Course 214
Section 6: Cauchy's Theorem
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David R. Wilkins

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6 Cauchy's Theorem

6.1 Path Integrals of Polynomial Functions

A function $f: \mathbb{C} \to \mathbb{C}$ is said to be a *polynomial function* of degree n if it can be represented by an expression of the form

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n,$$

where $a_0, a_1, \ldots, a_n$ are complex numbers and $a_n \neq 0$.

Let z_0 and z_1 be complex numbers, and let $\gamma: [a, b] \to \mathbb{C}$ be a piecewise continuously differentiable path from z_0 to z_1. If n is an integer and $n \neq -1$ then an immediate application of Lemma 5.5 shows that

$$\int_\gamma z^n\,dz = \int_\gamma \frac{d}{dz}\left(\frac{z^{n+1}}{n+1}\right)\,dz = \frac{z_1^{n+1} - z_0^{n+1}}{n+1}.$$

It follows from this that $\int_\gamma z^n\,dz = 0$ for all piecewise continuously differentiable closed paths γ, provided that n is an integer and $n \neq -1$. The following result is an immediate consequence of this.

Lemma 6.1 *Let $f: \mathbb{C} \to \mathbb{C}$ be a polynomial function. Then $\int_\gamma f(z)\,dz = 0$ for all piecewise continuously differentiable closed paths γ in the complex plane.*

6.2 Winding Numbers and Path Integrals

Proposition 6.2 *Let $\gamma: [a, b] \to \mathbb{C}$ be a piecewise continuously differentiable closed path in the complex plane, and let w be a complex number that does not lie on γ. Then the winding number $n(\gamma, w)$ of the closed path γ about w satisfies the identity*

$$n(\gamma, w) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - w}.$$

Proof There exists a path $\varphi: [a, b] \to \mathbb{C}$ such that $\gamma(t) - w = \exp(\varphi(t))$ for all $t \in [a, b]$. It is easy to see that this path φ is piecewise continuously differentiable. Now $\gamma'(t) = \exp(\varphi(t))\varphi'(t)$ for all $t \in [a, b]$. (This is an immediate consequence of Lemma 5.4, given that the derivative of the exponential function is the exponential function itself.) It follows from the definition of the winding number $n(\gamma, w)$ that

$$\begin{aligned}\int_\gamma \frac{dz}{z - w} &= \int_a^b \frac{\gamma'(t)}{\gamma(t) - w}\,dt = \int_a^b \frac{\gamma'(t)}{\exp(\varphi(t))}\,dt = \int_a^b \varphi'(t)\,dt \\ &= \varphi(b) - \varphi(a) = 2\pi i n(\gamma, w),\end{aligned}$$

as required. ∎

Corollary 6.3 *Let X be a closed polygonal set in the complex plane with interior* $\mathrm{int}(X)$*. Then*

$$\int_{\partial X} \frac{dz}{z-w} = \begin{cases} 2\pi i & \text{if } w \in \mathrm{int}(X); \\ 0 & \text{if } w \in \mathbb{C} \setminus X. \end{cases}$$

Proof We can represent the closed polygonal set X as the union of a finite collection $T_1, T_2, \ldots, T_r$ finite collection of distinct triangles in the complex plane that intersect regularly. Moreover these triangles may be chosen such that the complex number w belongs to the interior of exactly one of those triangles. Let $w \in \mathrm{int}(T_1)$. A straightforward application of Proposition 6.2 shows that

$$\int_{\partial T_1} \frac{dz}{z-w} = 2\pi i.$$

Also a closed path round the boundary of the triangle T_j has zero winding number about w when $j \neq 1$, and therefore

$$\int_{\partial T_j} \frac{dz}{z-w} = 0 \text{ when } j \neq 1.$$

It follows from Proposition 4.3 that

$$\int_{\partial X} \frac{dz}{z-w} = \sum_{j=1}^{r} \int_{\partial T_j} \frac{dz}{z-w} = 2\pi i,$$

as required. ∎

6.3 Cauchy's Theorem for a Triangle

Theorem 6.4 (Cauchy's Theorem for a Triangle) *Let $f\colon D \to \mathbb{C}$ be a holomorphic function defined over an open set D in $\mathbb{C}$, and let T be a closed triangle contained in D. Then*

$$\int_{\partial T} f(z)\, dz = 0.$$

Proof The line segments joining the midpoints of the three edges of the triangular region T divide T into four triangular regions S_1, S_2, S_3 and S_4. An edge of any one of the triangles S_j is half the length of the edge of T that is parallel to it, and the area of each triangle S_i is a quarter of the area of T. Moreover

$$\int_{\partial T} f(z)\,dz = \sum_{j=1}^{4} \int_{\partial S_i} f(z)\,dz,$$

and therefore

$$\left| \int_{\partial T} f(z)\,dz \right| \leq \sum_{j=1}^{4} \left| \int_{\partial S_i} f(z)\,dz \right|,$$

