Course 214
Section 2: Infinite Series
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2 Infinite Series

An infinite series is the formal sum of the form $a_1 + a_2 + a_3 + \cdots$, where each number a_n is real or complex. Such a formal sum is also denoted by $\sum_{n=1}^{+\infty} a_n$.

Sometimes it is appropriate to consider infinite series $\sum_{n=m}^{+\infty} a_n$ of the form $a_m + a_{m+1} + a_{m+2} + \cdots$, where $m \in \mathbb{Z}$. Clearly results for such sequences may be deduced immediately from corresponding results in the case $m = 1$.

Definition An infinite series $\sum_{n=1}^{+\infty} a_n$ is said to *converge* to some complex number s if and only if, given any $\varepsilon > 0$, there exists some natural number N such that $\left| \sum_{n=1}^{m} a_n - s \right| < \varepsilon$ for all natural numbers m satisfying $m \geq N$. If the infinite series $\sum_{n=1}^{+\infty} a_n$ converges to s then we write $\sum_{n=1}^{+\infty} a_n = s$. An infinite series is said to be *divergent* if it is not convergent.

For each natural number m, the *mth partial sum* s_m of the infinite series $\sum_{n=1}^{+\infty} a_n$ is given by $s_m = a_1 + a_2 + \cdots + a_m$. Note that $\sum_{n=1}^{+\infty} a_n$ converges to some complex number s if and only if $s_m \to s$ as $m \to +\infty$. The following proposition therefore follows immediately on applying the results of Proposition 1.2.

Proposition 2.1 *Let $\sum_{n=1}^{+\infty} a_n$ and $\sum_{n=1}^{+\infty} b_n$ be convergent infinite series. Then $\sum_{n=1}^{+\infty} (a_n + b_n)$ is convergent, and $\sum_{n=1}^{+\infty} (a_n + b_n) = \sum_{n=1}^{+\infty} a_n + \sum_{n=1}^{+\infty} b_n$. Also $\sum_{n=1}^{+\infty} (\lambda a_n) = \lambda \sum_{n=1}^{+\infty} a_n$ for any complex number λ.*

If $\sum_{n=1}^{+\infty} a_n$ is convergent then $a_n \to 0$ as $n \to +\infty$. Indeed

$$\lim_{n \to +\infty} a_n = \lim_{n \to +\infty} (s_n - s_{n-1}) = \lim_{n \to +\infty} s_n - \lim_{n \to +\infty} s_{n-1} = s - s = 0,$$

where $s_m = \sum_{n=1}^{m} a_n$ and $s = \sum_{n=1}^{+\infty} a_n = \lim_{m \to +\infty} s_m$. However the condition that $a_n \to 0$ as $n \to +\infty$ is not in itself sufficient to ensure convergence. For example, the series $\sum_{n=1}^{+\infty} 1/n$ is divergent.

Proposition 2.2 *Let $a_1, a_2, a_3, a_4, \ldots$ be an infinite sequence of real numbers. Suppose that $a_n \geq 0$ for all n. Then $\sum_{n=1}^{+\infty} a_n$ is convergent if and only if there exists some real number C such that $a_1 + a_2 + \cdots + a_n \leq C$ for all n.*

Proof The sequence $s_1, s_2, s_3, \ldots$ of partial sums of the series $\sum_{n=1}^{+\infty} a_n$ is non-decreasing, since $a_n \geq 0$ for all n. The result therefore is a consequence of the fact that a non-decreasing sequence of real numbers is convergent if and only if it is bounded above (see Theorem 1.3). ∎

Example Let z be a complex number. The infinite series $1 + z + z^2 + z^3 + \cdots$ is referred to as the *geometric series*. This series is clearly divergent whenever $|z| \geq 1$, since z^n does not converge to 0 as $n \to +\infty$. We claim that the series converges to $1/(1-z)$ whenever $|z| < 1$. Now

$$1 + z + z^2 + \ldots z^m = \frac{1 - z^{m+1}}{1 - z}.$$

(To see this, multiply both sides of the equation by $1 - z$.) Thus if $|z| < 1$ then

$$\lim_{m \to +\infty} (1 + z + z^2 + \ldots z^m) = \frac{1}{1 - z}.$$

2.1 The Comparison Test and Ratio Test

Proposition 2.3 *An infinite series $\sum_{n=1}^{+\infty} a_n$ of real or complex numbers is convergent if and only if, given any $\varepsilon > 0$, there exists some natural number N with the property that*

$$|a_m + a_{m+1} + \cdots + a_{m+k}| < \varepsilon$$

for all m and k satisfying $m \geq N$ and $k \geq 0$.

