Course 214
Section 1: Basic Theorems of Complex Analysis
Second Semester 2008

David R. Wilkins

Contents

1 Basic Theorems of Complex Analysis

1.1 The Complex Plane

A *complex number* is a number of the form $x + iy$, where x and y are real numbers, and $i^2 = -1$. The real numbers x and y are uniquely determined by the complex number $x + iy$, and are referred to as the *real* and *imaginary* parts of this complex number.

The algebraic operations of addition, subtraction and multiplication are defined on complex numbers according to the formulae

$$(x+yi)+(u+iv) = (x+u)+i(y+v), \quad (x+yi)-(u+iv) = (x-u)+i(y-v),$$

$$(x + yi) \times (u + iv) = (xu - yv) + i(xv + yu),$$

where x, y, u and v are real numbers.

We regard a real number x as coinciding with the complex number $x + i \times 0$. Note that the operations of addition, subtraction and multiplication of complex numbers defined as above extend the corresponding operations on the set of real numbers.

The set $\mathbb{C}$ of complex numbers, with the operations of addition and multiplication defined above, has the following properties:

(i) $z_1 + z_2 = z_2 + z_1$ for all $z_1, z_2 \in \mathbb{C}$;

(ii) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ for all $z_1, z_2, z_3 \in \mathbb{C}$;

(iii) there exists a complex number 0 with the property that $z+0 = 0+z = z$ for all complex numbers z;

(iv) given any complex number z, there exists a complex number $-z$ such that $z + (-z) = (-z) + z = 0$;

(v) $z_1 \times z_2 = z_2 \times z_1$ for all $z_1, z_2 \in \mathbb{C}$;

(vi) $z_1 \times (z_2 \times z_3) = z_1 \times (z_2 \times z_3)$ for all $z_1, z_2, z_3 \in \mathbb{C}$;

(vii) there exists a complex number 1 with the property that $z \times 1 = 1 \times z = z$ for all complex numbers z;

(viii) given any complex number z satisfying $z \neq 0$, there exists a complex number z^{-1} such that $z \times z^{-1} = z^{-1} \times z = 1$;

(ix) $z_1 \times (z_2 + z_3) = (z_1 \times z_2) + (z_1 \times z_3)$ and $(z_1 + z_2) \times z_3 = (z_1 \times z_3) + (z_2 \times z_3)$ for all $z_1, z_2, z_3 \in \mathbb{C}$.

To verify property (viii), we note that if z is a non-zero complex number, where $z = x + iy$ for some real numbers x and y, and if z^{-1} is given by the formula

$$z^{-1} = \frac{x}{x^2 + y^2} - i\,\frac{y}{x^2 + y^2},$$

then $z \times z^{-1} = z^{-1} \times z = 1$.

Given complex numbers z and w, with $w \neq 0$, we define the *quotient* z/w (i.e., z divided by w) by the formula $z/w = zw^{-1}$.

The *conjugate* $\overline{z}$ of a complex number z is defined such that $\overline{x + iy} = x - iy$ for all real numbers x and y. The *modulus* $|z|$ of a complex number z is defined such that $|x + iy| = \sqrt{x^2 + y^2}$ for all real numbers x and y. Note that $|\overline{z}| = |z|$ for all complex numbers z. Also $\overline{z + w} = \overline{z} + \overline{w}$ for all complex numbers z and w. The real part $\operatorname{Re} z$ of a complex number satisfies the formula $2 \operatorname{Re} z = z + \overline{z}$. Now $|\operatorname{Re} z| \leq |z|$. It follows that $|z + \overline{z}| \leq 2|z|$ for all complex numbers z.

Straightforward calculations show that $z\overline{z} = |z|^2$ for all complex numbers z, from which it easily follows that $z^{-1} = |z|^{-2}\overline{z}$ for all non-zero complex numbers z.

Let z and w be complex numbers, and let $z = x + iy$ and $w = u + iv$, where x, y, u and v are real numbers. Then

$$\begin{aligned} |zw|^2 &= (xu - yv)^2 + (xv + yu)^2 \\ &= (x^2u^2 + y^2v^2 - 2xyuv) + (x^2v^2 + y^2u^2 + 2xyuv) \\ &= (x^2 + y^2)(u^2 + v^2) = |z|^2\,|w|^2 \end{aligned}$$

It follows that $|zw| = |z|\,|w|$ for all complex numbers z and w.

Let z and w be complex numbers. Then

$$\begin{aligned} |z + w|^2 &= (z + w)\,(\overline{z} + \overline{w}) = z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \\ &= |z|^2 + 2 \operatorname{Re} z\overline{w} + |w|^2 \\ &\leq |z|^2 + 2|z\overline{w}| + |w|^2 = |z|^2 + 2|z|\,|w| + |w|^2 \\ &= (|z| + |w|)^2. \end{aligned}$$

It follows that $|z + w| \leq |z| + |w|$ for all complex numbers z and w.

We define the *distance* from a complex number z to a complex number w to be the quantity $|w - z|$. Thus if $z = x + iy$ and $w = u + iv$ then

$$|w - z| = \sqrt{(x - u)^2 + (y - v)^2}.$$

We picture the complex numbers as representing points of the Euclidean plane. A complex number $x + iy$, where x and y are real numbers, represents the point of the plane whose Cartesian coordinates (with respect to an appropriate origin) are (x, y). The fact that $|w - z|$ represents the distance between the points of the plane represented by the complex numbers z and w is an immediate consequence of Pythagoras' Theorem.

Let z_1, z_2 and z_3 be complex numbers. Then

$$|z_3 - z_1| = |(z_3 - z_2) + (z_2 - z_1)| \leq |z_3 - z_2| + |z_2 - z_1|.$$

This important inequality is known as the *Triangle Inequality*. It corresponds to the geometric statement that the length of any side of a triangle in the Euclidean plane is less than or equal to the sum of the lengths of the other two sides.

1.2 Infinite Sequences of Complex Numbers

Definition A sequence $a_1, a_2, a_3, \ldots$ of complex numbers is said to *converge* to some complex number l if the following criterion is satisfied:

> given any positive real number ε, there exists some natural number N such that $|a_j - l| < \varepsilon$ for all natural numbers j satisfying $j \geq N$.

The complex number l is referred to as the *limit* of the sequence $a_1, a_2, a_3, \ldots$, and is denoted by $\lim\limits_{j \to +\infty} a_j$.

A sequence $a_1, a_2, a_3, \ldots$ of complex numbers is said to be *bounded* if there exists some real number $R \geq 0$ such that $|a_j| \leq R$ for all positive integers j. Every convergent sequence of complex numbers is bounded.

Example Let w be a complex number satisfying $|w| < 1$. Then the infinite sequence $w, w^2, w^3, \ldots$ converges to 0. Indeed suppose that $\varepsilon > 0$ is given. We can choose some positive integer N large enough to ensure that $|w|^N < \varepsilon$. Then $|w^j| < \varepsilon$ whenever $j \geq N$.

Lemma 1.1 *Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of complex numbers, and, for each positive integer j, let $a_j = x_j + iy_j$, where x_j and y_j are real numbers. Then $\lim\limits_{j \to +\infty} a_j = l$ for some complex number l if and only if $\lim\limits_{j \to +\infty} x_j = p$ and $\lim\limits_{j \to +\infty} y_j = q$, where p and q are real numbers satisfying $p + iq = l$.*

Proof Let l be a complex number, and let $l = p + iq$, where p and q are real numbers. Suppose that $\lim\limits_{j \to +\infty} a_j = l$. Then, given any positive real number ε, there exists some natural number N such that $|a_j - l| < \varepsilon$ whenever $j \geq N$. But then $|x_j - p| < \varepsilon$ and $|y_j - q| < \varepsilon$ whenever $j \geq N$. We conclude that $x_j \to p$ and $y_j \to q$ as $j \to +\infty$.

