be a complex number, and let $f$ be a function defined on $D_{w,r} \setminus \{w\}$ for some $r > 0$, where $D_{w,r}$ is the open disk of radius $r$ about $w$. Suppose that $f$ is not identically zero throughout $D_{w,r} \setminus \{w\}$. Then the function $f$ is meromorphic at $w$ if and only if there exists an integer $m$ and complex numbers $a_m, a_{m+1}, a_{m+2}, \ldots$ such that*

$$f(z) = \sum_{n=m}^{+\infty} a_n (z-w)^n$$

*for all $z \in D_{w,r}$, in which case* $\mathrm{Res}_w(f) = a_{-1}$.

## 7.6 Zero Sets of Holomorphic Functions

Let $D$ be an open set in the complex plane, let $f\colon D \to \mathbb{C}$ be a holomorphic function on $D$, and let $w$ be a complex number belonging to the set $D$. We say that the function $f$ is *identically zero* throughout some neighbourhood of $w$ if there exists some positive real number $\delta$ such that $f(z) = 0$ for all $z \in D$ satisfying $|z - w| < \delta$. Also we say that $w$ is an *isolated zero* of $f$ if there exists some positive real number $\delta$ such that $f(z) \neq 0$ for all $z \in D$ satisfying $0 < |z - w| < \delta$. If $f$ is not identically zero throughout some neighbourhood of $w$ then there exists some non-negative integer $m$ and some holomorphic function $g$ such that $g(w) \neq 0$ and $f(z) = (z - w)^m g(z)$ for all $z \in D$. If $m = 0$ then the function $f$ is non-zero at $w$. If $m > 0$ then the function $f$ has an isolated zero at $w$. The following result follows immediately.

**Lemma 7.7** *Let $D$ be an open set in the complex plane, let $f\colon D \to \mathbb{C}$ be a holomorphic function on $D$, and let $w$ be a complex number belonging to the set $D$. Then either the function $f$ is non-zero at $w$, or $f$ has an isolated zero at $w$, or $f$ is identically zero throughout some neighbourhood of $w$.*

**Lemma 7.8** *Let $D$ be a path-connected open set in the complex plane, and let $U$ and $V$ be open sets in the complex plane. Suppose that $U \cup V = D$ and $U \cap V = \emptyset$. Then either $U = \emptyset$ or $V = \emptyset$.*

**Proof** Let $g\colon D \to \{0, 1\}$ be the function on $D$ defined such that $g(z) = 0$ for all $z \in U$ and $g(z) = 1$ for all $z \in V$. We first prove that this function $g$ is continuous on $D$. Let $w$ be a complex number belonging to the open set $D$. If $w \in U$ then there exists a positive real number $\delta$ such that $\{z \in C : |z - w| < \delta\} \subset U$, because $U$ is an open set. Similarly if $w \in V$ then there exists a positive real number $\delta$ such that $\{z \in C : |z - w| < \delta\} \subset V$. It follows that, given any element $w$ of $D$, there exists some positive real number $\delta$ such that $z \in D$ and $g(z) = g(w)$ for all complex numbers $z$ satisfying $|z - w| < \delta$. It follows directly from this that the function $g\colon D \to \{0, 1\}$ is continuous on the path-connected open set $D$.

Suppose that the sets $U$ and $V$ were both non-empty. Let $z_0 \in U$ and $z_1 \in V$. Now the open set $D$ is path-connected. Therefore there would exist a path $\gamma\colon [0, 1] \to D$ with $\gamma(0) = z_0$ and $\gamma(1) = z_1$. The function $g \circ \gamma\colon [0, 1] \to \{0, 1\}$ would then be a non-constant integer-valued continuous function on the interval $[0, 1]$. But this is impossible, since every continuous integer-valued function on $[0, 1]$ is constant (Proposition 1.17). It follows that at least one of the sets $U$ and $V$ must be empty, as required. ∎

**Theorem 7.9** *Let $D$ be a path-connected open set in the complex plane, and let $f: D \to \mathbb{C}$ be a holomorphic function on $D$. Suppose there exists some non-empty open subset $D_1$ of $D$ such that $f(z) = 0$ for all $z \in D_1$. Then $f(z) = 0$ for all $z \in D$.*

**Proof** Let $U$ be the set of all complex numbers $w$ belonging to $D$ with the property that the function $f$ is identically zero in a neighbourhood of $w$. Now the set $U$ is an open set in the complex plane, for if $w$ is a complex number belonging to $U$ then there exists some real number $\delta$ such that $z \in D$ and $f(z) = 0$ for all complex numbers $z$ satisfying $|z - w| < 2\delta$. But then the function $f$ is identically zero in a neighbourhood of $w_1$ for all complex numbers $w_1$ satisfying $|w_1 - w| < \delta$, for if $z$ is a complex number satisfying $|z - w_1| < \delta$ then $|z - w| < 2\delta$ and therefore $f(z) = 0$. It follows from this that the set $U$ is an open set in the complex plane.

Now let $V$ be the complement $D \setminus U$ of $U$ in $D$, and let $w$ be a complex number belonging to $V$. Now the function $f$ is not identically zero in a neighbourhood of $w$. It therefore follows from Lemma 7.7 that either $f(z) \neq 0$, or else the function $f$ has an isolated zero at $w$. It follows that there exists some positive real number $\delta$ such that the function $f$ is defined and non-zero throughout the set $\{z \in C : 0 < |z - w| < \delta\}$. But then $\{z \in C : 0 < |z - w| < \delta\} \subset V$. We conclude from this that $V$ is an open set. Now $D$ is the union of the open sets $U$ and $V$, and $U \cap V = \emptyset$. It follows from Lemma 7.8 that either $U = \emptyset$ or $V = \emptyset$.