Proof The stated criterion is equivalent to the condition that the sequence of partial sums of the series be a Cauchy sequence. The required result thus follows immediately from Cauchy's Criterion for convergence (Theorem 1.9). ∎

Proposition 2.4 (Comparison Test) *Suppose that $0 \leq |a_n| \leq b_n$ for all n, where a_n is complex, b_n is real, and $\sum_{n=1}^{+\infty} b_n$ is convergent. Then $\sum_{n=1}^{+\infty} a_n$ is convergent.*

Proof Let $\varepsilon > 0$ be given. Then there exists some natural number N such that $b_m + b_{m+1} + \cdots + b_{m+k} < \varepsilon$ for all m and k satisfying $m \geq N$ and $k \geq 0$. But then

$$\begin{aligned} |a_m + a_{m+1} + \cdots + a_{m+k}| &\leq |a_m| + |a_{m+1}| + \cdots + |a_{m+k}| \\ &\leq b_m + b_{m+1} + \cdots + b_{m+k} < \varepsilon \end{aligned}$$

when $m \geq N$ and $k \geq 0$. Thus $\sum_{n=1}^{+\infty} a_n$ is convergent, by Proposition 2.3. ∎

Let us apply the Comparison Test in the case when a_n and b_n are non-negative real numbers satisfying $0 \leq a_n \leq b_n$ for all n. If $\sum_{n=1}^{+\infty} b_n$ is convergent, then so is $\sum_{n=1}^{+\infty} a_n$. Thus if $\sum_{n=1}^{+\infty} a_n$ is divergent then so is $\sum_{n=1}^{+\infty} b_n$. These results also follow directly from Proposition 2.2.

Example Comparison with the geometric series shows that the infinite series $\sum_{n=1}^{+\infty} z^n/n$ is convergent whenever $|z| < 1$.

Proposition 2.5 (Ratio Test) *Let $a_1, a_2, a_3 \ldots$ be complex numbers. Suppose that $r = \lim_{n \to +\infty} \frac{a_{n+1}}{a_n}$ exists and satisfies $|r| < 1$. Then $\sum_{n=1}^{+\infty} a_n$ is convergent.*

Proof Choose ρ satisfying $|r| < \rho < 1$. Then there exists some natural number N such that $|a_{n+1}/a_n| < \rho$ for all $n \geq N$. Let

$$K = \text{maximum}\left(\frac{|a_1|}{\rho}, \frac{|a_2|}{\rho^2}, \frac{|a_3|}{\rho^3}, \ldots \frac{|a_N|}{\rho^N}\right).$$

Now $|a_{n+1}| \leq \rho |a_n|$ whenever $n \geq N$. Therefore $|a_n| \leq \rho^{n-N}|a_N| \leq K\rho^n$ whenever $n \geq N$. But the choice of K also ensures that $|a_n| \leq K\rho^n$ when $n < N$. Moreover $\sum_{n=1}^{+\infty} K\rho^n$ converges, since $\rho < 1$. The desired result therefore follows on applying the Comparison Test (Proposition 2.4). ∎

Example Let z be a complex number. Then $\displaystyle\sum_{n=0}^{+\infty} \frac{z^n}{n!}$ converges for all values of z. For if $a_n = z^n/n!$ then $a_{n+1}/a_n = z/(n+1)$, and hence $a_{n+1}/a_n \to 0$ as $n \to +\infty$. The result therefore follows on applying the Ratio Test.