Conversely suppose that $\lim_{j\to+\infty} x_j = p$ and $\lim_{j\to+\infty} y_j = q$. Let some positive real number ε be given. Then there exist natural numbers N_1 and N_2 such that $|x_j - p| < \varepsilon/\sqrt{2}$ whenever $j \geq N_1$ and $|y_j - q| < \varepsilon/\sqrt{2}$ whenever $j \geq N_2$. Let N be the maximum of N_1 and N_2. If $j \geq N$ then

$$|a_j - l|^2 = |x_j - p|^2 + |y_j - q|^2 < \tfrac{1}{2}\varepsilon^2 + \tfrac{1}{2}\varepsilon^2 = \varepsilon^2,$$

and thus $|a_j - l| < \varepsilon$. This shows that $a_j \to l$ as $j \to +\infty$, as required. ∎

Proposition 1.2 *Let (a_j) and (b_j) be convergent infinite sequences of complex numbers. Then the sequences (a_j+b_j), (a_j-b_j) and (a_jb_j) are convergent, and*

$$\begin{aligned}
\lim_{j\to+\infty}(a_j + b_j) &= \lim_{j\to+\infty} a_j + \lim_{j\to+\infty} b_j,\\
\lim_{j\to+\infty}(a_j - b_j) &= \lim_{j\to+\infty} a_j - \lim_{j\to+\infty} b_j,\\
\lim_{j\to+\infty}(a_jb_j) &= \left(\lim_{j\to+\infty} a_j\right)\left(\lim_{j\to+\infty} b_j\right).
\end{aligned}$$

If in addition $b_j \neq 0$ for all $n \in \mathbb{N}$ and $\lim_{j\to+\infty} b_j \neq 0$, then the sequence (a_j/b_j) is convergent, and

$$\lim_{j\to+\infty}\frac{a_j}{b_j} = \frac{\lim_{j\to+\infty} a_j}{\lim_{j\to+\infty} b_j}.$$

Proof Let $l = \lim_{j\to+\infty} a_j$ and $m = \lim_{j\to+\infty} b_j$.

Let some positive real number ε be given. It follows from the definition of limits that there exist natural numbers N_1 and N_2 such that $|a_j - l| < \frac{1}{2}\varepsilon$ whenever $j \geq N_1$ and $|b_j - m| < \frac{1}{2}\varepsilon$ whenever $j \geq N_2$. Let N be the maximum of N_1 and N_2. If $j \geq N$ then

$$|a_j + b_j - (l + m)| \leq |a_j - l| + |b_j - m| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

Thus $\lim_{j\to+\infty}(a_j + b_j) = l + m$.

Let c be some complex number. We show that $\lim_{j\to+\infty}(cb_j) = cm$. Now, given any positive real number ε, we can choose a positive number δ small enough to ensure that $|c|\delta < \varepsilon$. Now $\lim_{j\to+\infty} b_j = m$, and therefore there exists some natural number N such that $|b_j - m| < \delta$ whenever $j \geq N$. But then $|cb_j - cm| = |c|\,|b_j - m| \leq |c|\delta < \varepsilon$ whenever $j \geq N$. This shows that $\lim_{j\to+\infty}(cb_j) = cm$.

On applying this result with $c = -1$, we see that $\lim_{j\to+\infty}(-b_j) = -m$. It follows that $\lim_{j\to+\infty}(a_j - b_j) = l - m$.

Next we prove that if $u_1, u_2, u_3, \ldots$ and $v_1, v_2, v_3, \ldots$ are infinite sequences of complex numbers, and if $\lim_{j\to+\infty} u_j = 0$ and $\lim_{j\to+\infty} v_j = 0$ then $\lim_{j\to+\infty}(u_j v_j) = 0$. Let some positive number ε be given. Then there exist positive integers N_3 and N_4 such that $|u_j| < \sqrt{\varepsilon}$ whenever $j \geq N_3$ and $|v_j| < \sqrt{\varepsilon}$ whenever $j \geq N_4$. Let N be the maximum of N_3 and N_4. If $j \geq N$ then $|u_j| < \sqrt{\varepsilon}$ and $|v_j| < \sqrt{\varepsilon}$, and therefore $|u_j v_j| < \varepsilon$. It follows that $\lim_{j\to+\infty}(u_j v_j) = 0$.

We can apply this result with $u_j = a_j - l$ and $v_j = b_j - m$ for all positive integers j, where Let $l = \lim_{j\to+\infty} a_j$ and $m = \lim_{j\to+\infty} b_j$. Now $\lim_{j\to+\infty} u_j = 0$ and $\lim_{j\to+\infty} v_j = 0$. It follows that

$$\begin{aligned} 0 &= \lim_{j\to+\infty}(u_j v_j) = \lim_{j\to+\infty}((a_j - l)(b_j - m)) \\ &= \lim_{j\to+\infty}(a_j b_j - l b_j - m a_j + lm) \\ &= \lim_{j\to+\infty} a_j b_j - l \lim_{j\to+\infty} b_j - m \lim_{j\to+\infty} a_j + lm = \lim_{j\to+\infty} a_j b_j - lm. \end{aligned}$$

and therefore $\lim_{j\to+\infty}(a_j b_j) = lm$.

Finally suppose that $b_j \neq 0$ for all positive integers j, and that $m \neq 0$. Then, given any positive real number ε there exists some natural number N_5 such that

$$|b_j - m| < \tfrac{1}{2}|m|^2\varepsilon \text{ and } |b_j - m| < \tfrac{1}{2}|m|$$

whenever $j \geq N_5$. But then if $j \geq N_5$ then $|b_j| \geq |m| - |b_j - m| > \frac{1}{2}|m|$, and therefore

$$\left|\frac{1}{b_j} - \frac{1}{m}\right| = \left|\frac{m - b_j}{m b_j}\right| \leq \frac{2}{|m|^2}|b_j - m| < \varepsilon.$$

Thus $\lim_{j\to+\infty}(1/b_j) = 1/m$. It follows that $\lim_{j\to+\infty}(a_j/b_j) = l/m$, as required. ∎

1.3 The Least Upper Bound Principle

A widely-used basic principle of real analysis, from which many important theorems ultimately derive, is the Least Upper Bound Principle.

Let S be a subset of the set $\mathbb{R}$ of real numbers. A real number u is said to be an *upper bound* of the set S of $x \leq u$ for all $x \in S$. The set S is said to be *bounded above* if such an upper bound exists.

Definition Let S be some set of real numbers which is bounded above. A real number s is said to be the *least upper bound* (or *supremum*) of S (denoted by $\sup S$) if s is an upper bound of S and $s \leq u$ for all upper bounds u of S.

Example The real number 2 is the least upper bound of the sets $\{x \in \mathbb{R} : x \leq 2\}$ and $\{x \in \mathbb{R} : x < 2\}$. Note that the first of these sets contains its least upper bound, whereas the second set does not.

The *Least Upper Bound Principle* may be stated as follows:

> given any non-empty S subset of $\mathbb{R}$ which is bounded above, there exists a *least upper bound* $\sup S$ for the set S.

A *lower bound* of a set S of real numbers is a real number l with the property that $l \leq x$ for all $x \in S$. A set S of real numbers is said to be *bounded below* if such a lower bound exists. If S is bounded below, then there exists a greatest lower bound (or *infimum*) $\inf S$ of the set S. Indeed $\inf S = -\sup\{x \in \mathbb{R} : -x \in S\}$.

1.4 Monotonic Sequences

An infinite sequence $a_1, a_2, a_3, \ldots$ of real numbers is said to be *strictly increasing* if $a_{j+1} > a_j$ for all j, *strictly decreasing* if $a_{j+1} < a_j$ for all j, *non-decreasing* if $a_{j+1} \geq a_j$ for all j, or *non-increasing* if $a_{j+1} \leq a_j$ for all j. A sequence satisfying any one of these conditions is said to be *monotonic*; thus a monotonic sequence is either non-decreasing or non-increasing.

Theorem 1.3 *Any bounded non-decreasing sequence of real numbers is is convergent. Similarly any bounded non-increasing sequence of real numbers is convergent.*

Proof Let $a_1, a_2, a_3, \ldots$ be a bounded non-decreasing sequence of real numbers. It follows from the Least Upper Bound Principle that there exists a least upper bound l for the set $\{a_j : j \in \mathbb{N}\}$. We claim that the sequence converges to l.

Let $\varepsilon > 0$ be given. We must show that there exists some natural number N such that $|a_j - l| < \varepsilon$ whenever $j \geq N$. Now $l - \varepsilon$ is not an upper bound for the set $\{a_j : j \in \mathbb{N}\}$ (since l is the least upper bound), and therefore there must exist some natural number N such that $a_N > l - \varepsilon$. But then $l - \varepsilon < a_j \leq l$ whenever $j \geq N$, since the sequence is non-decreasing and bounded above by l. Thus $|a_j - l| < \varepsilon$ whenever $j \geq N$. Therefore $a_j \to l$ as $j \to +\infty$, as required.

