Course 214
Section 5: Holomorphic Functions
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5 Holomorphic Functions

5.1 Holomorphic Functions and Derivatives

Definition A function $f\colon D \to \mathbb{C}$ defined on an open set D in the complex plane is said to be *holomorphic* on D if the limit

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

is defined for all $z \in D$. The value of this limit is denoted by $f'(z)$, or by $\dfrac{df(z)}{dz}$, and is referred to as the *derivative* of the function f at z.

Note that if $f\colon D \to \mathbb{C}$ is a holomorphic function defined on an open set $\mathbb{C}$ in the complex plane then f is continuous on D. For let $z \in D$. Then

$$\begin{aligned} \lim_{h \to 0} f(z+h) &= \lim_{h \to 0} \left(f(z) + h \times \frac{f(z+h) - f(z)}{h} \right) \\ &= f(z) + \left(\lim_{h \to 0} h \right) \left(\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \right) \\ &= f(z), \end{aligned}$$

and thus the function f is continuous at z, as required.

Lemma 5.1 *A function $f\colon D \to \mathbb{C}$, defined on an open set D in the complex plane, is holomorphic on D if and only if, given any complex number w belonging to D, and given any positive real number ε, there exists some real positive number δ such that $|f(z) - f(w) - (z-w)f'(w)| \leq \varepsilon |z-w|$ whenever $|z-w| < \delta$.*

Proof The function f has a well-defined derivative $f'(w)$ at a point w of D if and only if

$$f'(w) = \lim_{h \to 0} \frac{f(w+h) - f(w)}{h} = \lim_{z \to w} \frac{f(z) - f(w)}{z - w}.$$

This limit exists if and only if, given any positive real number ε, there exists some positive real number δ such that

$$\left| \frac{f(z) - f(w) - (z-w)f'(w)}{z - w} \right| \leq \varepsilon$$

whenever $0 < |z-w| < \delta$. The required result follows directly on rearranging the above inequality. ∎

Proposition 5.2 *Let $f\colon D \to \mathbb{C}$ and $g\colon D \to \mathbb{C}$ be holomorphic functions defined over an open set D in the complex plane. Then the sum $f + g$, difference $f - g$ and product $f \cdot g$ of the functions f and g are holomorphic, where $(f + g)(z) = f(z) + g(z)$, $(f - g)(z) = f(z) - g(z)$ and $(f \cdot g)(z) = f(z)g(z)$ for all $z \in D$. Moreover $(f + g)'(z) = f'(z) + g'(z)$, $(f - g)'(z) = f'(z) - g'(z)$ and $(f \cdot g)'(z) = f'(z)g(z) + f(z)g'(z)$ for all $z \in D$.*

Proof The results for $f + g$ and $f - g$ follow easily from the fact that the limit of a sum or difference of two complex-valued functions is the sum or difference of the limits of those functions (Proposition 1.11). The limit of a product of complex-valued functions is the product of the limits of those functions, and therefore

$$\begin{aligned}(f \cdot g)'(z) &= \lim_{h\to 0} \frac{f(z+h)g(z+h) - f(z)g(z)}{h} \\ &= \lim_{h\to 0} \left(\frac{(f(z+h) - f(z)}{h}\, g(z+h) \right) \\ &\quad + \lim_{h\to 0} \left(f(z)\, \frac{g(z+h) - g(z)}{h} \right) \\ &= f'(z)g(z) + f(z)g'(z),\end{aligned}$$

as required. ∎

5.2 The Cauchy-Riemann Equations

Let $f\colon D \to \mathbb{C}$ be a holomorphic function defined over an open set D in the complex plane, and let $\tilde{D}$ denote the open set in $\mathbb{R}^2$ defined by

$$\tilde{D} = \{(x, y) \in \mathbb{R}^2 : x + iy \in D\}.$$

Then the holomorphic function f on D determines differentiable real-valued functions u and v on $\tilde{D}$ such that $f(x + iy) = u(x, y) + iv(x, y)$ for all $(x, y) \in \tilde{D}$. Now if g is a function of a complex variable, defined in the neighbourhood of zero, then $\lim_{h\to 0} g(h)$, if it exists, has the same value whether h tends to 0 along the real axis or along the imaginary axis. It follows that if $\lim\limits_{h\to 0} g(h)$ exists then

$$\lim_{h\to 0} g(h) = \lim_{t\to 0} g(t) = \lim_{t\to 0} g(it),$$

where t tends to zero through real values only. On applying this principle to the holomorphic function f, we find that the limit

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{u(x+h,y)+iv(x+h,y)-u(x,y)-iv(x,y)}{h} \\
&= \frac{\partial u(x,y)}{\partial x} + i\frac{\partial v(x,y)}{\partial x}
\end{aligned}
$$

and also

$$
\begin{aligned}
f'(z) &= \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} \\
&= \lim_{h\to 0} \frac{u(x,y+h)+iv(x,y+h)-u(x,y)-iv(x,y)}{ih} \\
&= \frac{\partial v(x,y)}{\partial y} - i\frac{\partial u(x,y)}{\partial y}
\end{aligned}
$$

It follows that the functions u and v must satisfy the partial differential equations

$$
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.
$$

These equations are referred to as the *Cauchy-Riemann equations*. Thus to each holomorphic function f there corresponds a pair of differentiable real-valued functions u and v, defined over an open subset of $\mathbb{R}^2$ and satisfying the above system of partial differential equations. The converse is true (provided that the partial derivatives of the functions u and v are continuous).