Suppose that $\int_{\partial T} f(z)\,dz \neq 0$. Then there would exist some real number ε_0 satisfying $\varepsilon_0 > 0$ such that

$$\left| \int_{\partial T} f(z)\,dz \right| \geq \varepsilon_0 \operatorname{area}(T).$$

There would then exist at least one triangle S_j for which

$$\left| \int_{\partial S_j} f(z)\,dz \right| \geq \frac{1}{4} \varepsilon_0 \operatorname{area}(T) = \varepsilon_0 \operatorname{area}(S_j).$$

It follows from repeated applications of this observation that there would exist an infinite sequence $T_0, T_1, T_2, T_3, \ldots$ of closed triangles, where $T_0 = T$, such that, for each positive integer j, the triangle T_j is one of the four triangles obtained by dividing the triangle T_{j-1} along the line segments joining the midpoints of its edges, and

$$\left| \int_{\partial T_j} f(z)\,dz \right| \geq \varepsilon_0 \operatorname{area}(T_j).$$

Moreover $\operatorname{area}(T_j) = \operatorname{area}(T)/4^j$, and the length of any side of the triangle T_j is half the length of the side of the triangle T_{j-1} that is parallel to it. Therefore the sides of the triangle T_j would have length at most $L/2^j$, where L is the length of the longest side of the triangle T, and therefore the length l_j of the perimeter of the triangle T_j would satisfy $l_j \leq 3L/2^j$.

Let $(w_j : j \in \mathbb{N})$ be a sequence of complex numbers satisfying $w_j \in T_j$ for each positive integer j. Then $w_k \in T_j$ whenever $k > j$, and therefore $|w_k - w_j| \leq L/2^j$ whenever $k \geq j$. It follows from this that the sequence $(w_j : j \in \mathbb{N})$ would be a Cauchy sequence, and would therefore converge to some limit w, where $w \in T_j$ for all positive integers j. The function f would be holomorphic at w, and therefore, given any positive real number ε_1, there would exist some positive number δ such that $|f(z) - f(w) - (z - w)f'(w)| \leq \varepsilon_1 |z - w|$ whenever $|z - w| < \delta$ (Lemma 5.1). But the triangle T_j would be wholly contained within the open disk of radius δ centred on w, provided that j were chosen sufficiently large, in which case $|z - w| \leq L/2^j$ and $|f(z) - f(w) - (z - w)f'(w)| \leq \varepsilon_1 L/2^j$ for all $z \in T_j$. Now

$$\int_{\partial T_j} (f(w) + (z - w)f'(w))\, dz = 0,$$

(see Lemma 6.1). Thus if j were chosen sufficiently large then

$$\begin{aligned} \left| \int_{\partial T_j} f(z)\, dz \right| &= \left| \int_{\partial T_j} (f(z) - f(w) - (z - w)f'(w))\, dz \right| \\ &\leq l_j \sup_{z \in T_j} |f(z) - f(w) - (z - w)f'(w)| \leq \frac{3L^2 \varepsilon_1}{4^j}. \end{aligned}$$

But this would not be possible, had ε_1 been chosen small enough to ensure that $3L^2\varepsilon_1 < \varepsilon_0 \operatorname{area}(T)$. Thus the assumption that

$$\left| \int_{\partial T} f(z)\, dz \right| \geq \varepsilon_0 \operatorname{area}(T)$$

for some real number ε_0 satisfying $\varepsilon_0 > 0$ leads to a contradiction. We conclude that $\int_{\partial T} f(z)\, dz = 0$, as required. ∎

6.4 Cauchy's Theorem for Star-Shaped Domains

We recall that an open set D in the complex plane is *star-shaped* if there exists some element z_0 of D chosen such that the line segment $\{(1 - t)z_0 + tz : t \in [0, 1]\}$ joining z_0 to z is contained in D for all $z \in D$.

Proposition 6.5 *Let $f\colon D \to \mathbb{R}$ be a continuous function defined over a star-shaped open set D in $\mathbb{C}$. Suppose that*

$$\int_{\partial T} f(z)\, dz = 0$$

for all closed triangles T contained in D. Then there exists a holomorphic function $F\colon D \to \mathbb{R}$ such that $f(z) = F'(z)$ for all $z \in D$.