If the sequence $a_1, a_2, a_3, \ldots$ is bounded and non-increasing then the sequence $-a_1, -a_2, -a_3, \ldots$ is bounded and non-decreasing, and is therefore convergent. It follows that the sequence $a_1, a_2, a_3, \ldots$ is also convergent. ∎

1.5 Upper and Lower Limits of Bounded Sequences of Real Numbers

Let $a_1, a_2, a_3, \ldots$ be a bounded infinite sequence of real numbers, and, for each positive integer j, let

$$S_j = \{a_j, a_{j+1}, a_{j+2}, \ldots\} = \{a_k : k \geq j\}.$$

The sets $S_1, S_2, S_3, \ldots$ are all bounded. It follows that there exist well-defined infinite sequences $u_1, u_2, u_3, \ldots$ and $l_1, l_2, l_3, \ldots$ of real numbers, where $u_j = \sup S_j$ and $l_j = \inf S_j$ for all positive integers j. Now S_{j+1} is a subset of S_j for each positive integer j, and therefore $u_{j+1} \leq u_j$ and $l_{j+1} \geq l_j$ for each positive integer j. It follows that the bounded infinite sequence $(u_j : j \in \mathbb{N})$ is a non-increasing sequence, and is therefore convergent (Theorem 1.3). Similarly the bounded infinite sequence $(l_j : j \in \mathbb{N})$ is a non-decreasing sequence, and is therefore convergent. We define

$$\begin{aligned} \limsup_{j \to +\infty} a_j &= \lim_{j \to +\infty} u_j = \lim_{j \to +\infty} \sup\{a_j, a_{j+1}, a_{j+2}, \ldots\}, \\ \liminf_{j \to +\infty} a_j &= \lim_{j \to +\infty} l_j = \lim_{j \to +\infty} \inf\{a_j, a_{j+1}, a_{j+2}, \ldots\}. \end{aligned}$$

The quantity $\limsup_{j \to +\infty} a_j$ is referred to as the *upper limit* of the sequence $a_1, a_2, a_3, \ldots$. The quantity $\liminf_{j \to +\infty} a_j$ is referred to as the *lower limit* of the sequence $a_1, a_2, a_3, \ldots$.

Note that every bounded infinite sequence $a_1, a_2, a_3, \ldots$ of real numbers has a well-defined upper limit $\limsup_{j \to +\infty} a_j$ and a well-defined lower limit $\liminf_{j \to +\infty} a_j$.

Lemma 1.4 *Let $a_1, a_2, a_3, \ldots$ be a bounded infinite sequence of real numbers, and let $u = \limsup_{j \to +\infty} a_j$. Then, given any positive real number ε, and given any positive integer N, there exists a positive integer j such that $j \geq N$ and $|a_j - u| < \varepsilon$.*

Proof It follows from the definition of upper limits that $u = \lim_{j \to +\infty} u_j$, where $u_j = \sup\{a_k : k \geq j\}$. Moreover the infinite sequence $u_1, u_2, u_3, \ldots$ is non-increasing. Therefore there exists some positive integer m such that $m \geq N$ and $u \leq u_m < u + \varepsilon$. Then $a_j \leq u_m < u + \varepsilon$ whenever $j \geq m$. Moreover $u - \varepsilon$ is not an upper bound on the set $\{a_j : j \geq m\}$, because u_m is the least upper bound of this set. Therefore there exists some positive integer j such that $j \geq m$ and $a_j > u - \varepsilon$. Then $j \geq N$ and $|a_j - u| < \varepsilon$, as required. ∎

1.6 The Bolzano-Weierstrass Theorem

Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of real or complex numbers. A *subsequence* of this sequence is a sequence that is of the form $a_{m_1}, a_{m_2}, a_{m_3}, \ldots$, where $m_1, m_2, m_3, \ldots$ are positive integers satisfying $m_1 < m_2 < m_3 < \cdots$. Thus, for example, $a_2, a_4, a_6, \ldots$ and $a_1, a_4, a_9, \ldots$ are subsequences of the given sequence.

Proposition 1.5 *Any bounded infinite sequence $a_1, a_2, a_3, \ldots$ of real numbers has a subsequence which converges to the upper limit $\limsup_{j \to +\infty} a_j$ of the given sequence.*

Proof Let $u = \limsup_{j \to +\infty} a_j$. It follows from Lemma 1.4 that, given positive integers j and k_j for which $|a_{k_j} - u| < 1/j$, there exists some positive integer k_{j+1} satisfying $k_{j+1} > k_j$ for which $|a_{k_{j+1}} - u| < 1/(j+1)$. Therefore there exists an increasing sequence $k_1, k_2, k_3, \ldots$ of positive integers such that $|a_{k_j} - u| < 1/j$ for all positive integers j. Then $a_{k_1}, a_{k_2}, a_{k_3}$ is a subsequence of $a_1, a_2, a_3, \ldots$ which converges to the upper limit of the sequence, as required. ∎

The following theorem, known as the *Bolzano-Weierstrass Theorem*, is an immediate consequence of Proposition 1.5.

Theorem 1.6 (Bolzano-Weierstrass) *Every bounded sequence of real numbers has a convergent subsequence.*

Theorem 1.7 (Bolzano-Weierstrass Theorem for Complex Sequences) *Every bounded sequence of complex numbers has a convergent subsequence*

Proof Let $a_1, a_2, a_3, \ldots$ be a bounded sequence of complex numbers, and, for each positive integer j, let $a_j = x_j + i y_j$, where x_j are y_j are real numbers. The Bolzano-Weierstrass Theorem for sequences of real numbers (Theorem 1.6) guarantees the existence of a subsequence $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ of the given sequence such that the real parts $x_{j_1}, x_{j_2}, x_{j_3}, \ldots$ converge. A further application of Theorem 1.6 then allows to replace this subsequence by a subsequence of itself in order to ensure that the imaginary parts $y_{j_1}, y_{j_2}, y_{j_3}, \ldots$ also converge. But then $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ is a convergent subsequence of $a_1, a_2, a_3, \ldots$, by Lemma 1.1. ∎

1.7 Cauchy's Criterion for Convergence

Definition A sequence $a_1, a_2, a_3, \ldots$ of complex numbers is said to be a *Cauchy sequence* if the following condition is satisfied:

> for every real number ε satisfying $\varepsilon > 0$ there exists some natural number N such that $|a_j - a_k| < \varepsilon$ for all natural numbers j and k satisfying $j \geq N$ and $k \geq N$.

Lemma 1.8 *Every Cauchy sequence of complex numbers is bounded.*

Proof Let $a_1, a_2, a_3, \ldots$ be a Cauchy sequence. Then there exists some natural number N such that $|a_j - a_k| < 1$ whenever $j \geq N$ and $k \geq N$. In particular, $|a_j| \leq |a_N| + 1$ whenever $j \geq N$. Therefore $|a_j| \leq R$ for all positive integers j, where R is the maximum of the real numbers $|a_1|, |a_2|, \ldots, |a_{N-1}|$ and $|a_N| + 1$. Thus the sequence is bounded, as required. ∎

The following important result is known as *Cauchy's Criterion for convergence*, or as the *General Principle of Convergence.*

Theorem 1.9 (Cauchy's Criterion for Convergence) *An infinite sequence of complex numbers is convergent if and only if it is a Cauchy sequence.*

Proof First we show that convergent sequences are Cauchy sequences. Let $a_1, a_2, a_3, \ldots$ be a convergent sequence, and let $l = \lim_{n \to +\infty} a_j$. Let $\varepsilon > 0$ be given. Then there exists some natural number N such that $|a_j - l| < \frac{1}{2}\varepsilon$ for all $j \geq N$. Thus if $j \geq N$ and $k \geq N$ then $|a_j - l| < \frac{1}{2}\varepsilon$ and $|a_k - l| < \frac{1}{2}\varepsilon$, and hence

$$|a_j - a_k| = |(a_j - l) - (a_k - l)| \leq |a_j - l| + |a_k - l| < \varepsilon.$$

Thus the sequence $a_1, a_2, a_3, \ldots$ is a Cauchy sequence.

Conversely we must show that any Cauchy sequence $a_1, a_2, a_3, \ldots$ is convergent. Now Cauchy sequences are bounded, by Lemma 1.8. The sequence $a_1, a_2, a_3, \ldots$ therefore has a convergent subsequence $a_{k_1}, a_{k_2}, a_{k_3}, \ldots$, by the Bolzano-Weierstrass Theorem (Theorem 1.7). Let $l = \lim_{m \to +\infty} a_{k_m}$. We claim that the sequence $a_1, a_2, a_3, \ldots$ itself converges to l.

