Course 214
Section 3: Winding Numbers of Closed Paths in the Complex Plane
Second Semester 2008

David R. Wilkins

Contents

3 Winding Numbers of Closed Paths in the Complex Plane

3.1 Paths in the Complex Plane

Let D be a subset of the complex plane $\mathbb{C}$. We define a *path* in D to be a continuous complex-valued function $\gamma\colon [a,b] \to D$ defined over some closed interval $[a,b]$. We shall denote the range $\gamma([a,b])$ of the function γ defining the path by $[\gamma]$. It follows from Theorem 1.32 that $[\gamma]$ is a closed bounded subset of the complex plane.

A path $\gamma\colon [a,b] \to \mathbb{C}$ in the complex plane is said to be *closed* if $\gamma(a) = \gamma(b)$. (This use of the technical term *closed* has no relation to the notions of open and closed sets.) Thus a *closed path* is a path that returns to its starting point.

Let $\gamma\colon [a,b] \to \mathbb{C}$ be a path in the complex plane. We say that a complex number w *lies on* the path γ if $w \in [\gamma]$, where $[\gamma] = \gamma([a,b])$.

Lemma 3.1 *Let $\gamma\colon [a,b] \to \mathbb{C}$ be a path in the complex plane, and let w be a complex number that does not lie on the path γ. Then there exists some positive real number ε_0 such that $|\gamma(t) - w| \geq \varepsilon_0 > 0$ for all $t \in [a,b]$.*

Proof The closed unit interval $[a,b]$ is a closed bounded subset of $\mathbb{R}$. It follows from Lemma 1.31 that there exists some positive real number M such that $|\gamma(t) - w|^{-1} \leq M$ for all $t \in [a,b]$. Let $\varepsilon_0 = M^{-1}$. Then the positive real number ε_0 has the required property. ∎

3.2 The Path Lifting Theorem

Theorem 3.2 (Path Lifting Theorem) *Let $\gamma\colon [a,b] \to \mathbb{C} \setminus \{0\}$ be a path in the set $\mathbb{C} \setminus \{0\}$ of non-zero complex numbers. Then there exists a path $\tilde{\gamma}\colon [a,b] \to \mathbb{C}$ in the complex plane which satisfies $\exp(\tilde{\gamma}(t)) = \gamma(t)$ for all $t \in [a,b]$.*

Proof The complex number $\gamma(t)$ is non-zero for all $t \in [a,b]$, and therefore there exists some positive number ε_0 such that $|\gamma(t)| \geq \varepsilon_0$ for all $t \in [a,b]$. (Lemma 3.1). Moreover it follows from Theorem 1.33 that the function $\gamma\colon [a,b] \to \mathbb{C} \setminus \{0\}$ is uniformly continuous, since the domain of this function is a closed bounded subset of $\mathbb{R}$, and therefore there exists some positive real number δ such that $|\gamma(t) - \gamma(s)| < \varepsilon_0$ for all $s, t \in [a,b]$ satisfying $|t - s| < \delta$. Let m be a natural number satisfying $m > |b - a|/\delta$, and let $t_j = a + j(b-a)/m$ for $j = 0, 1, 2, \ldots, m$. Then $|t_j - t_{j-1}| < \delta$ for

$j = 1, 2, \ldots, m$. It follows from this that $|\gamma(t) - \gamma(t_j)| < \varepsilon_0 \leq |\gamma(t_j)|$ for all $t \in [t_{j-1}, t_j]$, and thus $\gamma([t_{j-1}, t_j]) \subset D_{\gamma(t_j),|\gamma(t_j)|}$ for $j = 1, 2, \ldots, n$, where $D_{w,|w|} = \{z \in \mathbb{C} : |z - w| < |w|\}$ for all $w \in \mathbb{C}$. Now it follows from Corollary 2.13 that there exist continuous functions $F_j \colon D_{\gamma(t_j),|\gamma(t_j)|} \to \mathbb{C}$ with the property that $\exp(F_j(z)) = z$ for all $z \in D_{\gamma(t_j),|\gamma(t_j)|}$. Let $\tilde{\gamma}_j(t) = F_j(\gamma(t))$ for all $t \in [t_{j-1}, t_j]$. Then, for each integer j between 1 and m, the function $\tilde{\gamma}_j \colon [t_{j-1}, t_j] \to \mathbb{C}$ is continuous, and is thus a path in the complex plane with the property that $\exp(\tilde{\gamma}_j(t)) = \gamma(t)$ for all $t \in [t_{j-1}, t_j]$.