Proof Let z_0 be an element of D chosen such that the line segment joining z_0 to z is contained in D for all $z \in D$, and let $F\colon D \to \mathbb{C}$ be the function defined by

$$F(z) = \int_{[z_0, z]} f(\zeta)\, d\zeta.$$

Let z be some complex number belonging to D. Then there exists some positive number δ_0 such that $z + h \in D$ for all complex numbers h satisfying $|h| < \delta_0$. The choice of z_0 then ensures that if $|h| < \delta_0$ then the closed triangle with vertices z_0, z and $z + h$ is contained in D. But then

$$\int_{[z_0,z]} f(\zeta)\,d\zeta + \int_{[z,z+h]} f(\zeta)\,d\zeta + \int_{[z+h,z_0]} f(\zeta)\,d\zeta = 0,$$

and thus

$$F(z+h) - F(z) = \int_{[z,z+h]} f(\zeta)\,d\zeta = h\int_0^1 f(z+th)\,dt.$$

Now the continuity of the function f ensures that, given any positive real number ε, there exists some real number δ satisfying $0 < \delta \leq \delta_0$ such that $|f(z+h) - f(z)| \leq \varepsilon$ whenever $|h| < \delta$. But then

$$\begin{aligned} \left|\frac{F(z+h) - F(z)}{h} - f(z)\right| &= \left|\int_0^1 (f(z+th) - f(z))\,dt\right| \\ &\leq \int_0^1 |f(z+th) - f(z)|\,dt \leq \varepsilon \end{aligned}$$

whenever $|h| < \delta$. It follows that the function F has a well-defined derivative $F'(z)$ at z, and

$$F'(z) = \lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = f(z).$$

Thus the function $F: D \to \mathbb{C}$ is holomorphic on D, and $F'(z) = f(z)$ for all $z \in D$, as required. ∎

Theorem 6.6 (Cauchy's Theorem for Star-Shaped Domains) *Let $f: D \to \mathbb{R}$ be a holomorphic function defined over a star-shaped open set D in $\mathbb{C}$. Then*

$$\int_\gamma f(z)\,dz = 0.$$

for all piecewise continuously differentiable closed paths γ in D.

Proof Cauchy's Theorem for a Triangle (Theorem 6.4) ensures that

$$\int_{\partial T} f(z)\,dz = 0$$

for all closed triangles T contained in D. It then follows from Proposition 6.5 that there exists a holomorphic function $F\colon D \to \mathbb{R}$ such that $f(z) = F'(z)$ for all $z \in D$. But then it follows from Lemma 5.5 that

$$\int_\gamma f(z)\,dz = \int_\gamma F'(z)\,dz = F(\gamma(b)) - F(\gamma(a)) = 0$$

for any piecewise continuously differentiable closed path $\gamma\colon [a,b] \to D$ in D. ∎

6.5 Cauchy's Theorem for Closed Polygonal Sets

Lemma 6.7 *Let D be an open set in the complex plane, let X be a closed polygonal set contained in D, and let $f\colon D \to \mathbb{C}$ be a holomorphic function defined throughout D. Then $\int_{\partial X} f(z)\,dz = 0$.*

Proof Any closed polygonal set the complex plane may be represented as the union of a finite collection $T_1, T_2, \ldots, T_r$ finite collection of distinct triangles in the complex plane that intersect regularly. Moreover

$$\int_{\partial X} f(z)\,dz = \sum_{j=1}^{r} \int_{\partial T_j} f(z)\,dz.$$

(Proposition 4.3). The result is therefore an immediate consequence of Cauchy's Theorem for a Triangle (Theorem 6.4). ∎

Lemma 6.8 *Let D be an open set in the complex plane, let X be a closed polygonal set contained in D, let w be a complex number in the interior of X, and let $f\colon D \to \mathbb{C}$ be a complex-valued function that is continuous on D and holomorphic on $D \setminus \{w\}$. Then $\int_{\partial X} f(z)\,dz = 0$.*

Proof The continuity of the function f at w ensures that there exist some positive real numbers M and δ_0 such that the open disk of radius δ_0 about w is contained in the interior of X and $|f(z)| \leq M$ for all complex numbers z satisfying $|z - w| < \delta_0$. Moreover, given any positive real number ε there exists a closed triangle T_ε such that $w \in \mathrm{int}(T_\varepsilon)$, T_ε is contained in the open disk of radius δ_0 about w, and the perimeter of T_ε is of length less than ε. Then $\left|\int_{\partial T_\varepsilon} f(z)\,dz\right| \leq M\varepsilon$ (see Lemma 4.2). Let $Y = X \setminus \mathrm{int}(T_\varepsilon)$, where $\mathrm{int}(T_\varepsilon)$ denotes the interior of the triangle T_ε. Then Y is a closed polygonal set, and the function f is holomorphic over an open set that contains Y, and therefore $\int_{\partial Y} f(z)\,dz = 0$. But the boundary of Y is the disjoint union of the boundary of X and the boundary of T_ε, and

$$\int_{\partial Y} f(z)\,dz = \int_{\partial X} f(z)\,dz - \int_{\partial T_\varepsilon} f(z)\,dz,$$