Let $\varepsilon > 0$ be given. Then there exists some natural number N such that $|a_j - a_k| < \frac{1}{2}\varepsilon$ whenever $j \geq N$ and $k \geq N$ (since the sequence is a Cauchy sequence). Let m be chosen large enough to ensure that $k_m \geq N$ and $|a_{k_m} - l| < \frac{1}{2}\varepsilon$. Then

$$|a_j - l| \leq |a_j - a_{k_m}| + |a_{k_m} - l| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon$$

whenever $j \geq N$, and thus $a_j \to l$ as $j \to +\infty$, as required. ∎

1.8 Limits of Functions of a Complex Variable

Let D be a subset of the set $\mathbb{C}$ of complex numbers A complex number w is said to be a *limit point* of D if and only if, given any $\delta > 0$, there exists $z \in D$ satisfying $0 < |z - w| < \delta$.

A complex number w belonging to some subset D of the complex plane is said to be an *isolated point* of D if it is not a limit point of D. Thus an element w of D is an isolated point of D if and only if there exists some $\delta > 0$ such that $\{z \in D : |z - w| < \delta\} = \{w\}$.

Definition Let $f \colon D \to \mathbb{C}$ be a function defined over some subset D of $\mathbb{C}$. Let w be a limit point of D. A complex number l is said to be the *limit* of the function f as z tends to w in D if, given any real number ε satisfying $\varepsilon > 0$, there exists some real number δ satisfying $\delta > 0$ such that $|f(z) - l| < \varepsilon$ for all $z \in D$ satisfying $0 < |z - w| < \delta$.

Lemma 1.10 *Let $f \colon D \to \mathbb{C}$ be a complex-valued function defined over some subset D of the complex plane, and let w be a limit point of D. Then the limit $\lim_{z \to w} f(z)$, if it exists, is unique.*

Proof Suppose that $\lim_{z \to w} f(z) = l$ and $\lim_{z \to w} f(z) = m$. We must show that $l = m$. Let $\varepsilon > 0$ be given. Then there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $|f(z) - l| < \varepsilon$ whenever $z \in D$ satisfies $0 < |z - w| < \delta_1$ and $|f(z) - m| < \varepsilon$ whenever $z \in D$ satisfies $0 < |z - w| < \delta_2$. Choose $z \in D$ satisfying $0 < |z - w| < \delta$, where δ is the minimum of δ_1 and δ_2. (This is possible since w is a limit point of D.) Then $|f(z) - l| < \varepsilon$ and $|f(z) - m| < \varepsilon$, and hence

$$|l - m| \leq |l - f(z)| + |f(z) - m| < 2\varepsilon$$

by the Triangle Inequality. Since $|l - m| < 2\varepsilon$ for all $\varepsilon > 0$, we conclude that $l = m$, as required. ∎

Proposition 1.11 *Let $f \colon D \to \mathbb{C}$ and $g \colon D \to \mathbb{C}$ be functions defined over some subset D of $\mathbb{C}$. Let w be a limit point of D. Suppose that $\lim_{z \to w} f(z)$ and $\lim_{z \to w} g(z)$ exist. Then $\lim_{z \to w} (f(z) + g(z))$, $\lim_{z \to w} (f(z) - g(z))$ and $\lim_{z \to w} (f(z)g(z))$ exist, and*

$$\begin{aligned}
\lim_{z \to w} (f(z) + g(z)) &= \lim_{z \to w} f(z) + \lim_{z \to w} g(z), \\
\lim_{z \to w} (f(z) - g(z)) &= \lim_{z \to w} f(z) - \lim_{z \to w} g(z), \\
\lim_{z \to w} (f(z)g(z)) &= \lim_{z \to w} f(z) \lim_{z \to w} g(z).
\end{aligned}$$

If in addition $g(z) \neq 0$ *for all* $z \in D$ *and* $\lim_{z \to w} g(z) \neq 0$*, then* $\lim_{z \to w} f(z)/g(z)$ *exists, and*

$$\lim_{z \to w} \frac{f(z)}{g(z)} = \frac{\lim_{z \to w} f(z)}{\lim_{z \to w} g(z)}.$$

Proof Let $l = \lim_{z \to w} f(z)$ and $m = \lim_{z \to w} g(z)$.

First we prove that $\lim_{z \to w}(f(z) + g(z)) = l + m$. Let $\varepsilon > 0$ be given. We must prove that there exists some $\delta > 0$ such that $|f(z) + g(z) - (l + m)| < \varepsilon$ for all $z \in D$ satisfying $0 < |z - w| < \delta$. Now there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $|f(z) - l| < \frac{1}{2}\varepsilon$ for all $z \in D$ satisfying $0 < |z - w| < \delta_1$, and $|g(z) - m| < \frac{1}{2}\varepsilon$ for all $z \in D$ satisfying $0 < |z - w| < \delta_2$, since $l = \lim_{z \to w} f(z)$ and $m = \lim_{z \to w} g(z)$. Let δ be the minimum of δ_1 and δ_2. If $z \in D$ satisfies $0 < |z - w| < \delta$ then $|f(z) - l| < \frac{1}{2}\varepsilon$ and $|g(z) - m| < \frac{1}{2}\varepsilon$, and hence

$$|f(z) + g(z) - (l + m)| \leq |f(z) - l| + |g(z) - m| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon$$

This shows that $\lim_{z \to w} (f(z) + g(z)) = l + m$.

Let c be some complex number. We show that $\lim_{z \to w} (cg(z)) = cm$. Let some positive number ε be given. Choose a positive number ε_1 small enough to ensure that $|c|\varepsilon_1 < \varepsilon$. Then there exists some real number $\delta > 0$ such that $|g(z) - m| < \varepsilon_1$ whenever $0 < |z - w| < \delta$. But then

$$|cg(z) - cm| = |c||g(z) - m| \leq |c|\varepsilon_1 < \varepsilon$$

whenever $0 < |z - w| < \delta$. Thus

$$\lim_{z \to w} (cg(z)) = cm.$$

If we combine this result, for $c = -1$, with the previous result, we see that $\lim_{z \to w} (-g(z)) = -m$, and therefore $\lim_{z \to w} (f(z) - g(z)) = l - m$.

Next we show that if $p: D \to \mathbb{R}$ and $q: D \to \mathbb{R}$ are functions with the property that $\lim_{z \to w} p(z) = \lim_{z \to w} q(z) = 0$, then $\lim_{z \to w} (p(z)q(z)) = 0$. Let $\varepsilon > 0$ be given. Then there exist real numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that $|p(z)| < \sqrt{\varepsilon}$ whenever $0 < |z - w| < \delta_1$ and $|q(z)| < \sqrt{\varepsilon}$ whenever $0 < |z - w| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. If $0 < |z - w| < \delta$ then $|p(z)q(z)| < \varepsilon$. We deduce that $\lim_{z \to w} (p(z)q(z)) = 0$.

We can apply this result with $p(z) = f(z) - l$ and $q(z) = g(z) - m$ for all $z \in D$. Using the results we have already obtained, we see that

$$\begin{aligned} 0 &= \lim_{z \to w}(p(z)q(z)) = \lim_{z \to w}(f(z)g(z) - f(z)m - lg(z) + lm) \\ &= \lim_{z \to w}(f(z)g(z)) - m \lim_{z \to w} f(z) - l \lim_{z \to w} g(z) + lm = \lim_{z \to w}(f(z)g(z)) - lm. \end{aligned}$$

Thus $\lim_{z \to w} (f(z)g(z)) = lm$.

Next we show that if $h: D \to R$ is a function that is non-zero throughout D, and if $\lim_{z \to w} h(z) \to 1$ then $\lim_{z \to w} (1/h(z)) = 1$. Let $\varepsilon > 0$ be given. Let ε_0 be the minimum of $\frac{1}{2}\varepsilon$ and $\frac{1}{2}$. Then there exists some $\delta > 0$ such that $|h(z) - 1| < \varepsilon_0$ whenever $0 < |z - w| < \delta$. Thus if $0 < |z - w| < \delta$ then $|h(z) - 1| < \frac{1}{2}\varepsilon$ and $|h(z)| \geq 1 - |1 - h(z)| > \frac{1}{2}$. But then

$$\left| \frac{1}{h(z)} - 1 \right| = \left| \frac{h(z) - 1}{h(z)} \right| = \frac{|h(z) - 1|}{|h(z)|} < 2|h(z) - 1| < \varepsilon.$$

We deduce that $\lim_{z \to w} 1/h(z) = 1$. If we apply this result with $h(z) = g(z)/m$, where $m \neq 0$, we deduce that $\lim_{z \to w} m/g(z) = 1$, and thus $\lim_{z \to w} 1/g(z) = 1/m$. The result we have already obtained for products of functions then enables us to deduce that

$$\lim_{z \to w} (f(z)/g(z)) = l/m.$$

1.9 Continuous Functions of a Complex Variable.

Definition Let D be a subset of $\mathbb{C}$, and let $f: D \to \mathbb{C}$ be a function on D. Let w be an element of D. The function f is said to be *continuous* at w if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that $|f(z) - f(w)| < \varepsilon$ for all $z \in D$ satisfying $|z - w| < \delta$. If f is continuous at every element of D then we say that f is continuous on D.