Now $\exp(\tilde{\gamma}_j(t_j)) = \gamma(t_j) = \exp(\tilde{\gamma}_{j+1}(t_j))$ for each integer j between 1 and $m - 1$. The periodicity properties of the exponential function (Lemma 2.11) therefore ensure that there exist integers $k_1, k_2, \ldots, k_{m-1}$ such that $\tilde{\gamma}_{j+1}(t_j) = \tilde{\gamma}_j(t_j) + 2\pi i k_j$ for $j = 1, 2, \ldots, m-1$. It follows from this that there is a well-defined function $\tilde{\gamma} \colon [a, b] \to \mathbb{C}$, where $\tilde{\gamma}(t) = \tilde{\gamma}_1(t)$ whenever $t \in [a, t_1]$, and

$$\tilde{\gamma}(t) = \tilde{\gamma}_j(t) - 2\pi i \sum_{r=1}^{j-1} k_r$$

whenever $t \in [t_{j-1}, t_j]$ for some integer j between 2 and m. This function $\tilde{\gamma}$ is continuous on each interval $[t_{j-1}, t_j]$, and is therefore continuous throughout $[a, b]$. Moreover $\exp(\tilde{\gamma}(t)) = \gamma(t)$ for all $t \in [a, b]$. We have thus proved the existence of a path $\tilde{\gamma}$ in the complex plane with the required properties. ∎

3.3 Winding Numbers

Let $\gamma \colon [a, b] \to \mathbb{C}$ be a closed path in the complex plane, and let w be a complex number that does not lie on γ. It follows from the Path Lifting Theorem (Theorem 3.2) that there exists a path $\tilde{\gamma}_w \colon [a, b] \to \mathbb{C}$ in the complex plane such that $\exp(\tilde{\gamma}_w(t)) = \gamma(t) - w$ for all $t \in [a, b]$. Now the definition of closed paths ensures that $\gamma(b) = \gamma(a)$. Also two complex numbers z_1 and z_2 satisfy $\exp z_1 = \exp z_2$ if and only if $(2\pi i)^{-1}(z_2 - z_1)$ is an integer (Lemma 2.11). It follows that there exists some integer $n(\gamma, w)$ such that $\tilde{\gamma}_w(b) = \tilde{\gamma}_w(a) + 2\pi i n(\gamma, w)$.

Now let $\varphi \colon [a, b] \to \mathbb{C}$ be any path with the property that $\exp(\varphi(t)) = \gamma(t) - w$ for all $t \in [a, b]$. Then the function sending $t \in [a, b]$ to $(2\pi i)^{-1}(\varphi(t) - \tilde{\gamma}_w(t))$ is a continuous integer-valued function on the interval $[a, b]$, and is therefore constant on this interval (Proposition 1.17). It follows that

$$\varphi(b) - \varphi(a) = \tilde{\gamma}_w(b) - \tilde{\gamma}_w(a) = 2\pi i n(\gamma, w).$$

It follows from this that the value of the integer $n(\gamma, w)$ depends only on the choice of γ and w, and is independent of the choice of path $\tilde{\gamma}_w$ satisfying $\exp(\tilde{\gamma}_w(t)) = \gamma(t) - w$ for all $t \in [a, b]$.

Definition Let $\gamma\colon [a,b] \to \mathbb{C}$ be a closed path in the complex plane, and let w be a complex number that does not lie on γ. The *winding number* of γ about w is defined to be the unique integer $n(\gamma, w)$ with the property that $\varphi(b) - \varphi(a) = 2\pi i n(\gamma, w)$ for all paths $\varphi\colon [a,b] \to \mathbb{C}$ in the complex plane that satisfy $\exp(\varphi(t)) = \gamma(t) - w$ for all $t \in [a,b]$.

Example Let n be an integer, and let $\gamma_n\colon [0,1] \to \mathbb{C}$ be the closed path in the complex plane defined by $\gamma_n(t) = \exp(2\pi i n t)$. Then $\gamma_n(t) = \exp(\varphi_n(t))$ for all $t \in [0,1]$ where $\varphi_n\colon [0,1] \to \mathbb{C}$ is the path in the complex plane defined such that $\varphi_n(t) = 2\pi i n t$ for all $t \in [0,1]$. It follows that $n(\gamma_n, 0) = (2\pi i)^{-1}(\varphi_n(1) - \varphi_n(0)) = n$.

Given a closed path γ, and given a complex number w that does not lie on γ, the winding number $n(\gamma, w)$ measures the number of times that the path γ winds around the point w of the complex plane in the anticlockwise direction.