(The contribution to $\int_{\partial Y} f(z)\,dz$ arising from the boundary of the triangle T_ε is $-\int_{\partial T_\varepsilon} f(z)\,dz$ on account of the fact that the closed polygonal set Y lies to the right of the line segments making up the boundary of the triangle as those line segments are traversed in an anticlockwise direction.) It follows that

$$\int_{\partial X} f(z)\,dz = \int_{\partial T_\varepsilon} f(z)\,dz$$

and therefore

$$\left|\int_{\partial X} f(z)\,dz\right| \leq M\varepsilon.$$

This inequality holds no matter how small the value of ε that was chosen. It follows that $\int_{\partial X} f(z)\,dz = 0$, as required. ∎

Lemma 6.9 *Let D be an open set in the complex plane, let X be a closed polygonal set contained in D, let w be a complex number in the interior of X, and let $f\colon D \to \mathbb{C}$ be a holomorphic function defined throughout D. Then*

$$\int_{\partial X} \frac{f(z)}{z-w}\,dz = 2\pi i f(w)$$

Proof Let $g\colon D \to \mathbb{C}$ be the function defined such that

$$g(z) = \begin{cases} \dfrac{f(z)-f(w)}{z-w} & \text{if } z \in D \setminus \{w\}; \\ f'(w) & \text{if } z = w. \end{cases}$$

Then the function $g\colon D \to \mathbb{C}$ is continuous on D and holomorphic on $D\setminus\{w\}$. It follows from Lemma 6.8 that $\int_{\partial X} g(z)\,dz = 0$. Therefore

$$\int_{\partial X} \frac{f(z)}{z-w}\,dz = f(w)\int_{\partial X} \frac{dz}{z-w}.$$

But $\displaystyle\int_{\partial X} \frac{dz}{z-w} = 2\pi i$. (Corollary 6.3). The result follows. ∎

6.6 Cauchy's Theorem for Arbitrary Domains

Lemma 6.10 *Let K be a closed bounded subset of $\mathbb{C}$, and let D be an open set in $\mathbb{C}$. Suppose that $K \subset D$. Then there exists a closed polygonal set X such that $X \subset D$ and $K \subset \operatorname{int} X$, where $\operatorname{int} X$ denotes the interior of X.*

Proof There exists some positive number δ such that $|z - w| \geq 2\delta$ for all $z \in K$ and $w \in \mathbb{C} \setminus D$ (see Lemma 1.27). Now the complex plane may be subdivided by lines parallel to the real and imaginary axes into closed squares that are of the form

$$\{x + iy \in \mathbb{C} : k\delta \leq x \leq (k+1)\delta \text{ and } l\delta \leq y \leq (l+1)\delta\}$$

for some integers k and l. The number of such squares that intersect the closed bounded set K is finite; let $X_1, X_2, \ldots, X_r$ be the squares of the above form that intersect K, and let $X = X_1 \cup X_2 \cup \cdots \cup X_r$. Then X is a closed polygonal set. Moreover $K \subset \operatorname{int} X$ and $X \subset D$, as required. ∎

The following theorem is a very general form of Cauchy's Theorem.

Theorem 6.11 *Let D be an open set in $\mathbb{C}$, and let $\gamma_1, \gamma_2, \ldots, \gamma_m$ be piecewise continuously differentiable closed paths in D. Suppose that $\sum_{j=0}^{m} n(\gamma_j, w) = 0$ for all $w \in \mathbb{C} \setminus D$, where $n(\gamma_j, w)$ denotes the winding number of the closed path γ_j about w. Then*

$$\sum_{j=1}^{m} \int_{\gamma_j} f(z)\, dz = 0$$

for all holomorphic functions $f\colon D \to \mathbb{C}$ defined throughout the open set D.

