Course 214
Section 4: Path Integrals in the Complex Plane
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4 Path Integrals in the Complex Plane

4.1 Path Integrals of Continuous Functions

We say that a path $\gamma\colon [a,b] \to \mathbb{C}$ in the complex plane is *continuously differentiable* (or C^1) if the real and imaginary parts of the function γ are the restrictions to $[a,b]$ of continuously differentiable real-valued functions defined over some open interval that contains the domain $[a,b]$ of γ.

Let $f\colon D \to \mathbb{C}$ be a continuous function defined over a subset D of the complex plane, and let $\gamma\colon [a,b] \to D$ be a continuously differentiable path in D. Let $f(z) = u(z) + iv(z)$ and let $\gamma(t) = \alpha(t) + i\beta(t)$, where $u\colon D \to \mathbb{R}$, $v\colon D \to \mathbb{R}$, $\alpha\colon [a,b] \to \mathbb{R}$ and $\beta\colon [a,b] \to \mathbb{R}$ are continuous real-valued functions. Let

$$\gamma'(t) = \frac{d\gamma(t)}{dt} = \frac{d\alpha(t)}{dt} + i\frac{d\beta(t)}{dt},$$

for all $t \in [a,b]$. The *path integral* $\int_\gamma f(z)\,dz$ of f along γ is then defined by the formulae

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_a^b f(\gamma(t))\,\gamma'(t)\,dt \\
&= \int_a^b \left(u(\gamma(t))\,\frac{d\alpha(t)}{dt} - v(\gamma(t))\,\frac{d\beta(t)}{dt} \right) dt \\
&\quad + i\int_a^b \left(u(\gamma(t))\,\frac{d\beta(t)}{dt} + v(\gamma(t))\,\frac{d\alpha(t)}{dt} \right) dt.
\end{aligned}$$

Lemma 4.1 *Let $f\colon D \to \mathbb{C}$ be a continuous function defined over a subset D of the complex plane, let $\gamma\colon [a,b] \to D$ be a continuously differentiable path in D, and let $p\colon [c,d] \to [a,b]$ be a strictly increasing continuously differentiable function mapping some closed interval $[c,d]$ onto the domain $[a,b]$ of the function γ. Then $\int_{\gamma\circ p} f(z)\,dz = \int_\gamma f(z)\,dz$.*

Proof Let $\theta\colon [c,d] \to D$ be the path $\gamma \circ p$ defined by $\theta(s) = \gamma(p(s))$ for all $s \in [c,d]$. Then

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_c^d f(\gamma(p(s)))\gamma'(p(s))\frac{dp(s)}{ds}\,ds \\
&= \int_c^d f(\theta(s))\theta'(s)\,ds = \int_\theta f(z)\,dz,
\end{aligned}$$

as required. ∎

The *length* $L(\gamma)$ of a continuously differentiable path $\gamma\colon [a,b] \to C$ is defined by the formula

$$L(\gamma) = \int_a^b |\gamma'(t)|\,dt$$

Definition We say that a path $\gamma\colon [a,b] \to \mathbb{C}$ is *piecewise continuously differentiable* (or *piecewise* C^1) if γ is continuous and there exist real numbers $s_0, s_1, \ldots, s_m$, where $a = s_0 < s_1 < \ldots < s_m = b$, such that the restriction $\gamma|[s_{j-1}, s_j]\colon [s_{j-1}, s_j] \to \mathbb{C}$ of γ to each interval $[s_{j-1}, s_j]$ is C_1. We say that the piecewise continuously differentiable path γ is the *concatenation* of the continuously differentiable paths $\gamma_1, \gamma_2, \ldots, \gamma_m$, where γ_j is the restriction $\gamma|[s_{j-1}, s_j]$ of γ to the interval $[s_{j-1}, s_j]$.

Let $\gamma\colon [a,b] \to \mathbb{C}$ be a piecewise continuously differentiable path in $\mathbb{C}$ that is the concatenation of continously differentiable paths $\gamma_1, \gamma_2, \ldots, \gamma_m$. We define the *length* of γ to be the sum of the lengths of the continuously differentiable paths $\gamma_1, \gamma_2, \ldots, \gamma_m$, and we define

$$\int_\gamma f(z)\,dz = \sum_{j=1}^m \int_{\gamma_j} f(z)\,dz,$$

for any continuous complex-valued function f whose domain is a subset of the complex plane that contains $[\gamma]$.