Let $\sum_{n=1}^{+\infty} a_n$ be an infinite series for which $r = \lim_{n \to +\infty} a_{n+1}/a_n$ is well-defined. The series clearly diverges if $|r| > 1$, since $|a_n|$ increases without limit as $n \to +\infty$. If however $|r| = 1$ then the Ratio Test is of no help in deciding whether or not the series converges, and one must try other more sensitive tests.

2.2 Absolute Convergence

Definition An infinite series $\sum_{n=1}^{+\infty} a_n$ is said to be *absolutely convergent* if the infinite series $\sum_{n=1}^{+\infty} |a_n|$ is convergent. A convergent series which is not absolutely convergent is said to be *conditionally convergent*.

An absolutely convergent infinite series is convergent, and the sum of any two absolutely convergent series is itself absolutely convergent. These results follow on applying the Comparison Test (Proposition 2.4). Moreover the following criterion for absolute convergence follows directly from Proposition 2.3.

Proposition 2.6 *An infinite series* $\sum_{n=1}^{+\infty} a_n$ *is absolutely convergent if and only if, given any* $\varepsilon > 0$, *there exists some natural number* N *such that*

$$|a_m| + |a_{m+1}| + \cdots + |a_{m+k}| < \varepsilon$$

for all m *and* k *satisfying* $m \geq N$ *and* $k \geq 0$.

Many of the tests for convergence described above do in fact test for absolute convergence; these include the Comparison Test and the Ratio Test.

2.3 The Cauchy Product of Infinite Series

The *Cauchy product* of two infinite series $\sum_{n=0}^{+\infty} a_n$ and $\sum_{n=0}^{+\infty} b_n$ is defined to be the series $\sum_{n=0}^{+\infty} c_n$, where

$$c_n = \sum_{j=0}^{n} a_j b_{n-j} = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0.$$

The convergence of $\sum_{n=0}^{+\infty} a_n$ and $\sum_{n=0}^{+\infty} b_n$ is not in itself sufficient to ensure the convergence of the Cauchy product of these series. Convergence is however assured provided that the series $\sum_{n=0}^{+\infty} a_n$ and $\sum_{n=0}^{+\infty} b_n$ are absolutely convergent.

Theorem 2.7 *The Cauchy product* $\sum_{n=0}^{+\infty} c_n$ *of two absolutely convergent infinite series* $\sum_{n=0}^{+\infty} a_n$ *and* $\sum_{n=0}^{+\infty} b_n$ *is absolutely convergent, and*

$$\sum_{n=0}^{+\infty} c_n = \left(\sum_{n=0}^{+\infty} a_n\right)\left(\sum_{n=0}^{+\infty} b_n\right).$$

Proof For each non-negative integer m, let

$$\begin{aligned} S_m &= \{(j,k) \in \mathbb{Z}\times\mathbb{Z} : 0 \le j \le m, \quad 0 \le k \le m\}, \\ T_m &= \{(j,k) \in \mathbb{Z}\times\mathbb{Z} : j \ge 0,\ k \ge 0,\ 0 \le j+k \le m\}. \end{aligned}$$

Now $\sum_{n=0}^{m} c_n = \sum_{(j,k)\in T_m} a_j b_k$ and $\left(\sum_{n=0}^{m} a_n\right)\left(\sum_{n=0}^{m} b_n\right) = \sum_{(j,k)\in S_m} a_j b_k$. Also

$$\sum_{n=0}^{m} |c_n| \le \sum_{(j,k)\in T_m} |a_j||b_k| \le \sum_{(j,k)\in S_m} |a_j||b_k| \le \left(\sum_{n=0}^{+\infty} |a_n|\right)\left(\sum_{n=0}^{+\infty} |b_n|\right),$$

since $|c_n| \le \sum_{j=0}^{n} |a_j||b_{n-j}|$ and the infinite series $\sum_{n=0}^{+\infty} a_n$ and $\sum_{n=0}^{+\infty} b_n$ are absolutely convergent. It follows from Proposition 2.2 that the Cauchy product $\sum_{n=0}^{+\infty} c_n$ is absolutely convergent, and is thus convergent. Moreover