Lemma 1.12 *A complex-valued function $f: D \to \mathbb{C}$ defined on some subset D of $\mathbb{C}$ is continuous on D if and only if $\lim_{z \to w} f(z) = f(w)$ for all limit points w of D that belong to D.*

Proof Every element w of the domain D of the function f is either a limit point of D or an isolated point of D. If w is an isolated point of D then it follows from the definition of continuity that every complex-valued function with domain D is continuous at w. If w is a limit point D, then comparison of the relevant definitions shows that the function f is continuous at w if and only if $\lim_{z \to w} f(z) = f(w)$. The result follows. ∎

Given functions $f: D \to \mathbb{C}$ and $g: D \to \mathbb{C}$ defined over some subset D of $\mathbb{C}$, we denote by $f + g$, $f - g$, $f \cdot g$ and f/g the functions on D defined by

$$(f + g)(z) = f(z) + g(z), \qquad (f - g)(z) = f(z) - g(z),$$

$$(f \cdot g)(z) = f(z)g(z), \qquad (f/g)(z) = f(z)/g(z).$$

Proposition 1.13 *Let $f\colon D \to \mathbb{C}$ and $g\colon D \to \mathbb{C}$ be functions defined over some subset D of $\mathbb{C}$. Suppose that f and g are continuous at some element w of D. Then the functions $f+g$, $f-g$ and $f\cdot g$ are also continuous at w. If moreover the function g is everywhere non-zero on D then the function f/g is continuous at w.*

Proof This result follows directly using Proposition 1.11 and the relationship between continuity and limits described above. ∎

Proposition 1.14 *Let $f\colon D \to \mathbb{C}$ and $g\colon E \to \mathbb{C}$ be functions defined on D and E respectively, where D and E are subsets of the complex plane satisfying $f(D) \subset E$. Let w be an element of D. Suppose that the function f is continuous at w and that the function g is continuous at $f(w)$. Then the composition $g \circ f$ of f and g is continuous at w.*

Proof Let $\varepsilon > 0$ be given. Then there exists some $\eta > 0$ such that $|g(z) - g(f(w))| < \varepsilon$ for all $z \in E$ satisfying $|z - f(w)| < \eta$. But then there exists some $\delta > 0$ such that $|f(z) - f(w)| < \eta$ for all $z \in D$ satisfying $|z - w| < \delta$. Thus if $|z - w| < \delta$ then $|f(z) - f(w)| < \eta$, and therefore $|g(f(z)) - g(f(w))| < \varepsilon$. Hence $g \circ f$ is continuous at w. ∎

Lemma 1.15 *Let $f\colon D \to \mathbb{C}$ be a function defined on some subset D of $\mathbb{C}$, and let $a_1, a_2, a_3, \ldots$ be a sequence of complex numbers belonging to D. Suppose that $\lim\limits_{j\to+\infty} a_j = w$, where $w \in D$, and that f is continuous at w. Then $\lim\limits_{j\to+\infty} f(a_j) = f(w)$.*

Proof Let $\varepsilon > 0$ be given. Then there exists some $\delta > 0$ such that $|f(z) - f(w)| < \varepsilon$ for all $z \in D$ satisfying $|z - w| < \delta$. But then there exists some positive integer N such that $|a_j - w| < \delta$ for all j satisfying $j \geq N$. Thus $|f(a_j) - f(w)| < \varepsilon$ for all $j \geq N$. Hence $f(a_j) \to f(w)$ as $j \to +\infty$. ∎

Proposition 1.16 *Let $f\colon D \to \mathbb{C}$ and $g\colon E \to \mathbb{C}$ be functions defined on D and E respectively, where D and E are subsets of $\mathbb{C}$ satisfying $f(D) \subset E$. Let w be a limit point of D, and let l be an element of E. Suppose that $\lim\limits_{z\to w} f(z) = l$ and that the function g is continuous at l. Then $\lim\limits_{z\to w} g(f(z)) = g(l)$.*

Proof Let $\varepsilon > 0$ be given. Then there exists some $\eta > 0$ such that $|g(z) - g(l)| < \varepsilon$ for all $z \in E$ satisfying $|z - l| < \eta$. But then there exists $\delta > 0$ such that $|f(z) - l| < \eta$ for all $z \in D$ satisfying $0 < |z - w| < \delta$. Thus if $0 < |z - w| < \delta$ then $|f(z) - l| < \eta$, and therefore $|g(f(z)) - g(l)| < \varepsilon$. Hence $\lim\limits_{z\to w} g(f(z)) = g(l)$. ∎

1.10 The Intermediate Value Theorem

Proposition 1.17 *Let $f\colon [a,b] \to \mathbb{Z}$ continuous integer-valued function defined on a closed interval $[a,b]$. Then the function f is constant.*

Proof Let

$$S = \{x \in [a,b] : f \text{ is constant on the interval } [a,x]\},$$

and let $s = \sup S$. Now $s \in [a,b]$, and therefore the function f is continuous at s. Therefore there exists some real number δ satisfying $\delta > 0$ such that $|f(x) - f(s)| < \frac{1}{2}$ for all $x \in [a,b]$ satisfying $|x - s| < \delta$. But the function f is integer-valued. It follows that $f(x) = f(s)$ for all $x \in [a,b]$ satisfying $|x - s| < \delta$. Now $s - \delta$ is not an upper bound for the set S. Therefore there exists some element x_0 of S satisfying $s - \delta < x_0 \leq s$. But then $f(x) = f(s) = f(x_0) = f(a)$ for all $x \in [a,b]$ satsifying $s \leq x < s + \delta$, and therefore the function f is constant on the interval $[a,x]$ for all $x \in [a,b]$ satisfying $s \leq x < s + \delta$. Thus $x \in [a,b] \cap [s, s+\delta) \subset S$. In particular $s \in S$. Now S cannot contain any elements x of $[a,b]$ satisfying $x > s$. Therefore $[a,b] \cap [s, s+\delta) = \{s\}$, and therefore $s = b$. This shows that $b \in S$, and thus the function f is constant on the interval $[a,b]$, as required. ∎

Theorem 1.18 (The Intermediate Value Theorem) *Let a and b be real numbers satisfying $a < b$, and let $f\colon [a,b] \to \mathbb{R}$ be a continuous function defined on the interval $[a,b]$. Let c be a real number which lies between $f(a)$ and $f(b)$ (so that either $f(a) \leq c \leq f(b)$ or else $f(a) \geq c \geq f(b)$.) Then there exists some $s \in [a,b]$ for which $f(s) = c$.*

Proof Let c be a real number which lies between $f(a)$ and $f(b)$, and let $g_c\colon \mathbb{R} \setminus \{c\} \to \mathbb{Z}$ be the continuous integer-valued function on $\mathbb{R} \setminus \{c\}$ defined such that $g_c(x) = 0$ whenever $x < c$ and $g_c(x) = 1$ if $x > c$. Suppose that c were not in the range of the function f. Then the composition function $g_c \circ f\colon [a,b] \to \mathbb{R}$ would be a continuous integer-valued function defined throughout the interval $[a,b]$. This function would not be constant, since $g_c(f(a)) \neq g_c(f(b))$. But every continuous integer-valued function on the interval $[a,b]$ is constant (Proposition 1.17). It follows that every real number c lying between $f(a)$ and $f(b)$ must belong to the range of the function f, as required. ∎

Corollary 1.19 *Let $f\colon [a,b] \to [c,d]$ be a strictly increasing continuous function mapping an interval $[a,b]$ into an interval $[c,d]$, where a, b, c and d are real numbers satisfying $a < b$ and $c < d$. Suppose that $f(a) = c$ and $f(b) = d$. Then the function f has a continuous inverse $f^{-1}\colon [c,d] \to [a,b]$.*

Proof Let x_1 and x_2 be distinct real numbers belonging to the interval $[a, b]$ then either $x_1 < x_2$, in which case $f(x_1) < f(x_2)$ or $x_1 > x_2$, in which case $f(x_1) > f(x_2)$. Thus $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. It follows that the function f is injective. The Intermediate Value Theorem (Theorem 1.18) ensures that f is surjective. It follows that the function f has a well-defined inverse $f^{-1}: [c, d] \to [a, b]$. It only remains to show that this inverse function is continuous.