Lemma 4.2 *Let $f\colon D \to \mathbb{C}$ be a continuous function defined over a subset D of the complex plane, and let $\gamma\colon [a,b] \to D$ be a piecewise continuously differentiable path in D. Then*

$$\left| \int_\gamma f(z)\,dz \right| \leq L(\gamma) \sup_{z\in[\gamma]} |f(z)|,$$

where $L(\gamma)$ denotes the length of the path γ.

Proof We first prove the result for paths $\gamma\colon [a,b] \to \mathbb{C}$ that are continuously differentiable. Let $M = \sup_{z\in[\gamma]} |f(z)|$, and let $\gamma(t) = \alpha(t) + i\beta(t)$ for all $t \in [a,b]$, where α and β are real-valued functions on $[a,b]$. Now there exists a complex number θ satisfying $|\theta| = 1$ for which $\theta \int_\gamma f(z)\,dz$ is a non-negative real number. Then

$$\begin{aligned} \left| \int_\gamma f(z)\,dz \right| &= \theta \int_\gamma f(z)\,dz = \int_a^b \theta f(\gamma(t))\gamma'(t)\,dt \\ &\leq \int_a^b |\theta f(\gamma(t))\gamma'(t)|\,dt = \int_a^b |\theta||f(\gamma(t))||\gamma'(t)|\,dt \\ &\leq M \int_a^b |\gamma'(t)|\,dt = ML(\gamma), \end{aligned}$$

The result therefore holds for continuously differentiable paths.

Now let $\gamma\colon [a, b] \to \mathbb{C}$ be a piecewise continuously differentiable path. Then there exist real numbers $s_0, s_1, \ldots, s_m$ satisfing $a = s_0 < s_1 < \cdots < s_n$ such that the restriction γ_j of γ to the interval $[s_{j-1}, s_j]$ is continuously differentiable for $j = 1, 2, \ldots, m$. Then

$$\left|\int_\gamma f(z)\,dz\right| = \left|\sum_{j=1}^m \int_{\gamma_j} f(z)\,dz\right| \leq \sum_{j=1}^m \left|\int_\gamma f(z)\,dz\right| \leq \sum_{j=1}^m M_j L(\gamma_j) \leq M L(\gamma),$$

where $M_j = \sup_{z\in[\gamma_j]} |f(z)|$ for $j = 1, 2, \ldots, m$ and $M = \sup_{z\in[\gamma]} |f(z)|$. ∎

Let $z_0, z_1, z_2, \ldots, z_n$ be complex numbers, and let $f\colon D \to \mathbb{C}$ be a continuous complex-valued function defined of some subset D of the complex plane that contains the line segments joining z_{j-1} to z_j for $j = 1, 2, \ldots, n$. We define

$$\int_{[z_0,z_1,\ldots,z_n]} f(z)\,dz = \sum_{j=1}^n \int_{\gamma_j} f(z)\,dz = \int_\gamma f(z)\,dz,$$

where $\gamma_j\colon [0, 1] \to D$ is the path defined by $\gamma_j(t) = (1 - t)z_{j-1} + tz_j$ for all $t \in [0, 1]$, and $\gamma\colon [0, n] \to \mathbb{C}$ is the path defined such that $\gamma(t) = \gamma_j(t - j + 1)$ when $j - 1 \leq t \leq j$ for some integer j between 1 and n. The length of the path γ_j is $|z_j - z_{j-1}|$ for $j = 1, 2, \ldots, n$. It follows from Lemma 4.2 that

$$\left|\int_{[z_0,z_1,\ldots,z_n]} f(z)\,dz\right| \leq M \sum_{j=1}^n |z_j - z_{j-1}|,$$

where $M = \sup_{z\in[\gamma]} |f(z)|$.

Note that if r is any integer satisfying $0 < r < n$ then

$$\int_{[z_0,z_1,\ldots,z_n]} f(z)\,dz = \int_{[z_0,z_1,\ldots,z_r]} f(z)\,dz + \int_{[z_r,z_{r+1},\ldots,z_n]} f(z)\,dz.$$

Thus

$$\int_{[z_0,z_1,\ldots,z_n]} f(z)\,dz = \sum_{j=1}^n \int_{[z_{i-1},z_i]} f(z)\,dz.$$

Also

$$\int_{[z_0,z_1]} f(z)\,dz = -\int_{[z_1,z_0]} f(z)\,dz,$$

for all complex numbers z_0 and z_1, and for all continuous functions f defined along the line segment joining z_0 to z_1.