$$\begin{aligned} &\left|\sum_{n=0}^{2m} c_n - \left(\sum_{n=0}^{m} a_n\right)\left(\sum_{n=0}^{m} b_n\right)\right| \\ &\qquad = \left|\sum_{(j,k)\in T_{2m}\setminus S_m} a_j b_k\right| \\ &\qquad \le \sum_{(j,k)\in T_{2m}\setminus S_m} |a_j b_k| \le \sum_{(j,k)\in S_{2m}\setminus S_m} |a_j b_k| \\ &\qquad = \left(\sum_{n=0}^{2m} |a_n|\right)\left(\sum_{n=0}^{2m} |b_n|\right) - \left(\sum_{n=0}^{m} |a_n|\right)\left(\sum_{n=0}^{m} |b_n|\right), \end{aligned}$$

since $S_m \subset T_{2m} \subset S_{2m}$. But

$$\begin{aligned} \lim_{m\to+\infty} \left(\sum_{n=0}^{2m} |a_n|\right)\left(\sum_{n=0}^{2m} |b_n|\right) &= \left(\sum_{n=0}^{+\infty} |a_n|\right)\left(\sum_{n=0}^{+\infty} |b_n|\right) \\ &= \lim_{m\to+\infty} \left(\sum_{n=0}^{m} |a_n|\right)\left(\sum_{n=0}^{m} |b_n|\right), \end{aligned}$$

since the infinite series $\sum_{n=0}^{+\infty} a_n$ and $\sum_{n=0}^{+\infty} b_n$ are absolutely convergent. It follows that

$$\lim_{m\to+\infty}\left(\sum_{n=0}^{2m} c_n - \left(\sum_{n=0}^{m} a_n\right)\left(\sum_{n=0}^{m} b_n\right)\right) = 0,$$

and hence

$$\sum_{n=0}^{+\infty} c_n = \lim_{m\to+\infty}\sum_{n=0}^{2m} c_n = \left(\sum_{n=0}^{+\infty} a_n\right)\left(\sum_{n=0}^{+\infty} b_n\right),$$

as required. ∎

2.4 Uniform Convergence for Infinite Series

Let $f_1, f_2, f_2, \ldots$ be complex-valued functions defined over a subset D of $\mathbb{C}$. The infinite series series $\sum_{n=1}^{+\infty} f_n(z)$ is said to converge *uniformly* on D to some function s if, given any $\varepsilon > 0$, there exists some natural number N (which does not depend on the value of z) such that $\left|s(z) - \sum_{m=0}^{n} f_m(z)\right| < \varepsilon$ whenever $z \in D$ and $n \geq N$.

Note that an infinite series $\sum_{n=0}^{+\infty} f_n(z)$ of functions converges uniformly if and only if the partial sums of this series converge uniformly. It follows immediately from Theorem 1.20 that if the functions f_n are continuous on D, and if the series $\sum_{n=0}^{+\infty} f_n(z)$ converges uniformly on D to some function, then that function is also continuous on D.

Proposition 2.8 (The Weierstrass M-Test) *Let D be a subset of $\mathbb{C}$ and let $f_1, f_2, f_3, \ldots$ be a sequence of functions from D to $\mathbb{C}$, let $M_1, M_2, M_3, \ldots$ be non-negative real numbers satisfying $|f_n(z)| \leq M_n$ for all natural numbers n and $z \in D$. Suppose that $\sum_{n=1}^{+\infty} M_n$ converges. Then $\sum_{n=1}^{+\infty} f_n(z)$ converges absolutely and uniformly on D.*

Proof It follows immediately from the Comparison Test (Proposition 2.4) that the series $\sum_{n=1}^{+\infty} f_n(z)$ is absolutely convergent for all $z \in D$. We must show that the convergence is uniform.