Let y be a real number satisfying $c < y < d$, and let x be the unique real number such that $a < x < b$ and $f(x) = y$. Let $\varepsilon > 0$ be given. We can then choose $x_1, x_2 \in [a, b]$ such that $x - \varepsilon < x_1 < x < x_2 < x + \varepsilon$. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then $y_1 < y < y_2$. Choose $\delta > 0$ such that $\delta < y - y_1$ and $\delta < y_2 - y$. If $v \in [c, d]$ satisfies $|v - y| < \delta$ then $y_1 < v < y_2$ and therefore $x_1 < f^{-1}(v) < x_2$. But then $|f^{-1}(v) - f^{-1}(y)| < \varepsilon$. We conclude that the function $f^{-1}: [c, d] \to [a, b]$ is continuous at all elements in the interior of the interval $[a, b]$. A similar argument shows that it is continuous at the endpoints of this interval. Thus the function f has a continuous inverse, as required. ∎

1.11 Uniform Convergence

Let D be a subset of $\mathbb{C}$ and let $f_1, f_2, f_3, \ldots,$ be a sequence of functions mapping D into $\mathbb{C}$. We say that the infinite sequence $f_1, f_2, f_3, \ldots$ converges *uniformly* on D to a function $f: D \to \mathbb{C}$ if, given any $\varepsilon > 0$, there exists some natural number N such that $|f_j(z) - f(z)| < \varepsilon$ for all $z \in D$ and for all natural numbers j satisfying $j \geq N$, where the value of N chosen does not depend on the value of z.

Theorem 1.20 *Let D be a subset of $\mathbb{C}$, and let $f_1, f_2, f_3, \ldots$ be a sequence of continuous functions mapping D into $\mathbb{C}$ which is uniformly convergent on D to some function $f: D \to \mathbb{C}$. Then the function f is continuous on D.*

Proof Let w be an element of D. We wish to show that the function f is continuous at w. Let $\varepsilon > 0$ be given. We must show that there exists some $\delta > 0$ such that $|f(z) - f(w)| < \varepsilon$ whenever $z \in D$ satisfies $|z - w| < \delta$. Now we can find some value of N, independent of z, with the property that $|f_j(z) - f(z)| < \frac{1}{3}\varepsilon$ for all $z \in D$ and for all $j \geq N$. Choose any j satisfying $j \geq N$. We can find some $\delta > 0$ such that $|f_j(z) - f_j(w)| < \frac{1}{3}\varepsilon$ whenever $z \in D$ satisfies $|z - w| < \delta$, since the function f_j is continuous at w. But then

$$\begin{aligned} |f(z) - f(w)| &\leq |f(z) - f_j(z)| + |f_j(z) - f_j(w)| + |f_j(w) - f(w)| \\ &< \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon \end{aligned}$$

whenever $|z-w| < \delta$. Thus the function f is continuous at w, as required. ∎

1.12 Open Sets in the Complex Plane

Let w be a complex number, and let r be a non-negative real number. We define the *open disk* $D(w,r)$ of *radius* r about w to be the subset

$$\{z \in \mathbb{C} : |z - w| < r\}$$

of the complex plane consisting of all complex numbers that lie within a distance r of w.

Definition A subset V of the complex plane is said to be *open* if and only if, given any element v of V, there exists some $\delta > 0$ such that $D(v,\delta) \subset V$, where $D(w,\delta)$ is the open disc of radius δ about v.

By convention, we regard the empty set $\emptyset$ as being an open subset of the complex plane. (The criterion given above is satisfied vacuously in the case when V is the empty set.)

Example Let c be a real number, and let $H = \{z \in \mathbb{C} : \operatorname{Re} z > c\}$. Given $w \in H$, let $\delta = \operatorname{Re} w - c$. Then $\delta > 0$. Now $\operatorname{Re} z - \operatorname{Re} w > -|z - w|$ for all complex numbers z, and therefore $\operatorname{Re} z > c$ for all complex numbers z satisfying $|z - w| < \delta$. Thus $D(w,\delta) \subset H$, where $D(w,\delta)$ denotes the open disk of radius δ about w. Thus H is an open set in the complex plane. Similarly $\{z \in \mathbb{C} : \operatorname{Re} z < c\}$, $\{z \in \mathbb{C} : \operatorname{Im} z > c\}$ and $\{z \in \mathbb{C} : \operatorname{Im} z < c\}$ are open sets in the complex plane.

Lemma 1.21 *Given any complex number w and any positive real number r, the open disk $D(w,r)$ of radius r about w is an open set in the complex plane.*

Proof Let z be an element of the open disk $D(w,r)$. We must show that there exists some $\delta > 0$ such that $D(z,\delta) \subset D(w,r)$. Let $\delta = r - |z - w|$. Then $\delta > 0$, since $|z - w| < r$. Moreover if $z_1 \in D(z,\delta)$ then

$$|z_1 - w| \leq |z_1 - z| + |z - w| < \delta + |z - w| = r,$$

by the Triangle Inequality, and hence $z_1 \in D(w,r)$. Thus $D(z,\delta) \subset D(w,r)$. This shows that $D(w,r)$ is an open set, as required. ∎

Lemma 1.22 *Given any complex number w and any positive real number r, the set $\{z \in \mathbb{C} : |z - w| > r\}$ is an open set in the complex plane.*

Proof Let z be a complex number satisfying $|z - w| > r$, and let z_1 be a complex number satisfying $|z_1 - z| < \delta$, where $\delta = |z - w| - r$. Then

$$|z - w| \leq |z - z_1| + |z_1 - w|,$$

by the Triangle Inequality, and therefore

$$|z_1 - w| \geq |z - w| - |z_1 - z| > |z - w| - \delta = r.$$

Thus the open disk $D(z, \delta)$ of radius δ about z is contained in the given set. The result follows. ∎

Proposition 1.23 *The collection of open sets in the complex plane has the following properties:—*

(i) *the empty set $\emptyset$ and the whole complex plane $\mathbb{C}$ are both open sets;*

(ii) *the union of any collection of open sets is itself an open set;*

(iii) *the intersection of any* finite *collection of open sets is itself an open set.*

Proof The empty set $\emptyset$ is an open set by convention. Moreover the definition of an open set is satisfied trivially by the whole complex plane. This proves (i).

Let $\mathcal{A}$ be any collection of open sets in the complex plane, and let W denote the union of all the open sets belonging to $\mathcal{A}$. We must show that W is itself an open set. Let $z \in W$. Then $z \in V$ for some set V belonging to the collection $\mathcal{A}$. It follows that there exists some $\delta > 0$ such that $D(z, \delta) \subset V$, where $D(z, \delta)$ denotes the open disk of radius δ about z. But $V \subset W$, and thus $D(z, \delta) \subset W$. This shows that W is an open set. This proves (ii).

Finally let $V_1, V_2, V_3, \ldots, V_k$ be a *finite* collection of open sets in the complex plane, and let V denote the intersection $V_1 \cap V_2 \cap \cdots \cap V_k$ of these sets. Let $z \in V$. Now $z \in V_j$ for $j = 1, 2, \ldots, k$, and therefore there exist strictly positive real numbers $\delta_1, \delta_2, \ldots, \delta_k$ such that $D(z, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. (This is where we need the fact that we are dealing with a finite collection of sets.) Now $D(z, \delta) \subset D(z, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$, and thus $D(z, \delta) \subset V$. Thus the intersection V of the sets $V_1, V_2, \ldots, V_k$ is itself an open set. This proves (iii). ∎

Example The set $\{z \in \mathbb{C} : |z - 3| < 2 \text{ and } \operatorname{Re} z > 1\}$ is an open set in the complex plane, as it is the intersection of the open disk of radius 2 about 3 with the open set $\{z \in \mathbb{C} : \operatorname{Re} z > 1\}$.