4.2 Path Integrals and Boundaries

Let us define a *closed polygonal set* in the complex plane to be a closed bounded subset X of the complex plane such that X is the closure of its interior $\mathrm{int}(X)$ and the boundary $X \setminus \mathrm{int}(X)$ of X (i.e., the set consisting of those points of X that do not lie in the interior of X) is a finite union of line segments. We may express the boundary of such a set X as a union of a finite collection $E_1, E_2, \ldots, E_r$ of line segments in such a way that any two distinct line segments belonging to this collection intersect, if at all, at some endpoint common to both line segments. We shall refer to the line segments in this collection as the *edges* of the closed polygonal set X.

Let X be a closed polygonal set in the complex plane, and let z_0 and z_1 be the endpoints of an edge of X. We say that X lies to the *left* of the directed edge from z_0 to z_1 if the ratio $(z - z_0)/(z_1 - z_0)$ has positive imaginary part for all points of X that lie sufficiently close to the midpoint of the edge; we say that X lies to the *right* of the directed edge from z_0 to z_1 if these ratios have negative imaginary part. It follows easily from this that the region X lies either to the left or to the right of the directed edge from z_0 to z_1 (and cannot lie on both sides of this edge). Also if the region lies to the left of the directed edge from z_0 to z_1 then it lies to the right of the directed edge from z_1 to z_0.

Let X be a closed polygonal set in $\mathbb{C}$ with edges $E_1, E_2, \ldots, E_r$, and let $z_-^{(j)}$ and $z_+^{(j)}$ denote the endpoints of the edge E_j for $j = 1, 2, \ldots, r$, where these endpoints are ordered such that X lies to the left of the directed edge from $z_-^{(j)}$ to $z_+^{(j)}$. Given any continuous complex-valued function f defined throughout the boundary of X, we define

$$\int_{\partial X} f(z)\,dz = \sum_{j=1}^{r} \int_{[z_-^{(j)}, z_+^{(j)}]} f(z)\,dz.$$

Example Let z_0, z_1 and z_2 be distinct complex numbers that are not colinear. Then z_0, z_1 and z_2 determine a closed bounded subset T of the complex plane whose boundary consists of the three line segments that join pairs of points in $\{z_0, z_1, z_2\}$. This closed bounded set consists of all points of the complex plane that may be expressed in the form $t_0 z_0 + t_1 z_1 + t_2 z_2$ for some real numbers t_0, t_1, t_2 that satisfy $0 \leq t_j \leq 1$ for $j = 0, 1, 2$ and $t_0 + t_1 + t_2 = 1$. We refer to this closed bounded subset T of the complex plane as the *closed triangle* with vertices z_0, z_1 and z_2. If z_0, z_1 and z_2 are ordered so that the ratio $(z_2 - z_0)/(z_1 - z_0)$ has positive imaginary part, then z_0, z_1 and z_2 occur in anticlockwise order around the boundary of the triangle, the triangle lies to the left of the directed edges, from z_0 to z_1, from z_1 to z_2, and from z_2 to

z_0, and therefore

$$\int_{\partial T} f(z)\,dz = \int_{[z_0,z_1]} f(z)\,dz + \int_{[z_0,z_2]} f(z)\,dz + \int_{[z_2,z_0]} f(z)\,dz.$$

Suppose that we are given a collection of closed triangles in the complex plane. We say that the triangles in this collection *intersect regularly* if the intersection of any two distinct triangles in the collection is a common edge, a common vertex, or the empty set.

Proposition 4.3 *Let $T_1, T_2, \ldots, T_r$ be a finite collection of distinct triangles in the complex plane that intersect regularly, and let X be the closed polygonal set in the complex plane that is the union of the triangles $T_1, T_2, \ldots, T_r$. Then*

$$\int_{\partial X} f(z)\,dz = \sum_{j=1}^{r} \int_{\partial T_j} f(z)\,dz.$$

Proof Let z_0 and z_1 be the endpoints of an edge of a triangle in the collection. Now this edge can be an edge of at most two triangles in the collection. If it is the edge of exactly two triangles T_j and T_k then one of these triangles lies to the left and the other to the right of the directed edge from z_0 and z_1. It follows that the contribution to $\int_{\partial T_j} f(z)\,dz$ deriving from the edge E is the negative of the corresponding contribution $\int_{\partial T_k} f(z)\,dz$. It follows therefore that $\sum_{j=1}^{r} \int_{\partial T_j} f(z)\,dz$ reduces to a sum of path integrals taken along line segments that are edges of exactly one triangle in the collection $T_1, T_2, \ldots, T_k$. The collection of such edges constitutes the boundary of X, and the definition of $\int_{\partial X} f(z)\,dz$ ensures that this quantity is the sum of the contributions to $\sum_{j=1}^{r} \int_{\partial T_j} f(z)\,dz$ deriving from those edges that bound exactly one triangle in the collection $T_1, T_2, \ldots, T_r$. ∎