Proposition 3.3 *Let $\gamma_1\colon [a,b] \to \mathbb{C}$ and $\gamma_2\colon [a,b] \to \mathbb{C}$ be closed paths in the complex plane, and let w be a complex number that does not lie on γ_1. Suppose that $|\gamma_2(t) - \gamma_1(t)| < |\gamma_1(t) - w|$ for all $t \in [a,b]$. Then $n(\gamma_2, w) = n(\gamma_1, w)$.*

Proof Note that the inequality satisfied by the functions γ_1 and γ_2 ensures that w does not lie on the path γ_2. Let $\varphi_1\colon [0,1] \to \mathbb{C}$ be a path in the complex plane such that $\exp(\varphi_1(t)) = \gamma_1(t) - w$ for all $t \in [a,b]$, and let

$$\rho(t) = \frac{\gamma_2(t) - w}{\gamma_1(t) - w}$$

for all $t \in [a,b]$ Then $|\rho(t) - 1| < 1$ for all $t \in [a,b]$, and therefore $[\rho]$ does not intersect the set $\{x \in \mathbb{R} : x \leq 0\}$. It follows that

$$\log\colon \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\} \to \mathbb{C},$$

the principal branch of the logarithm function, is defined and continuous throughout $[\rho]$ (see Proposition 2.12). Let $\varphi_2\colon [0,1] \to \mathbb{C}$ be the path in the complex plane defined such that $\varphi_2(t) = \log(\rho(t)) + \varphi_1(t)$ for all $t \in [a,b]$. Then

$$\exp(\varphi_2(t)) = \exp(\log(\rho(t)))\exp(\varphi_1(t)) = \rho(t)(\gamma_1(t) - w) = \gamma_2(t) - w.$$

Now $\rho(b) = \rho(a)$. It follows that

$$\begin{aligned} 2\pi i n(\gamma_2, w) &= \varphi_2(b) - \varphi_2(a) = \log(\rho(b)) + \varphi_1(b) - \log(\rho(a)) - \varphi_1(a) \\ &= \varphi_1(b) - \varphi_1(a) = 2\pi i n(\gamma_1, w), \end{aligned}$$

as required. ∎

Corollary 3.4 *Let $\gamma\colon [a,b] \to \mathbb{C}$ be a closed path in the complex plane and let W be the set $\mathbb{C} \setminus [\gamma]$ of all points of the complex plane that do not lie on the curve γ. Then the function that sends $w \in W$ to the winding number $n(\gamma, w)$ of γ about w is a continuous function on W.*

Proof Let $w \in W$. It then follows from Lemma 3.1 that there exists some positive real number ε_0 such that $|\gamma(t) - w| \geq \varepsilon_0 > 0$ for all $t \in [a,b]$. Let w_1 be a complex number satisfying $|w_1 - w| < \varepsilon_0$, and let $\gamma_1\colon [a,b] \to \mathbb{C}$ be the closed path in the complex plane defined such that $\gamma_1(t) = \gamma(t) + w - w_1$ for all $t \in [a,b]$. Then $\gamma(t) - w_1 = \gamma_1(t) - w$ for all $t \in [a,b]$, and therefore $n(\gamma, w_1) = n(\gamma_1, w)$. Also $|\gamma_1(t) - \gamma(t)| < |\gamma(t) - w|$ for all $t \in [a,b]$. It follows from Proposition 3.3 that $n(\gamma, w_1) = n(\gamma_1, w) = n(\gamma, w)$. This shows that the function sending $w \in W$ to $n(\gamma, w)$ is continuous on W, as required. ∎

Corollary 3.5 *Let $\gamma\colon [a,b] \to \mathbb{C}$ be a closed path in the complex plane, and let R be a positive real number with the property that $|\gamma(t)| < R$ for all $t \in [a,b]$. Then $n(\gamma, w) = 0$ for all complex numbers w satisfying $|w| \geq R$.*

Proof Let $\gamma_0\colon [a,b] \to \mathbb{C}$ be the constant path defined by $\gamma_0(t) = 0$ for all $[a,b]$. If $|w| > R$ then $|\gamma(t) - \gamma_0(t)| = |\gamma(t)| < |w| = |\gamma_0(t) - w|$. It follows from Proposition 3.3 that $n(\gamma, w) = n(\gamma_0, w) = 0$, as required. ∎