5.3 The Chain Rule for Holomorphic Functions

Proposition 5.3 *Let $f\colon D \to \mathbb{C}$ and $g\colon E \to \mathbb{C}$ be holomorphic functions defined over open sets D and E in the complex plane. Suppose that $f(D) \subset E$. Then the composition function $g \circ f\colon D \to \mathbb{C}$ is a holomorphic function on D, and $(g\circ f)'(z) = g'(f(z))f'(z)$ for all $z \in D$.*

Proof Let z_0 be a point of D and let $w_0 = f(z_0)$. Then

$$
\frac{g(f(z)) - g(f(z_0))}{z - z_0} = G(f(z))\frac{f(z)-f(z_0)}{z-z_0},
$$

for all $z \in D$ satisfying $z \neq z_0$, where

$$
G(w) = \begin{cases} \dfrac{g(w)-g(w_0)}{w-w_0} & \text{if } w \neq w_0; \\ g'(w_0) & \text{if } w = w_0, \end{cases}.
$$

Now $\lim_{w\to w_0} G(w) = G(w_0)$, and therefore the function G is continuous at w_0 (Lemma 1.12). It follows that the composition function $G \circ f\colon D \to \mathbb{C}$ is continuous at at z_0 (Lemma 1.14), and therefore $\lim_{z \to z_0} G((f(z)) = G(f(z_0)) = G(w_0) = g'(f(z_0))$. It follows from standard properties of limits of complex-valued functions (Proposition 1.11) that the limit defining the derivative $(g \circ f)'(z_0)$ of $g \circ f$ at z_0 exists, and

$$\begin{aligned}
(g \circ f)'(z_0) &= \lim_{z \to z_0} \frac{g(f(z)) - g(f(z_0))}{z - z_0} \\
&= \left(\lim_{z \to z_0} G(f(z)) \right) \left(\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \right) \\
&= g'(f(z_0)) f'(z_0),
\end{aligned}$$

as required. ∎

Let $\gamma \colon (a, b) \to \mathbb{C}$ be a complex-valued function defined on an open interval (a, b). Then there are real-valued functions α and β defined on (a, b) such that $\gamma(t) = \alpha(t) + i\beta(t)$. for all $t \in (a, b)$. We say that the function γ is *differentiable* on (a, b) if the functions α and β are differentiable, in which case we define

$$\gamma'(t) = \frac{d\gamma(t)}{dt} = \frac{d\alpha(t)}{dt} + i \frac{d\beta(t)}{dt} = \lim_{h \to 0} \frac{\gamma(t + h) - \gamma(t)}{h}.$$

Lemma 5.4 *Let $f \colon D \to \mathbb{C}$ be a holomorphic function defined over a subset D of $\mathbb{C}$ and let $\gamma \colon [a, b] \to D$ be a continuously differentiable path in D. Then $(f \circ \gamma)'(t) = f'(\gamma(t))\gamma'(t)$ for all $t \in [a, b]$.*

Proof The method of proof is the same at that used to prove Proposition 5.3. Given $t_0 \in (a, b)$ let $z_0 = \gamma(t_0)$, and let

$$F(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} & \text{if } z \neq z_0; \\ f'(z_0)) & \text{if } z = z_0, \end{cases}.$$

for all $z \in D$. Then the function F is continuous at z_0, and therefore

$$\begin{aligned}
(f \circ \gamma)'(t_0) &= \lim_{h \to 0} \frac{f(\gamma(t_0 + h)) - f(\gamma(t_0))}{h} \\
&= \lim_{h \to 0} \left(F(\gamma(t_0 + h)) \, \frac{\gamma(t_0 + h) - \gamma(t_0)}{h} \right) \\
&= F(\gamma(t_0))\gamma'(t_0) = f'(\gamma(t_0))\gamma'(t_0),
\end{aligned}$$

as required. ∎

Lemma 5.5 *Let $f\colon D \to \mathbb{C}$ be a holomorphic function defined over a subset D of $\mathbb{C}$ and let $\gamma\colon [a,b] \to D$ be a piecewise continuously differentiable path in D. Then*

$$\int_\gamma f'(z)\,dz = f(\gamma(b)) - f(\gamma(a))$$

Proof First suppose that the path $\gamma\colon [a,b] \to D$ is continuously differentiable. It follows from Lemma 5.4 and the definition of the path integral that

$$\int_\gamma f'(z)\,dz = \int_a^b f'(\gamma(t))\gamma'(t)\,dt = \int_a^b (f\circ\gamma)'(t)\,dt = f(\gamma(b)) - f(\gamma(a)).$$