Proof Let $K = F \cup G$, where $F = [\gamma_1] \cup [\gamma_2] \cup \cdots \cup [\gamma_m]$ and

$$G = \left\{ z \in \mathbb{C} \setminus F : \sum_{j=1}^{m} n(\gamma_j, z) \neq 0 \right\}.$$

Then

$$\mathbb{C} \setminus K = \left\{ z \in \mathbb{C} \setminus F : \sum_{j=1}^{m} n(\gamma_j, z) = 0 \right\}.$$

Now, for each closed path γ_j, the function mapping $z \in \mathbb{C} \setminus F$ to $n(\gamma_j, z)$ is a continuous integer-valued function on $\mathbb{C} \setminus F$. (Corollary 3.4). It follows that, given any $z_0 \in \mathbb{C} \setminus K$, there exists an open subset V of $\mathbb{C} \setminus F$ such that $n(\gamma_j, z) = n(\gamma_j, z_0)$ for all $z \in V$. But then $\sum_{j=1}^{m} n(\gamma_j, z) = 0$ for all $z \in V$, and thus $V \subset \mathbb{C} \setminus K$. This shows that $\mathbb{C} \setminus K$ is an open set, and thus K is a closed set.

Next we show that the set K is bounded. Now there exists some real number R_0 such that $[\gamma_j]$ is contained in the open disk of radius R_0 about the origin for $j = 1, 2, \ldots, m$ (see Lemma 1.31). It then follows from Corollary 3.5 that $n(\gamma_j, w) = 0$ when $|w| \geq R_0$. Thus the closed set K is contained in the open disk of radius R_0 about zero, and is therefore bounded. Moreover the conditions in the statement of the theorem ensure that $K \subset D$. It follows from Lemma 6.10 that there exists a closed polygonal set X such that $K \subset \operatorname{int} X$ and $X \subset D$. Now $[\gamma_j] \subset K \subset \operatorname{int} X$ for $j = 1, 2, \ldots, m$, and

$$f(z) = \frac{1}{2\pi i} \int_{\partial X} \frac{f(w)}{w - z}\, dw$$

for all $z \in \operatorname{int} X$ (Lemma 6.9). Therefore

$$\begin{aligned} \int_{\gamma_j} f(z)\, dz &= \frac{1}{2\pi i} \int_{\gamma_j} \left(\int_{\partial X} \frac{f(w)}{w - z}\, dw \right) dz \\ &= \frac{1}{2\pi i} \int_{\partial X} \left(f(w) \int_{\gamma_j} \frac{dz}{w - z} \right) dw \\ &= \int_{\partial X} f(w) n(\gamma_j, w)\, dw \end{aligned}$$

Now the boundary of the closed polygonal set X is contained in the complement $\mathbb{C} \setminus K$ of K, and therefore $\sum_{j=1}^{m} n(\gamma_j, w) = 0$ at all points of the boundary of X. It follows that

$$\sum_{j=1}^{m} \int_{\gamma_j} f(z)\, dz = \int_{\partial X} \left(f(w) \sum_{j=1}^{m} n(\gamma_j, w) \right) dw = 0,$$

as required. ∎

Corollary 6.12 *Let D be an open set in $\mathbb{C}$, let $\gamma_1, \gamma_2, \ldots, \gamma_s$ be piecewise continuously differentiable closed paths in D, and let r be an integer satisfying $1 \leq r < s$. Suppose that $\sum_{j=0}^{r} n(\gamma_j, w) = \sum_{j=r+1}^{s} n(\gamma_j, w)$ for all $w \in \mathbb{C} \setminus D$. Then*

$$\sum_{j=1}^{r} \int_{\gamma_j} f(z)\, dz = \sum_{j=r+1}^{s} \int_{\gamma_j} f(z)\, dz$$

for all holomorphic functions $f\colon D \to \mathbb{C}$ defined throughout the open set D.

Proof For each of the paths $\gamma_j\colon [a_j, b_j] \to \mathbb{C}$ with $j > r$, let $\theta_j\colon [a_j, b_j] \to \mathbb{C}$ be the reversed path defined so that $\theta_j(t) = \gamma_j(a_j + b_j - t)$ for all $t \in [a_j, b_j]$. Then $n(\theta_j, w) = -n(\gamma_j, w)$ for all $w \in \mathbb{C} \setminus [\gamma_j]$, and $\int_{\theta_j} f(z)\,dz = -\int_{\gamma_j} f(z)\,dz$ for all continuous functions f defined along $[\gamma_j]$.