Now the open set $U$ is non-empty, since $D_1 \subset U$. Therefore $V = \emptyset$, and thus $U = D$. It follows immediately from this that the function $f$ is identically zero throughout $D$ as required. ∎

**Corollary 7.10** *Let $D$ be a path-connected open set in the complex plane, and let $f: D \to \mathbb{C}$ and $g: D \to \mathbb{C}$ be holomorphic functions on $D$. Suppose there exists some non-empty open subset $D_1$ of $D$ such that $f(z) = g(z)$ for all $z \in D_1$. Then $f(z) = g(z)$ for all $z \in D$.*

**Proof** The result follows immediately on applying Theorem 7.9 to the function $f - g$. ∎

## 7.7 The Maximum Modulus Principle

**Proposition 7.11** (Maximum Modulus Principle) *Let $f: D \to \mathbb{C}$ be a holomorphic function defined over a path-connected open set $D$ in the complex plane. Suppose that the real-valued function on $D$ sending $z \in D$ to $|f(z)|$ attains a local maximum at some point $w$ of $D$. Then $f$ is constant throughout $D$.*

**Proof** Suppose that $f$ is not constant throughout $D$. It follows from Corollary 7.10 that $f$ cannot be constant over any open subset of $D$.

Let $u(z) = |f(z)|$ for all $z \in D$, and let $w$ be an element of $D$. Then the holomorphic function that sends $z \in D$ to $f(z) - f(w)$ has a zero at $w$. This zero is an isolated zero of order $m$ for some positive integer $m$, and there exists a holomorphic function $g$ on $D$ such that $g(w) \neq 0$ and $f(z) = f(w) + (z - w)^m g(z)$ for all $z \in D$. If $f(w) = 0$ then $w$ is not a local maximum for the function $u$, since $f(z) \neq 0$ for all complex numbers $z$ that are distinct from $w$ but sufficiently close to $w$. Suppose therefore that $f(w) \neq 0$. Then there exists a complex number $\alpha$ such that $|\alpha| = 1$ and $\alpha^m g(w) f(w)^{-1}$ is a positive real number. It then follows from the continuity of $g$ that $\alpha^m g(z) f(w)^{-1}$ has a positive real part when $z$ is sufficiently close to $w$. But then $|1 + t^m \alpha^m g(w + t\alpha) f(w)^{-1}| > 1$ for all sufficiently small positive real numbers $t$. It follows that $|f(w + t\alpha)| > |f(w)|$ for all sufficiently small positive real numbers $t$, and therefore the function $u$ does not have a local maximum at $w$. Thus if $f$ is not constant on $D$ then the function $u$ that sends $z \in D$ to $|f(z)|$ does not have a local maximum at any element of $D$. The result follows. ∎

## 7.8 The Argument Principle

**Theorem 7.12** (The Argument Principle) *Let $D$ be a simply-connected open set in the complex plane and let $f$ be a meromorphic function on $D$ whose zeros and poles are located at $w_1, w_2, \ldots, w_s$. Let $m_1, m_2, \ldots, m_s$ be integers, determined such that $m_j = k$ if $f$ has a zero of order $k$ at $w_j$, and $m_j = -k$ if $f$ has a pole of order $k$ at $w_j$. Let $\gamma\colon [a, b] \to D$ be a piecewise continuously differentiable closed path in $D$ which does not pass through any zero or pole of $f$. Then*

$$n(f \circ \gamma, 0) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_{j=1}^{s} m_j n(\gamma, w_j).$$

**Proof** It follows from Proposition 6.2 that

$$\begin{aligned} n(f \circ \gamma, 0) &= \frac{1}{2\pi i} \int_{f\circ\gamma} \frac{dz}{z} = \frac{1}{2\pi i} \int_a^b \frac{(f \circ \gamma)'(t)}{f(\gamma(t))}\, dt \\ &= \frac{1}{2\pi i} \int_a^b \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t))}\, dt = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz. \end{aligned}$$

Let $F(z) = f'(z) f(z)^{-1}$ for all $z \in D \setminus \{w_1, \ldots, w_s\}$. Suppose that $f(z) = (z - w_j)^{m_j} g_j(z)$, where $g_j$ is holomorphic over some open disk of positive radius centred on $w_j$ and $g_j(w_j) \neq 0$. Then

$$\frac{f'(z)}{f(z)} = \frac{m_j}{z - w_j} + \frac{g_j'(z)}{g_j(z)}$$

for all complex numbers $z$ that are not equal to $w$ but are sufficiently close to $w$. Moreover the function sending $z$ to $g'(z)g^{-1}(z)$ is holomorphic around $w$. It follows that the function $F$ has a simple pole at $w_j$, and that the residue of $F$ at $w_j$ is $m_j$. It therefore follows from Corollary 6.17 that

$$n(f \circ \gamma, 0) = \frac{1}{2\pi i} \int_\gamma F(z)\, dz = \sum_{j=1}^{s} m_j n(\gamma, w_j),$$

as required. ∎