Proposition 3.6 *Let $[a,b]$ and $[c,d]$ be closed bounded intervals, and, for each $s \in [c,d]$, let $\gamma_s\colon [a,b] \to \mathbb{C}$ be a closed path in the complex plane. Let w be a complex number that does not lie on any of the paths γ_s. Suppose that the function $H\colon [a,b] \times [c,d] \to \mathbb{C}$ is continuous, where $H(t,s) = \gamma_s(t)$ for all $t \in [a,b]$ and $s \in [c,d]$. Then $n(\gamma_c, w) = n(\gamma_d, w)$.*

Proof The rectangle $[a,b] \times [c,d]$ is a closed bounded subset of $\mathbb{R}^2$. It follows from Lemma 1.31 that the continuous function on the closed rectangle $[a,b] \times [c,d]$ that sends a point (t,s) of the rectangle to $|H(t,s) - w|^{-1}$ is a bounded function on the square, and therefore there exists some positive number ε_0 such that $|H(t,s) - w| \geq \varepsilon_0 > 0$ for all $t \in [a,b]$ and $s \in [c,d]$.

Now it follows from Theorem 1.33 that the function $H\colon [a,b] \times [c,d] \to \mathbb{C} \setminus \{w\}$ is uniformly continuous, since the domain of this function is a closed bounded set in $\mathbb{R}^2$. Therefore there exists some positive real number δ such that $|H(t,s) - H(t,u)| < \varepsilon_0$ for all $t \in [a,b]$ and for all $s, u \in [c,d]$ satisfying $|s - u| < \delta$. Let $s_0, s_1, \ldots, s_m$ be real numbers chosen such that $c = s_0 < s_1 < \ldots < s_m = d$ and $|s_j - s_{j-1}| < \delta$ for $j = 1, 2, \ldots, m$. Then

$$\begin{aligned} |\gamma_{s_j}(t) - \gamma_{s_{j-1}}(t)| &= |H(t, s_j) - H(t, s_{j-1})| \\ &< \varepsilon_0 \leq |H(t, s_{j-1}) - w| = |\gamma_{s_{j-1}}(t) - w| \end{aligned}$$

for all $t \in [a,b]$, and for each integer j between 1 and m. It therefore follows from Proposition 3.3 that $n(\gamma_{s_{j-1}}, w) = n(\gamma_{s_j}, w)$ for each integer j between 1 and m. But then $n(\gamma_c, w) = n(\gamma_d, w)$, as required. ∎

Definition Let D be a subset of the complex plane, and let $\gamma\colon [a,b] \to D$ be a closed path in D. The closed path γ is said to be *contractible* in D if and only if there exists a continuous function $H\colon [a,b] \times [0,1] \to D$ such that $H(t,1) = \gamma(t)$ and $H(t,0) = H(a,0)$ for all $t \in [a,b]$, and $H(a,s) = H(b,s)$ for all $s \in [0,1]$.

Corollary 3.7 *Let D be a subset of the complex plane, and let $\gamma\colon [a,b] \to D$ be a closed path in D. Suppose that γ is contractible in D. Then $n(\gamma, w) = 0$ for all $w \in \mathbb{C} \setminus D$, where $n(\gamma, w)$ denotes the winding number of γ about w.*

Proof Let $H\colon [a,b] \times [0,1] \to D$ be a continuous function such that $H(t,1) = \gamma(t)$ and $H(t,0) = H(a,0)$ for all $t \in [a,b]$, and $H(a,s) = H(b,s)$ for all $s \in [0,1]$, and, for each $s \in [0,1]$ let $\gamma_s\colon [a,b] \to D$ be the closed path in D defined such that $\gamma_s(t) = H(t,s)$ for all $t \in [a,b]$. Then γ_0 is a constant path, and therefore $n(\gamma_0, w) = 0$ for all points w that do not lie on γ_0. Let w be an element of $w \in \mathbb{C} \setminus D$. Then w does not lie on any of the paths γ_s. It follows from Proposition 3.6 that

$$n(\gamma, w) = n(\gamma_1, w) = n(\gamma_1, w) = n(\gamma_0, w) = 0,$$

as required. ∎

3.4 Path-Connected and Simply-Connected Subsets of the Complex Plane

Definition A subset D of the complex plane is said to be *path-connected* if, given any elements z_1 and z_2, there exists a path in D from z_1 and z_2.

Definition A path-connected subset D of the complex plane is said to be *simply-connected* if every closed loop in D is contractible.

Definition An subset D of the complex plane is said to be a *star-shaped* if there exists some complex number z_0 in D with the property that

$$\{(1-t)z_0 + tz : t \in [0,1]\} \subset D$$

for all $z \in D$. (Thus an open set in the complex plane is a star-shaped if and only if the line segment joining any point of D to z_0 is contained in D.)