Now let γ be a piecewise continuously differentiable path. Then there exist real numbers $s_0, s_1, \ldots, s_m$ satisfying $a = s_0 < s_1 < \cdots < s_m = b$ such that the restriction $\gamma_i\colon [s_{j-1}, s_j] \to D$ of γ to the interval $[s_{j-1}, s_j]$ is continuously differentiable for $j = 1, 2, \ldots, n$. Then

$$\begin{aligned}\int_\gamma f'(z)\,dz &= \sum_{j=1}^m \int_{\gamma_j} f'(z)\,dz = \sum_{j=1}^m \left(f(\gamma(s_j)) - f(\gamma(s_{j-1}))\right) \\ &= f(\gamma(b)) - f(\gamma(a)),\end{aligned}$$

as required. ∎

5.4 Differentiation of Power Series

Theorem 5.6 *Let $\sum\limits_{n=0}^{+\infty} a_n z^n$ be a power series which converges around $z = 0$, let R be a positive real number that does not exceed the radius of convergence of this power series, let*

$$D = \{z \in \mathbb{C} : |z| < R\}$$

and let $f(z) = \sum\limits_{n=0}^{+\infty} a_n z^n$ for all $z \in D$. Then $f\colon D \to \mathbb{C}$ is holomorphic on D, and $f'(z) = \sum\limits_{n=1}^{+\infty} n a_n z^{n-1}$.

Proof Let R_0 be a positive real number satisfying $R_0 < R$, and let z and w be complex numbers satisfying $|z| \le R_0$ and $|w| \le R_0$. Now

$$z^n - w^n = (z - w)\left(\sum_{j=0}^{n-1} z^j w^{n-1-j}\right).$$

It follows that

$$|z^n - w^n| = |z - w| \left| \sum_{j=0}^{n-1} z^j w^{n-1-j} \right| \leq n R_0^{n-1} |z - w|,$$

and

$$\begin{aligned} |z^n - w^n - n(z-w)w^{n-1}| &= \left| (z-w) \sum_{j=0}^{n-1} (z^j - w^j) w^{n-1-j} \right| \\ &\leq |z - w| \sum_{j=0}^{n-1} |z^j - w^j| R_0^{n-1-j} \\ &\leq |z - w|^2 R_0^{n-2} \sum_{j=0}^{n-1} j \\ &\leq \tfrac{1}{2} |z - w|^2 R_0^{n-2} n(n-1). \end{aligned}$$

Choose some real number R_1 satisfying $R_0 < R_1 < R$, and let $\rho = R_0/R_1$. Then there exists some positive real number M such that $|a_n| R_1^n \leq M$ for all non-negative integers n. But then

$$|n a_n z^{n-1}| \leq n M R_1^{-1} \rho^{n-1}$$

and

$$|a_n(z^n - w^n - n(z-w)w^{n-1})| \leq \tfrac{1}{2} M R_1^{-2} n(n-1) \rho^{n-2} |z - w|^2$$

for all non-negative integers n. Now $\rho < 1$, and therefore the infinite series $\sum_{n=1}^{+\infty} n\rho^{n-1}$ and $\sum_{n=2}^{+\infty} n(n-1)\rho^{n-2}$ are convergent. Indeed the convergence of these series may be verified by straightforward applications of the Ratio Test; moreover the first converges to $(1-\rho)^{-2}$ and the second converges to $2(1-\rho)^{-3}$. An immediate application of the Comparison Test shows that the infinite series $\sum_{n=1}^{+\infty} n a_n z^{n-1}$ converges to some complex number $g(z)$ whenever $|z| \leq R_0$. Moreover

$$\begin{aligned} |f(z) - f(w) - (z-w)g(w)| &= \left| \sum_{n=0}^{+\infty} a_n (z^n - w^n - n(z-w)w^{n-1}) \right| \\ &\leq \sum_{n=0}^{+\infty} |a_n (z^n - w^n - n(z-w)w^{n-1})| \\ &\leq C|z - w|^2 \end{aligned}$$

whenever $|z| \le R_0$ and $|w| \le R_0$ where

$$C = \tfrac{1}{2} M R_1^{-2} \sum_{n=2}^{+\infty} n(n-1)\rho^{n-2} = M R_1^{-2}(1-\rho)^{-3}.$$

It follows from this that if $|w| < R_0$ then

$$\left| \frac{f(z) - f(w)}{z - w} - g(w) \right| \le C|z - w|,$$

and therefore

$$\lim_{h \to 0} \frac{f(w+h) - f(w)}{h} = \lim_{z \to w} \frac{f(z) - f(w)}{z - w} = g(w),$$

It follows from this that the derivative of the function f at w exists, and its value is $g(w)$. Thus the function f is holomorphic throughout $\{z \in \mathbb{C} : |z| \le R_0\}$, and $f'(z) = g(z) = \sum\limits_{n=1}^{+\infty} n a_n z^{n-1}$.