If $\sum_{j=1}^{r} n(\gamma_j, w) = \sum_{j=r+1}^{s} n(\gamma_j, w)$ then $\sum_{j=1}^{r} n(\gamma_j, w) + \sum_{j=r+1}^{s} n(\theta_j, w) = 0$ for all $w \in \mathbb{C} \setminus D$. It therefore follows from Theorem 6.11 that

$$\sum_{j=1}^{m} \int_{\gamma_j} f(z)\,dz - \sum_{j=r+1}^{s} \int_{\gamma_j} f(z)\,dz = \sum_{j=1}^{m} \int_{\gamma_j} f(z)\,dz + \sum_{j=r+1}^{s} \int_{\theta_j} f(z)\,dz = 0$$

for all holomorphic functions f defined over D, as required. ∎

6.7 Cauchy's Theorem for Simply-Connected Domains

Corollary 6.13 (Cauchy's Theorem for Simply-Connected Domains) *Let $f\colon D \to \mathbb{R}$ be a holomorphic function defined over a simply-connected open set D in $\mathbb{C}$. Then*

$$\int_\gamma f(z)\,dz = 0.$$

for all piecewise continuously differentiable closed paths γ in D.

Proof The requirement that the open set D be simply-connected ensures that the winding number of any closed curve in D about any element of the complement of D is zero (see Proposition 3.9). The result thus follows immediately from Theorem 6.11. ∎

6.8 Residues

Proposition 6.14 *Let w be a complex number, let r be a positive real number, and let f be a complex-valued function that is defined and holomorphic throughout $D_{w,r} \setminus \{w\}$, where $D_{w,r}$ is the open disk of radius r about w. Then there exists some complex number $\operatorname{Res}_w(f)$ with the property that $\int_\gamma f(z)\,dz = 2\pi i n(\gamma, w) \operatorname{Res}_w(f)$ for all piecewise continuously differentiable closed paths γ in $D_{w,r} \setminus \{w\}$.*

Proof Let $D^*_{w,r} = D_{w,r} \setminus \{w\}$, and let $\gamma_1\colon [0, 1] \to D^*_{w,r}$ be the closed path defined such that $\gamma_1(t) = w + \frac{1}{2} r e^{2\pi i t}$ for all $t \in [0, 1]$. Now $n(\gamma, z) = 0$ and $n(\gamma_1, z) = 0$ whenever $|z - w| \geq r$ (see Corollary 3.5). Thus $n(\gamma, z) -$

$kn(\gamma_1, z) = 0$ for all $z \in \mathbb{C} \setminus D^*_{w,r}$, where $k = n(\gamma, w)$. It follows from Theorem 6.11 (and Corollary 6.12) that

$$\int_\gamma f(z)\,dz - k\int_{\gamma_1} f(z)\,dz = 0$$

for any holomorphic function f defined throughout $D^*_{w,r}$. Let

$$\operatorname{Res}_w(f) = \frac{1}{2\pi i}\int_{\gamma_1} f(z)\,dz.$$

Then $\int_\gamma f(z)\,dz = 2\pi i n(\gamma, w)\operatorname{Res}_w(f)$ for all holomorphic functions f defined throughout $D^*_{w,r}$ and for all piecewise continuously differentiable closed paths γ in $D^*_{w,r}$, as required. ∎

Definition Let w be a complex number, and let f be a complex-valued function that is defined and holomorphic throughout the open set $D_{w,r} \setminus \{w\}$ for some positive real number r, where $D_{w,r}$ denotes the open disk of radius r about w. The *residue* of f at w is defined to be the complex number $\operatorname{Res}_w(f)$ characterized by the property that that $\int_\gamma f(z)\,dz = 2\pi i n(\gamma, w)\operatorname{Res}_w(f)$ for all piecewise continuously differentiable closed paths γ in $D_{w,r} \setminus \{w\}$.

Let w be a complex number, let r be a positive real number, and let f be a complex-valued function that is holomorphic throughout $D_{w,r'} \setminus \{w\}$ for some real number r' satisfying $r' > r$. Then

$$\operatorname{Res}_w(f) = \frac{r}{2\pi}\int_0^{2\pi} f(w + re^{i\theta})e^{i\theta}\,d\theta.$$

This formula may be derived on evaluating the path integral of f along the closed path $\gamma_{w,r}\colon [0, 2\pi] \to \mathbb{C}$, where $\gamma_{w,r}(\theta) = w + re^{i\theta}$ for all $\theta \in [0, 2\pi]$.

Lemma 6.15 *Let w be a complex number, let r be a positive real number, and let f be a complex-valued function that is holomorphic throughout $D_{w,r} \setminus \{w\}$, where $D_{w,r}$ is the open disk of radius r about w. Suppose that there exist complex numbers $b_1, b_2, \ldots, b_m$ and a holomorphic function $g\colon D_{w,r} \to \mathbb{C}$ defined throughout the open disk $D_{w,r}$ such that*

$$f(z) = \sum_{j=1}^{m} \frac{b_j}{(z - w)^j} + g(z)$$

for all $z \in D_{w,r} \setminus \{w\}$. Then $\operatorname{Res}_w(f) = b_1$.