Lemma 3.8 *Star-shaped subsets of the complex plane are simply-connected.*

Proof Let D be a star-shaped subset of the complex plane. Then there exists some element z_0 of D such that the line segment joining z_0 to z is contained in D for all $z \in D$. The star-shaped set D is obviously path-connected. Let $\gamma\colon [a,b] \to D$ be a closed path in D, and let $H(t,s) = (1-s)z_0 + s\gamma(t)$ for all $t \in [a,b]$ and $s \in [0,1]$. Then $H(t,s) \in D$ for all $t \in [a,b]$ and $s \in [0,1]$, $H(t,1) = \gamma(t)$ and $H(t,0) = z_0$ for all $t \in [a,b]$. Also $\gamma(a) = \gamma(b)$, and therefore $H(a,s) = H(b,s)$ for all $s \in [0,1]$. It follows that the closed path γ is contractible. Thus D is simply-connected. ∎

The following result is an immediate consequence of Corollary 3.7

Proposition 3.9 *Let D be a simply-connected subset of the complex plane, and let γ be a closed path in D. Then $n(\gamma, w) = 0$ for all $w \in \mathbb{C} \setminus D$.*

3.5 The Fundamental Theorem of Algebra

Theorem 3.10 (The Fundamental Theorem of Algebra) *Let $P\colon \mathbb{C} \to \mathbb{C}$ be a non-constant polynomial with complex coefficients. Then there exists some complex number z_0 such that $P(z_0) = 0$.*

Proof We shall prove that any polynomial that is everywhere non-zero must be a constant polynomial.

Let $P(z) = a_0 + a_1 z + \cdots + a_m z^m$, where $a_1, a_2, \ldots, a_m$ are complex numbers and $a_m \neq 0$. We write $P(z) = P_m(z) + Q(z)$, where $P_m(z) = a_m z^m$ and $Q(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1}$. Let

$$R = \frac{|a_0| + |a_1| + \cdots + |a_m|}{|a_m|}.$$

If $|z| > R$ then $|z| \geq 1$, and therefore

$$\begin{aligned} \left| \frac{Q(z)}{P_m(z)} \right| &= \frac{1}{|a_m z|} \left| \frac{a_0}{z^{m-1}} + \frac{a_1}{z^{m-2}} + \cdots + a_{m-1} \right| \\ &\leq \frac{1}{|a_m|\,|z|} \left(\left| \frac{a_0}{z^{m-1}} \right| + \left| \frac{a_1}{z^{m-2}} \right| + \cdots + |a_{m-1}| \right) \\ &\leq \frac{1}{|a_m|\,|z|} (|a_0| + |a_1| + \cdots + |a_{m-1}|) \leq \frac{R}{|z|} < 1. \end{aligned}$$

It follows that $|P(z) - P_m(z)| < |P_m(z)|$ for all complex numbers z satisfying $|z| > R$.

For each non-zero real number r, let $\gamma_r\colon [0,1] \to \mathbb{C}$ and $\varphi_r\colon [0,1] \to \mathbb{C}$ be the closed paths defined such that $\gamma_r(t) = P(r\exp(2\pi i t))$ and $\varphi_r(t) = P_m(r\exp(2\pi i t)) = a_m r^m \exp(2\pi i m t)$ for all $t \in [0,1]$. If $r > R$ then $|\gamma_r(t) - \varphi_r(t)| < |\varphi_r(t)|$ for all $t \in [0,1]$. It then follows from Proposition 3.3 that $n(\gamma_r, 0) = n(\varphi_r, 0) = m$ whenever $r > R$.

Now if the polynomial P is everywhere non-zero then it follows on applying Proposition 3.6 that the function sending each non-negative real number r to the winding number $n(\gamma_r, 0)$ of the closed path γ_r about zero is a continuous function on the set of non-negative real numbers. But any continuous integer-valued function on a closed bounded interval is necessarily constant (Proposition 1.17). It follows that $n(\gamma_r, 0) = n(\gamma_0, 0)$ for all positive real-numbers r. But γ_0 is the constant path defined by $\gamma_0(t) = P(0)$ for all $t \in [0,1]$, and therefore $n(\gamma_0, 0) = 0$. It follows that is the polynomial P is everywhere non-zero then $n(\gamma_r, 0) = 0$ for all non-negative real numbers r. But we have shown that $n(\gamma_r, 0) = m$ for sufficiently large values of r, where m is the degree of the polynomial P. It follows that if the polynomial P is everywhere non-zero, then it must be a constant polynomial. The result follows. ∎