Let $\varepsilon > 0$ be given. Then there exists some natural number N such that $\sum_{n=N}^{+\infty} M_n < \frac{1}{2}\varepsilon$, since $\sum_{n=1}^{+\infty} M_n$ converges. Now if m and k are integers satisfying $m \geq N$ and $k \geq 0$ then

$$\left|\sum_{n=1}^{m+k} f_n(z) - \sum_{n=1}^{m} f_n(z)\right| = \left|\sum_{n=m+1}^{m+k} f_n(z)\right| \leq \sum_{n=m+1}^{m+k} M_n \leq \sum_{n=N}^{+\infty} M_n < \tfrac{1}{2}\varepsilon$$

for any $z \in D$. On taking the limit as $k \to +\infty$, we see that

$$\left|\sum_{n=1}^{+\infty} f_n(z) - \sum_{n=1}^{m} f_n(z)\right| \leq \tfrac{1}{2}\varepsilon < \varepsilon$$

for all $z \in D$ and $m \geq N$. However N has been chosen independently of z. Thus the infinite series converges uniformly on D, as required. ∎

2.5 Power Series

A *power series* is an infinite series of the form $\sum_{n=0}^{+\infty} a_n(z - z_0)^n$, where the coefficients $a_0, a_1, a_2, \ldots$ are complex numbers.

Definition Let $\sum_{n=0}^{+\infty} a_n(z - z_0)^n$ be a power series centred on some complex number z_0. Suppose that the set of complex numbers z for which the power series converges is bounded. Then the *radius of convergence* R_0 of the power series is defined to be the smallest non-negative real number with the property that every complex number z for which the power series converges satisfies $|z - z_0| \leq R_0$. The circle $\{z \in \mathbb{C} : |z - z_0| = R_0\}$ is then referred to as the *circle of convergence* of the power series. We set $R_0 = +\infty$ if the set of complex numbers z for which the power series converges is unbounded.

Theorem 2.9 *Let $\sum_{n=0}^{+\infty} a_n(z - z_0)^n$ be a power series with radius of convergence R_0, and let $s(z)$ denote the sum of the power series at those complex numbers z at which the series converges.*

(i) *If $R_0 = +\infty$ then $s(z)$ is a continuous function of z defined over the entire complex plane $\mathbb{C}$.*

(ii) *If $R_0 < +\infty$ then $s(z)$ is a continuous function of z defined over the whole of the disk*

$$\{z \in \mathbb{C} : |z - z_0| < R_0\}$$

bounded by the circle of convergence of the power series.

Proof Let z_1 be any complex number satisfying $|z_1 - z_0| < R_0$. Then we can choose R such that $|z_1 - z_0| < R < R_0$ and $R < +\infty$. Now it follows from the definition of the radius of convergence that there exists some complex number w such that $R < |w| < R_0$ and $\sum_{n=0}^{+\infty} a_n w^n$ converges. Choose some positive real number A with the property that $|a_n w^n| \leq A$ for all n, and set $\rho = R/|w|$ and $M_n = A\rho^n$. If $|z - z_0| < R$ then $|a_n(z - z_0)^n| \leq |a_n|R^n \leq A\rho^n = M_n$ for all n. Also $\sum_{n=0}^{+\infty} M_n$ converges to $A/(1-\rho)$. Thus we can apply the Weierstrass M-Test (Proposition 2.8) to deduce that the power series $\sum_{n=0}^{+\infty} a_n(z-z_0)^n$ converges uniformly on the disk $\{z \in \mathbb{C} : |z - z_0| < R\}$ of radius R about z_0. It then follows from Theorem 1.20 that the restriction of the function s to this disk is continuous on the disk, and, in particular, is continuous around z_1. We deduce that the function s is continuous throughout the complex plane when $R_0 = +\infty$, and is continuous inside the circle of convergence when $R_0 < +\infty$, as required. ∎

A power series with finite radius of convergence will converge everywhere within its circle of convergence, and will diverge everywhere outside this circle. However Theorem 2.9 provides no information concerning the behaviour of the power series on the circle of convergence itself.

2.6 The Exponential Function

Definition The *exponential function* $\exp\colon \mathbb{C} \to \mathbb{C}$ is defined for all complex numbers z by the formula

$$\exp(z) = \sum_{n=0}^{+\infty} \frac{z^n}{n!}.$$

A straightforward application of the Ratio Test shows that radius of convergence of the power series defining $\exp(z)$ is infinite, and it therefore follows from Theorem 2.9 that the exponential function is continuous on $\mathbb{C}$.