Proof Let $\gamma: [a,b] \to D_{w,r} \setminus \{w\}$ be a closed path in $D_{w,r} \setminus \{w\}$, where $D_{w,r} \setminus \{w\} = \{z \in \mathbb{C} : 0 < |z-w| < r\}$, and let $h: D_{w,r} \setminus \{w\} \to \mathbb{C}$ be defined by

$$h(z) = \sum_{j=2}^{m} \frac{b_j}{(z-w)^j}.$$

Then $h(z) = H'(z)$ for all $z \in D_{w,r} \setminus \{w\}$, where

$$H(z) = -\sum_{j=2}^{m} \frac{b_j}{(j-1)(z-w)^{j-1}}$$

for all $z \in D_{w,r} \setminus \{w\}$. Therefore

$$\int_\gamma h(z)\,dz = \int_\gamma H'(z)dz = H(\gamma(b)) - H(\gamma(a)) = 0.$$

(see Lemma 5.5). Also Cauchy's Theorem for simply-connected open sets ensures that

$$\int_\gamma g(z)\,dz = 0$$

(see Corollary 6.13). It follows that

$$\int_\gamma f(z)\,dz = \int_\gamma \left(h(z) + \frac{b_1}{z-w} + g(z) \right) dz = b_1 \int_\gamma \frac{dz}{z-w} = 2\pi i b_1 n(\gamma, w)$$

(see Proposition 6.2). Thus $\mathrm{Res}_w(f) = b_1$, as required. ∎

6.9 Cauchy's Residue Theorem

We prove a very general form of Cauchy's Residue Theorem.

Theorem 6.16 *Let D be an open set in $\mathbb{C}$, and let $\gamma_1, \gamma_2, \ldots, \gamma_m$ be piecewise continuously differentiable closed paths in D. Suppose that $\sum_{j=0}^{m} n(\gamma_j, w) = 0$ for all $w \in \mathbb{C} \setminus D$, where $n(\gamma_j, w)$ denotes the winding number of the closed path γ_j about w. Let $w_1, w_2, \ldots, w_s$ be complex numbers in D that do not lie on any of the paths $\gamma_1, \gamma_2, \ldots, \gamma_m$. Then*

$$\sum_{j=1}^{m} \int_{\gamma_j} f(z)\,dz = 2\pi i \sum_{j=1}^{m} \sum_{k=1}^{s} n(\gamma_j, w_k)\,\mathrm{Res}_{w_k}(f)$$

for all complex-valued functions f that are defined and holomorphic throughout the open set $D \setminus \{w_1, w_2, \ldots, w_s\}$, where $n(\gamma_j, w_k)$ denotes the winding number of the closed path γ_j about w_k, and $\mathrm{Res}_{w_k}(f)$ denotes the residue of f at w_k.

Proof Let r be a positive real number chosen such that $D_{w_k,r} \subset D$ and $D_{w_k,r} \cap \{w_1, w_2, \dots, w_s\} = \{w_k\}$ for $k = 1, 2, \dots, s$, where $D_{w_k,r}$ denotes the open disk $D_{w_k,r}$ of radius r about w_k. We can then find continuously differentiable closed paths $\varphi_1, \varphi_2, \dots, \dots, \varphi_s$ in D such that $[\varphi_k]$ is contained $D_{w_k,r} \setminus \{w_k\}$ and

$$n(\varphi_k, w_k) = -\sum_{j=1}^{m} n(\gamma_j, w_k).$$

Then

$$\sum_{j=1}^{m} n(\gamma_j, w) + \sum_{k=1}^{s} n(\varphi_k, w) = 0$$

for all complex numbers w that do not belong to $D \setminus \{w_1, w_2, \dots, w_s\}$. It follows from Cauchy's Theorem (Theorem 6.11) that

$$\sum_{j=1}^{m} \int_{\gamma_j} f(z)\,dz + \sum_{k=1}^{s} \int_{\varphi_k} f(z)\,dz = 0.$$

But it follows from Proposition 6.14 that

$$\int_{\varphi_k} f(z)\,dz = 2\pi i n(\varphi_k, w_k) \operatorname{Res}_{w_k}(f) = -2\pi i \sum_{j=1}^{m} n(\gamma_j, w_k) \operatorname{Res}_{w_k}(f).$$

Therefore

$$\sum_{j=1}^{m} \int_{\gamma_j} f(z)\,dz = 2\pi i \sum_{j=1}^{m} \sum_{k=1}^{s} n(\gamma_j, w_k) \operatorname{Res}_{w_k}(f),$$

as required. ∎

Corollary 6.17 *Let D be a simply-connected open set in $\mathbb{C}$, let γ be a piecewise continuously differentiable closed path in D, and let $w_1, w_2, \dots, w_s$ be complex numbers in D that do not lie on the path γ. Then*

$$\int_{\gamma} f(z)\,dz = 2\pi i \sum_{k=1}^{s} n(\gamma, w_k) \operatorname{Res}_{w_k}(f)$$

for all complex-valued functions f that are defined and holomorphic throughout $D \setminus \{w_1, w_2, \dots, w_s\}$.