Example The set $\{z \in \mathbb{C} : |z - 3| < 2 \text{ or } \operatorname{Re} z > 1\}$ is an open set in the complex plane, as it is the union of the open disk of radius 2 about 3 and the open set $\{z \in \mathbb{C} : \operatorname{Re} z > 1\}$.

Example The set

$$\{z \in \mathbb{C} : |z - n| < \tfrac{1}{2} \text{ for some } n \in \mathbb{Z}\}$$

is an open set in the complex plane, since it is the union of the open disks of radius $\frac{1}{2}$ centred on integers.

Example For each natural number k, let

$$D(0, 1/k) = \{z \in \mathbb{C} : k|z| < 1\}.$$

Now each set $D(0, 1/k)$ is an open disk of radius $1/k$ about the origin, and is therefore an open set in the complex plane. However the intersection of the sets $D(0, 1/k)$ for all natural numbers k is the set $\{0\}$, and thus the intersection of the open sets $D(0, 1/k)$ for all natural numbers k is not itself an open set in the complex plane. This example demonstrates that infinite intersections of open sets need not be open.

Lemma 1.24 *A sequence $z_1, z_2, z_3, \ldots$ of complex numbers converges to a complex number w if and only if, given any open set V which contains w, there exists some natural number N such that $z_j \in V$ for all j satisfying $j \geq N$.*

Proof Suppose that the sequence $z_1, z_2, z_3, \ldots$ has the property that, given any open set V which contains w, there exists some natural number N such that $z_j \in V$ whenever $j \geq N$. Let $\varepsilon > 0$ be given. The open disk $D(w, \varepsilon)$ of radius ε about w is an open set by Lemma 1.21. Therefore there exists some natural number N such that $z_j \in D(w, \varepsilon)$ whenever $j \geq N$. Thus $|z_j - w| < \varepsilon$ whenever $j \geq N$. This shows that the sequence converges to w.

Conversely, suppose that the sequence $z_1, z_2, z_3, \ldots$ converges to w. Let V be an open set which contains w. Then there exists some $\varepsilon > 0$ such that the open disk $D(w, \varepsilon)$ of radius ε about w is a subset of V. Thus there exists some $\varepsilon > 0$ such that V contains all complex numbers z that satisfy $|z - w| < \varepsilon$. But there exists some natural number N with the property that $|z_j - w| < \varepsilon$ whenever $j \geq N$, since the sequence converges to w. Therefore $z_j \in V$ whenever $j \geq N$, as required. ∎

1.13 Interiors

Definition Let A be a subset of the complex plane. The *interior* of A is the subset of A consisting of those complex numbers w for which there exists some positive real number δ such that $D(w, \delta) \subset A$, where $D(w, \delta)$ denotes the open disk of radius δ centred on w.

A straightforward application of Lemma 1.21 shows that if A is a subset of the complex plane then the interior of A is an open set.

1.14 Closed Sets in the Complex Plane

A subset F of the complex plane is said to be *closed* if its complement $\mathbb{C} \setminus F$ is open. (Recall that $\mathbb{C} \setminus F = \{z \in \mathbb{C} : z \notin F\}$.)

Example The sets $\{z \in \mathbb{C} : \operatorname{Re} z \geq c\}$, $\{z \in \mathbb{C} : \operatorname{Re} z \leq c\}$, and $\{z \in \mathbb{C} : z = c\}$ are closed sets in the complex plane for each real number c, since the complements of these sets are open subsets of the complex plane.

Example Let w be a complex number, and let r be a non-negative real number. Then $\{z \in \mathbb{C} : |z - w| \leq r\}$ and $\{z \in \mathbb{C} : |z - w| \geq r\}$ are closed sets in the complex plane. In particular, the set $\{w\}$ consisting of a single complex number w is a closed set. (These results follow immediately using Lemma 1.21 and Lemma 1.22 and the definition of closed sets.)

Let $\mathcal{A}$ be some collection of subsets of a set X. Then

$$X \setminus \bigcup_{S \in \mathcal{A}} S = \bigcap_{S \in \mathcal{A}} (X \setminus S), \qquad X \setminus \bigcap_{S \in \mathcal{A}} S = \bigcup_{S \in \mathcal{A}} (X \setminus S)$$

(i.e., the complement of the union of some collection of subsets of X is the intersection of the complements of those sets, and the complement of the intersection of some collection of subsets of X is the union of the complements of those sets). The following result therefore follows directly from Proposition 1.23.

Proposition 1.25 *The collection of closed sets in the complex plane has the following properties:*

(i) *the empty set $\emptyset$ and the whole complex plane $\mathbb{C}$ are both closed sets;*

(ii) *the intersection of any collection of closed sets is itself closed;*

(iii) *the union of any* finite *collection of closed sets is itself closed.*

Lemma 1.26 *Let F be a closed set in the complex plane, and let $z_1, z_2, z_3, \ldots$ be a sequence of complex numbers belonging to F which converges to a complex number w. Then $w \in F$.*

Proof The complement $\mathbb{C} \setminus F$ of F is open, since F is closed. Suppose that w were an element of $\mathbb{C} \setminus F$. It would then follow from Lemma 1.24 that $z_j \in \mathbb{C} \setminus F$ for all values of j greater than some positive integer N, contradicting the fact that $z_j \in F$ for all j. This contradiction shows that w must belong to F, as required. ∎

Lemma 1.27 *Let F be a closed bounded set in the complex plane $\mathbb{C}$, and let U be an open set in $\mathbb{C}$. Suppose that $F \subset U$. Then there exists positive real number δ such that $|z - w| \geq \delta > 0$ for all $z \in F$ and $w \in \mathbb{C} \setminus U$.*

Proof Suppose that such a positive real number δ did not exist. Then there would exist an infinite sequence $(z_j : j \in \mathbb{N})$ of elements of F and a correspondinding infinite sequence $(w_j : j \in \mathbb{N})$ of elements of $\mathbb{C} \setminus U$ such that $|z_j - w_j| < 1/j$ for all positive integers j. The sequence $(z_j : j \in \mathbb{N})$ would be a bounded sequence of complex numbers, and would therefore have a convergent subsequence $(z_{m_j} : j \in \mathbb{N})$ (Theorem 1.7). Let $c = \lim\limits_{j \to +\infty} z_{m_j}$. Then $c = \lim\limits_{j \to +\infty} w_{m_j}$, because $\lim\limits_{j \to +\infty} (z_{m_j} - w_{m_j}) = 0$. But then $c \in F$ and $c \in \mathbb{C} \setminus U$, because the sets F and $\mathbb{C} \setminus U$ are closed (Lemma 1.26). But this is impossible, as $F \subset U$. It follows that there must exist some positive real number δ with the required properties. ∎

1.15 Closures

Definition Let A be a subset of the complex plane. The *closure* of A is the subset of the complex plane consisting of all complex numbers z with the property that, given any real number δ satisfying $\delta > 0$, there exists some element a of A such that $|z - a| < \delta$. We denote the closure of A by $\overline{A}$.

Let A be a subset of the complex plane. Note that a complex number z belongs to the closure of A in X if and only if $D(z, \delta) \cap A$ is a non-empty set for all positive real numbers δ, where $D(z, \delta)$ denotes the open disk of radius δ centred on z.

Lemma 1.28 *Let A be a subset of the complex plane, and let w be an element of the closure $\overline{A}$ of A. Then there exists an infinite sequence of elements of A which converges to w.*

Proof For each positive integer j let z_j be an element of A satisfying $|w - z_j| < 1/j$. Then $\lim_{j \to +\infty} z_j = w$. ∎

Proposition 1.29 *Let A be a subset of the complex plane. Then the closure $\overline{A}$ of A is a closed set. Moreover if F is a closed set in the complex plane, and if $A \subset F$ then $\overline{A} \subset F$.*

Proof Let w be a complex number belonging to the complement $X \setminus \overline{A}$ of $\overline{A}$ in X, and, for any positive real number r let $D(w, r)$ denote the open disk of radius r about w. Then there exists some real number δ such that $D(w, 2\delta) \cap A = \emptyset$. Let z be an element of $\overline{A}$. Then there exists some element a of A such that $|z - a| < \delta$. Then

$$2\delta \leq |w - a| \leq |w - z| + |z - a| < |w - z| + \delta,$$

and therefore $|w - z| > \delta$. This shows that $D(w, \delta) \cap \overline{A} = \emptyset$. We deduce that the complement of $\overline{A}$ is an open set, and therefore $\overline{A}$ is a closed set.