Lemma 2.10 *The exponential function has the property that*

$$\exp(z + w) = \exp(z)\exp(w).$$

for all complex numbers z and w.

Proof Let z and w be complex numbers. The infinite series defining $\exp(z)$ and $\exp(w)$ are absolutely convergent. It therefore follows from Theorem 2.7

that the value of $\exp(z)\exp(w)$ is the sum of the Cauchy product of the infinite series defining $\exp(z)$ and $\exp(w)$, and therefore

$$\exp(z)\exp(w) = \sum_{n=0}^{+\infty} c_n$$

where

$$c_n = \sum_{j=0}^{n} \frac{z^j}{j!}\,\frac{w^{n-j}}{(n-j)!}$$

On applying the Binomial Theorem, we see that

$$c_n = \frac{1}{n!}\sum_{j=0}^{n} \frac{n!}{j!(n-j)!} z^j w^{n-j} = \frac{1}{n!}\sum_{j=0}^{n} \binom{n}{j} z^j w^{n-j} = \frac{1}{n!}(z+w)^n,$$

and thus $\exp(z)\exp(w) = \exp(z+w)$, as required. ∎

Let z be a complex number. Then $\exp(z)\exp(-z) = \exp(z-z) = \exp(0) = 1$. It follows that $\exp(z) \neq 0$ and $\exp(-z) = 1/\exp(z)$ for all complex numbers z.

Let $\sin\colon \mathbb{C} \to \mathbb{C}$ and $\cos\colon \mathbb{C} \to \mathbb{C}$ be the functions defined as sums of power series according to the formulae

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, \qquad \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}.$$

If z is real then the power series appearing on the right hand side of these identities correspond to the Taylor expansions of the standard sine and cosine functions. A routine application of the Ratio Test shows that these power series both have infinite radius of convergence, and therefore define continuous functions defined over the whole complex plane. These continuous functions therefore constitute a natural extension to the complex plane of the standard sine and cosine functions of basic trigonometry.

On comparing power series, we see that

$$\cos z = \frac{1}{2}(\exp(iz) + \exp(-iz)), \quad \sin z = \frac{1}{2i}(\exp(iz) - \exp(-iz)),$$

$$\cos z + i\sin z = \exp(iz), \quad \cos z - i\sin z = \exp(-iz)$$

for all complex numbers z. Thus if $z = x + iy$, where x and y are real numbers, then

$$\exp(z) = \exp(x+iy) = \exp(x)\exp(iy) = e^x(\cos y + i\sin y).$$

The exponential map $\exp\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is surjective. For, given any complex number w, there exist real numbers r and θ such that $r > 0$ and $w = r(\cos\theta + i\sin\theta)$. Moreover there exists a real number x such that $r = e^x$. (This real number x is the natural logarithm $\log_e r$ of r.) Then $w = \exp(x + i\theta)$.

Lemma 2.11 *Let z and w be complex numbers. Then $\exp(z) = \exp(w)$ if and only if $w = z + 2\pi i n$ for some integer n.*

Proof If $w = z + 2\pi i n$ for some integer n then

$$\exp(w) = \exp(z)\exp(2\pi i n) = \exp(z)(\cos 2\pi n + i\sin 2\pi n) = \exp(z).$$

Conversely suppose that $\exp(w) = \exp(z)$. Let $w - z = u + iv$, where $u, v \in \mathbb{R}$. Then

$$e^u(\cos v + i\sin v) = \exp(w - z) = \exp(w)\exp(z)^{-1} = 1.$$

Taking the modulus of both sides, we see that $e^u = 1$, and thus $u = 0$. Also $\cos v = 1$ and $\sin v = 0$, and therefore $v = 2\pi n$ for some integer n. The result follows. ∎

Proposition 2.12 *Let $D_0 = \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$. Then there exists a unique continuous function $\log\colon D_0 \to \mathbb{C}$ characterized by the properties that $\log(1) = 0$, $|\operatorname{Im}\log(z)| < \pi$ and $\exp(\log(z)) = z$ for all $z \in D_0$. This function has the property that $\log(re^{i\theta}) = \log_e r + i\theta$ for all real numbers r and θ satisfying $r > 0$ and $-\pi < \theta < \pi$, where $\log_e r$ denotes the natural logarithm of the real number r.*