Finally we note that given $z \in D$, we can choose R_0 so that $|z| < R_0 < R$. It follows that the function f is holomorphic throughout D as required. ∎

Let $f \colon D \to \mathbb{C}$ be a continuous function defined over an open set D in the complex plane. If there exist functions $f^{(1)}, f^{(2)}, \ldots, f^{(m)}$ such that $\dfrac{d}{dz} f(z) = f^{(1)}(z)$ and $\dfrac{d}{dz} f^{(j-1)}(z) = f^{(j)}(z)$ for values of j satisfying $1 < j \le m$ then we say that the function f is m *times differentiable* on D, and we refer to the function $f^{(m)}$ as the *mth derivative* of the function f.

Corollary 5.7 *Let $\sum\limits_{n=0}^{+\infty} a_n z^n$ be a power series which converges around $z = 0$, let R be a positive real number that does not exceed the radius of convergence of this power series, let*

$$D = \{z \in \mathbb{C} : |z| < R\}$$

and let $f(z) = \sum\limits_{n=0}^{+\infty} a_n z^n$ for all $z \in D$. Then $f \colon D \to \mathbb{C}$ is m times differentiable for all positive integers m, and the mth derivative of f on D is given by the formula

$$f^{(m)}(z) = \sum_{n=m}^{+\infty} \frac{d^m}{dz^m}(a_n z^n) = \sum_{n=m}^{+\infty} \frac{n!}{(n-m)!} a_n z^{n-m}.$$

Remark Corollary 5.7 shows that if a function may be represented as the sum of a power series, then it may be differentiated any number of times. Moreover the derivatives of the function may be calculated by differentiating the power series term by term. It is a remarkable fact that any holomorphic function may be represented in the neighbourhood of a complex number belonging to its domain as the sum of a power series. We shall prove this result using the theory of path integrals of holomorphic functions developed by Cauchy in the early decades of the nineteenth century. Given this fact, it follows from Corollary 5.7 that any holomorphic function defined over an open subset of the complex plane has derivatives of all orders.

Example Let $\exp\colon \mathbb{C} \to \mathbb{C}$ be the exponential function, defined by $\exp(z) = \sum_{n=0}^{+\infty} \frac{z^n}{n!}$. This function is holomorphic throughout the entire complex plane, and

$$\frac{d}{dz}(\exp(z)) = \sum_{n=0}^{+\infty} \frac{d}{dz}\left(\frac{z^n}{n!}\right) = \sum_{n=1}^{+\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{+\infty} \frac{z^n}{n!} = \exp(z).$$

Example The power series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges for all complex numbers z satisfing $|z| < 1$. Let $f(z)$ be the sum of this power series for all complex numbers z satisfying $|z| < 1$. Then f is a holomorphic function on the open unit disk about 0, and

$$f'(z) = \sum_{n=1}^{+\infty} z^{n-1} = \frac{1}{1-z},$$

Let $D_{1,1} = \{z \in \mathbb{C} : |z - 1| < 1\}$, and let $L\colon D_{1,1} \to \mathbb{C}$ be the holomorphic function defined by $L(z) = -f(1 - z)$ for all $z \in D_{1,1}$. Then $L(1) = 0$ and $L'(z) = z^{-1}$ for all $z \in D_{1,1}$.

Now the principal branch of the logarithm function satisfies $\exp(\log(z)) = z$ for all $z \in D_{1,1}$. On differentiating this identity we find that

$$1 = \frac{d}{dz}(\exp(\log(z))) = \exp(\log(z))\frac{d}{dz}(\log(z)) = z\frac{d}{dz}(\log(z)).$$

Thus $\log'(z) = z^{-1}$ for all $z \in D_{1,1}$. Thus the function that sends $z \in D_{1,1}$ to $L(z) - \log(z)$ has zero derivative throughout $D_{1,1}$. A straightforward application of Lemma 5.5 shows that this function must be constant along all continuously differentiable paths in $D_{1,1}$, and therefore must be constant throughout $D_{1,1}$. But $L(1) = 0 = \log 1$. We conclude therefore that $L(z) = \log(z)$ for all $z \in D_{1,1}$. Thus

$$\sum_{n=1}^{\infty} \frac{z^n}{n} = -\log(1 - z)$$

for all complex numbers z satisfying $|z| < 1$.