Now let F be a closed set in the complex plane. Suppose that $A \subset F$. Let w be a complex number belonging to the complement $\mathbb{C} \setminus F$ of F. Then there exists some real number δ satisfying $\delta > 0$ for which $D(w, \delta) \cap F = \emptyset$. But then $D(w, \delta) \cap A = \emptyset$ and therefore $w \notin \overline{A}$. Thus $X \setminus F \subset X \setminus \overline{A}$, and therefore $\overline{A} \subset F$, as required. ∎

1.16 Continuous Functions and Open and Closed Sets

Let U be an open subset of the complex plane, and let $f: U \to \mathbb{C}$ be a complex-valued function defined on U. We recall that the function f is continuous at an element w of U if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that $|f(z) - f(w)| < \varepsilon$ for all elements z of U satisfying $|z - w| < \delta$. Moreover we can choose δ small enough to ensure that the open disk of radius δ about w is contained within the open set U. It follows that the function $f: U \to \mathbb{C}$ is continuous at w if and only if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that the function f maps $D(w, \delta)$ into $D(f(w), \varepsilon)$ (where $D(w, \delta)$ and $D(f(w), \varepsilon)$ denote the open disks of radius δ and ε about w and $f(w)$ respectively).

Given any function $f: U \to \mathbb{C}$, we denote by $f^{-1}(V)$ the *preimage* of a subset V of $\mathbb{C}$ under the map f, defined by $f^{-1}(V) = \{z \in U : f(z) \in V\}$.

Proposition 1.30 *Let U be an open set in $\mathbb{C}$, and let $f: U \to \mathbb{C}$ be a complex-valued function on U. The function f is continuous if and only if $f^{-1}(V)$ is an open set for every open subset V of $\mathbb{C}$.*

Proof Suppose that $f: U \to \mathbb{C}$ is continuous. Let V be an open set in $\mathbb{C}$. We must show that $f^{-1}(V)$ is an open set. Let $w \in f^{-1}(V)$. Then $f(w) \in V$. But V is open, hence there exists some $\varepsilon > 0$ with the property that $D(f(w), \varepsilon) \subset V$. But f is continuous at w. Therefore there exists some $\delta > 0$ such that f maps $D(w, \delta)$ into $D(f(w), \varepsilon)$ (see the remarks above). Thus $f(z) \in V$ for all $z \in D(w, \delta)$, showing that $D(w, \delta) \subset f^{-1}(V)$. This shows that $f^{-1}(V)$ is an open set for every open set V in $\mathbb{C}$.

Conversely suppose that $f: U \to \mathbb{C}$ is a function with the property that $f^{-1}(V)$ is an open set for every open set V in $\mathbb{C}$. Let $w \in U$. We must show that f is continuous at w. Let $\varepsilon > 0$ be given. Then $D(f(w), \varepsilon)$ is an open set in $\mathbb{C}$, by Lemma 1.21, hence $f^{-1}(D(f(w), \varepsilon))$ is an open set which contains w. It follows that there exists some $\delta > 0$ such that $D(w, \delta) \subset f^{-1}(D(f(w), \varepsilon))$. Thus, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that f maps $D(w, \delta)$ into $D(f(w), \varepsilon)$. We conclude that f is continuous at w, as required. ∎

Let U be an open subset of the complex plane, let $f: U \to \mathbb{R}$ be continuous, and let c be some real number. Then the sets $\{z \in U : f(z) > c\}$ and $\{z \in U : f(z) < c\}$ are open sets, and, given real numbers a and b satisfying $a < b$, the set $\{z \in U : a < f(z) < b\}$ is an open set.

1.17 Continuous Functions on Closed Bounded Sets

We shall prove that continuous functions of a complex variable map closed bounded sets to closed bounded sets.

Lemma 1.31 *Let X be a closed bounded subset of the complex plane space $\mathbb{C}$, and let $f: X \to \mathbb{C}$ be a continuous complex-valued function defined on X. Then there exists some non-negative real number M such that $|f(z)| \leq M$ for all $z \in X$.*

Proof Suppose that the function f were not bounded on X. Then there would exist a sequence $(z_j : j \in \mathbb{N})$ of complex numbers in X such that $|f(z_j)| > j$ for all positive integers j. Now the sequence $(z_j : j \in \mathbb{N})$ would be a bounded sequence of complex numbers, since the set X is bounded, and every bounded sequence of complex numbers has a convergent subsequence (Theorem 1.7). Therefore there would exist a subsequence $(z_{m_j} : j \in \mathbb{N})$ of $(z_j : j \in \mathbb{N})$ converging to some complex number w. Moreover w would belong to X, since X is closed (Lemma 1.26). Also it would follow from the continuity of the function f that $\lim_{j \to +\infty} f(z_{m_j}) = f(w)$ (Lemma 1.15), and therefore $|f(z_{m_j})| \leq |f(w)| + 1$ for all sufficiently large positive integers j. But this is impossible because $|f(z_j)| > j$ for all positive integers j, and thus

$|f(z_{m_j})| > m_j$ for all positive integers j, where $m_j \to +\infty$ as $j \to +\infty$. Thus the assumption that the function f is unbounded on X leads to a contradiction. We conclude that the function f must be unbounded on X, as required. ∎

Theorem 1.32 *Let X be a closed bounded set in the complex plane, and let $f: X \to \mathbb{C}$ be a continuous complex-valued function on X. Then the function f maps X onto a closed bounded set $f(X)$ in the complex plane.*

Proof It follows from Lemma 1.31 that the set $f(X)$ must be bounded.

Let q be a complex number belonging to the closure $\overline{f(A)}$ of $f(A)$. Then there exists a sequence $(z_j : j \in \mathbb{N})$ of complex numbers in X such that $\lim_{j\to+\infty} f(z_j) = q$ (Lemma 1.28). Because the set X is both closed and bounded, this sequence is a bounded sequence in the complex plane, and therefore has a convergent subsequence $(z_{m_j} : j \in \mathbb{N})$ (Theorem 1.7). Let $w = \lim_{j\to+\infty} z_{m_j}$. Then $w \in X$, because X is closed (Lemma 1.26). But then $q = \lim_{j\to+\infty} f(z_{m_j}) = f(w)$, and therefore $q \in f(A)$. Thus every element of the closure of $f(A)$ belongs to $f(A)$ itself, and therefore $f(A)$ is closed, as required. ∎

1.18 Uniform Continuity

Definition Let X and Y be subsets of the complex plane. A function $f: X \to Y$ from X to Y is said to be to be *uniformly continuous* if, given any $\varepsilon > 0$, there exists some $\delta > 0$ (which does not depend on either z' or z) such that $|f(z') - f(z)| < \varepsilon$ for all elements z' and z of X satisfying $|z' - z| < \delta$.

Theorem 1.33 *Let X be a subset of $\mathbb{C}$ that is both closed and bounded. Then any continuous function $f: X \to \mathbb{C}$ is uniformly continuous.*

Proof Let $\varepsilon > 0$ be given. Suppose that there did not exist any $\delta > 0$ such that $|f(z') - f(z)| < \varepsilon$ for all complex numbers z' and z in X satisfying $|z' - z| < \delta$. Then, for each natural number j, there would exist elements u_j and v_j in X such that $|u_j - v_j| < 1/j$ and $|f(u_j) - f(v_j)| \geq \varepsilon$. But the sequence $u_1, u_2, u_3, \ldots$ would be bounded, since X is bounded, and thus would possess a subsequence $u_{j_1}, u_{j_2}, u_{j_3}, \ldots$ converging to some complex number w (Theorem 1.7). Moreover $w \in X$, since X is closed. The sequence $v_{j_1}, v_{j_2}, v_{j_3}, \ldots$ would also converge to w, since $\lim_{k\to+\infty} |v_{j_k} - u_{j_k}| = 0$. But then the sequences $f(u_{j_1}), f(u_{j_2}), f(u_{j_3}), \ldots$ and $f(v_{j_1}), f(v_{j_2}), f(v_{j_3}), \ldots$ would converge to $f(w)$, since f is continuous (Lemma 1.15), and thus

$\lim_{k \to +\infty} |f(u_{j_k}) - f(v_{j_k})| = 0$. But this is impossible, since u_j and v_j have been chosen so that $|f(u_j) - f(v_j)| \geq \varepsilon$ for all j. We conclude therefore that there must exist some $\delta > 0$ such that $|f(z') - f(z)| < \varepsilon$ for all complex numbers z' and z in X satisfying $|z' - z| < \delta$, as required. ∎