Proof Given any complex number w belonging to the open set D_0 there exist unique real numbers r and θ for which $r > 0$, $-\pi < \theta < \pi$ and $re^{i\theta} = w$. It follows that there is a well-defined function $\log\colon D_0 \to \mathbb{C}$ such that $\log(re^{i\theta}) = \log_e r + i\theta$ for all real numbers real numbers r and θ satisfying $r > 0$ and $-\pi < \theta < \pi$, where $\log_e r$, the natural logarithm of r, is the unique real number t satisfying $e^t = r$. Moreover

$$\exp(\log(re^{i\theta})) = \exp(\log_e r)\exp(i\theta) = re^{i\theta}$$

for all real numbers r and θ with $r > 0$. It follows that $\exp(\log(z)) = z$ for all $z \in D_0$.

Let x and y be real numbers. Then

$$\log(x+iy) = \begin{cases} \frac{1}{2}\log_e(x^2+y^2) + i\arccos\left(\dfrac{x}{\sqrt{x^2+y^2}}\right) & \text{if } y > 0, \\ \frac{1}{2}\log_e(x^2+y^2) - i\arccos\left(\dfrac{x}{\sqrt{x^2+y^2}}\right) & \text{if } y < 0, \\ \frac{1}{2}\log_e(x^2+y^2) + i\arcsin\left(\dfrac{y}{\sqrt{x^2+y^2}}\right) & \text{if } x > 0, \end{cases}$$

where $\arcsin\colon [-1,1] \to [-\pi/2, \pi/2]$ and $\arccos\colon [-1,1] \to [0,\pi]$ are the inverses of the restrictions of the sine and cosine functions to the intervals $[-\pi/2, \pi/2]$ and $[0,\pi]$ respectively. Now the functions $\log_e$, arcsin and arccos are continuous, since the inverse of any strictly increasing or strictly decreasing function defined over some interval in $\mathbb{R}$ is itself continuous. (see Corollary 1.19). We conclude that the restrictions of $\log\colon D_0 \to \mathbb{C}$ to each of the open sets

$$\{z \in \mathbb{C} : \operatorname{Im} z > 0\}, \qquad \{z \in \mathbb{C} : \operatorname{Im} z < 0\}, \qquad \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$$

is continuous, and the open set D_0 is the union of these three open sets. It follows that $\log\colon D_0 \to \mathbb{C}$ is continuous throughout D_0. A straightforward application of Lemma 2.11 then shows that this function log is the unique function from D_0 to $\mathbb{C}$ with the required properties. ∎

The function $\log\colon D_0 \to \mathbb{C}$ characterized by the properties set out in the statement of Proposition 2.12 is referred to as the *principal branch of the logarithm function.* It is impossible to define a continuous 'logarithm' function $L\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ with the property that $\exp(L(z)) = z$ for all $z \in \mathbb{C} \setminus \{0\}$. The best that one can achieve is to define continuous inverses of the exponential map over appropriately chosen open subsets of $\mathbb{C} \setminus \{0\}$. Such functions are referred to as 'branches of the logarithm function'.

Corollary 2.13 *Let w be a non-zero complex number, and let*

$$D_{w,|w|} = \{z \in \mathbb{C} : |z - w| < |w|\}.$$

Then there exists a continuous function $F_w\colon D_{w,|w|} \to \mathbb{C}$ with the property that $\exp(F_w(z)) = z$ for all $z \in D_{w,|w|}$.

Proof Let $D_0 = \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$, let $\log\colon D_0 \to \mathbb{C}$ be the principal branch of the logarithm function, and let ζ be a complex number satisfying $\exp\zeta = w$. Then $z/w \in D_0$ for all $z \in D_{w,|w|}$. A function $F_w\colon D_{w,|w|} \to \mathbb{C}$ with the required properties may therefore be obtained on defining $F_w(z) = \zeta + \log(z/w)$ for all $z \in D_{w,|w|}$. ∎