Proof The requirement that the open set D be simply-connected ensures that the winding number of any closed curve in D about any element of the complement of D is zero (see Proposition 3.9). The result therefore follows directly on applying Theorem 6.16. ∎

Corollary 6.18 *Let D be a simply-connected open set in $\mathbb{C}$, let $f\colon D \to \mathbb{C}$ be a holomorphic function defined throughout D, and let γ be a piecewise continuously differentiable closed path in D.*

$$\int_\gamma \frac{f(z)}{z-w}\,dz = 2\pi i n(\gamma, w) f(w)$$

for all $w \in D$.

Proof Let $w \in D$. The integrand $\dfrac{f(z)}{z-w}$ is a holomorphic function on $D \setminus \{w\}$ and its residue at w is $f(w)$. (This follows directly on applying Lemma 6.9 to some triangle in D that contains the point w in its interior.) The required result therefore follows on applying Corollary 6.17. ∎

Example Let $D = \mathbb{C} \setminus \{-ia, ia\}$, where a is a positive real number, and let f be the holomorphic function on D defined such that $f(z) = (z^2 + a^2)^{-1}$ for all $z \in D$. Let R be a real number satisfying $R > a$, let $\rho_R\colon [-R, R] \to \mathbb{C}$ and $\sigma_R\colon [0, 1] \to \mathbb{C}$ be the continuously differentiable paths defined such that $\rho_R(t) = t$ for all $t \in [-R, R]$ and $\sigma_R(t) = Re^{\pi i t}$ for all $t \in [0, 1]$, and let $\varphi_R\colon [-R, R+1]$ be the closed path defined such that

$$\varphi_R(t) = \begin{cases} \rho_R(t) & \text{if } -R \le t \le R; \\ \sigma_R(t-R) & \text{if } R \le t \le R+1. \end{cases}$$

Then φ_R is a piecewise continuously differentiable closed path which traverses the boundary of the semicircle

$$\{z \in \mathbb{C} : |z| \le R \text{ and } \operatorname{Im} z \ge 0\}.$$

Moreover

$$\int_{\varphi_R} f(z)\,dz = \int_{\rho_R} f(z)\,dz + \int_{\sigma_R} f(z)\,dz.$$

Now $|f(z)| \le (R^2 - a^2)^{-1}$ for all $z \in [\sigma_R]$, and the length of the path σ_R is πR. It follows from Lemma 4.2 that

$$\left| \int_{\sigma_R} f(z)\,dz \right| \le \frac{\pi R}{R^2 - a^2}.$$

Therefore

$$\lim_{R \to +\infty} \int_{\sigma_R} f(z)\,dz = 0.$$

It follows that

$$\lim_{R\to+\infty}\int_{\varphi_R} f(z)\,dz = \lim_{R\to+\infty}\int_{\rho_R} f(z)\,dz = \int_{-\infty}^{+\infty}\frac{dt}{t^2+a^2}.$$

The function f is holomorphic except at ia and $-ia$. The singularity ia lies in the interior of the semicircle bounded by $[\varphi_R]$, and the singularity $-ia$ lies outside this semicircle. The boundary of the semicircle is traversed by the closed path φ_R once in the anticlockwise direction. It easily follows from this that $n(\varphi_R, ia) = 1$ and $n(\varphi_R, -ia) = 0$, since $R > a$. It then follows from Corollary 6.17 that

$$\int_{\varphi_R} f(z)\,dz = 2\pi i\,\mathrm{Res}_{ia}(f).$$

Now

$$f(z) = \frac{1}{2ia(z-ia)} - \frac{1}{2ia(z+ia)},$$

and therefore $\mathrm{Res}_{ia}(f) = (2ia)^{-1}$. It follows that It follows from that

$$\int_{-\infty}^{+\infty}\frac{dt}{t^2+a^2} = \int_{\varphi_R} f(z)\,dz = 2\pi i\,\mathrm{Res}_{ia}(f) = \frac{\pi}{a}.$$