Course 212: Trinity Term 2001
Part IV: Topology in the Plane

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Contents

11 The Exponential Map

11.1 The Exponential Map on the Complex Plane

The exponential map $\exp\colon \mathbb{C} \to \mathbb{C}$ is defined for all $z \in \mathbb{C}$ by the formula

$$\exp(z) = \sum_{n=0}^{+\infty} \frac{z^n}{n!}.$$

A straightforward application of the Ratio Test shows that this power series converges for all $z \in \mathbb{C}$, and converges uniformly on any disk of finite radius, centred on 0. It follows from this that the exponential function is continuous on $\mathbb{C}$. A general result on products of absolutely convergent infinite series can be used to show that

$$\exp(z + w) = \exp(z) \exp(w)$$

for all $z, w \in \mathbb{C}$. It follows immediately from this that $\exp(z) \neq 0$ for all $z \in \mathbb{C}$, and $\exp(-z) = 1/\exp(z)$, since

$$\exp(z) \exp(-z) = \exp(z - z) = \exp(0) = 1.$$

Note that, for any real number y,

$$\begin{aligned} \exp(iy) &= \sum_{m=0}^{+\infty} \frac{(iy)^{2m}}{(2m)!} + \sum_{m=0}^{+\infty} \frac{(iy)^{2m+1}}{(2m+1)!} \\ &= \sum_{m=0}^{+\infty} \frac{(-1)^m y^{2m}}{(2m)!} + i \sum_{m=0}^{+\infty} \frac{(-1)^m y^{2m+1}}{(2m+1)!} \\ &= \cos y + i \sin y \end{aligned}$$

(where we have split up the infinite series defining $\exp(iy)$ into separate summations over the even and the odd powers of z, and used the Taylor expansions for the sine and cosine functions derived using Taylor's Theorem). This identity is known as *de Moivre's Theorem*. It follows that

$$\exp(x + iy) = e^x (\cos y + i \sin y)$$

for all $x, y \in \mathbb{R}$. Also

$$\cos y = \frac{1}{2}(\exp(iy) + \exp(-iy)), \qquad \sin y = \frac{1}{2i}(\exp(iy) - \exp(-iy)).$$

Lemma 11.1 *Let z and w be complex numbers. Then $\exp(z) = \exp(w)$ if and only if $w = z + 2\pi i n$ for some integer n.*

Proof If $w = z + 2\pi i n$ for some integer n then

$$\exp(w) = \exp(z)\exp(2\pi i n) = \exp(z)(\cos 2\pi n + i \sin 2\pi n) = \exp(z).$$

Conversely suppose that $\exp(w) = \exp(z)$. Let $w - z = u + iv$, where $u, v \in \mathbb{R}$. Then

$$e^u(\cos v + i \sin v) = \exp(w - z) = \exp(w)\exp(z)^{-1} = 1.$$

Taking the modulus of both sides, we see that $e^u = 1$, and thus $u = 0$. Also $\cos v = 1$ and $\sin v = 0$, and therefore $v = 2\pi n$ for some integer n. The result follows. ▮

Proposition 11.2 *The exponential map* $\exp \mathbb{C} \to \mathbb{C}$ *maps the open strip*

$$\{z \in \mathbb{C} : \alpha < \operatorname{Im} z < \alpha + 2\pi\}$$

homeomorphically onto $\mathbb{C} \setminus L_\alpha$ *for each* $\alpha \in \mathbb{R}$*, where* $L_\alpha = \{te^{i\alpha} : t \geq 0\}$.

Proof Without loss of generality, we may assume that $\alpha = 0$, since $\exp(z) = \exp(i\alpha)\exp(z - i\alpha)$ for all $z \in \mathbb{C}$ and $\alpha \in \mathbb{R}$. Let

$$D = \mathbb{C} \setminus \{t \in \mathbb{R} : t \geq 0\}, \qquad E = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < 2\pi\},$$

and let $F\colon D \to E$ be the function sending $re^{i\theta}$ to $\log r + i\theta$ for all real numbers r and θ satisfying $r > 0$ and $0 < \theta < 2\pi$. Then $F\colon D \to E$ is the inverse of the restriction $\exp|E\colon E \to D$ of the exponential map to E, and thus $\exp|E$ is a bijection. Therefore it only remains to show that the function $F\colon D \to E$ is continuous. Now

$$F(x + iy) = \begin{cases} \frac{1}{2}\log(x^2 + y^2) + i \arccos\left(\dfrac{x}{\sqrt{x^2 + y^2}}\right) & \text{if } y > 0, \\ \frac{1}{2}\log(x^2 + y^2) + 2\pi i - i \arccos\left(\dfrac{x}{\sqrt{x^2 + y^2}}\right) & \text{if } y < 0, \\ \frac{1}{2}\log(x^2 + y^2) + \pi i - i \arcsin\left(\dfrac{y}{\sqrt{x^2 + y^2}}\right) & \text{if } x < 0, \end{cases}$$

where $\arcsin\colon [-1, 1] \to [-\pi/2, \pi/2]$ and $\arccos\colon [-1, 1] \to [0, \pi]$ are the inverses of the restrictions of the sine and cosine functions to the intervals $[-\pi/2, \pi/2]$ and $[0, \pi]$ respectively. Also the functions log, arcsin and arccos are continuous, since the inverse of any strictly increasing or strictly decreasing function defined over some interval in $\mathbb{R}$ is itself continuous. We conclude that the restrictions of $F\colon D \to E$ to each of the open sets

$$\{z \in \mathbb{C} : \operatorname{Im} z > 0\}, \qquad \{z \in \mathbb{C} : \operatorname{Im} z < 0\}, \qquad \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$$

is continuous, and the domain D of F is the union of these three open sets. It follows that $F\colon D \to E$ is itself continuous. Thus $\exp|E\colon E \to D$ is a homeomorphism, as required. ▮

11.2 Evenly Covered Open Sets

Definition We say that an open U in $\mathbb{C} \setminus \{0\}$ is *evenly covered* by the exponential map if the preimage $\exp^{-1}(U)$ of U is a disjoint union of open sets in $\mathbb{C}$, each of which is mapped homeomorphically onto U by the exponential map.

Corollary 11.3 *Let U be an open set in $\mathbb{C} \setminus \{0\}$. Suppose that $U \cap L_\alpha = \emptyset$ for some $\alpha \in \mathbb{R}$, where $L_\alpha = \{te^{i\alpha} : t \geq 0\}$. Then U is evenly covered by the exponential map.*

Proof For each integer n let

$$V_n = \exp^{-1}(U) \cap \{z \in \mathbb{C} : \alpha + 2\pi n < \operatorname{Im} z < \alpha + 2\pi(n+1)\}.$$

Then each V_n is an open set in $\mathbb{C}$ (since it is an intersection of open sets). Moreover $\exp^{-1}(U)$ is the union of the sets V_n, since $U \cap L_\alpha = \emptyset$ and therefore no point z of $\exp^{-1}(U)$ satisfies $\operatorname{Im} z = \alpha + 2\pi n$ for any integer n. Now it follows from Proposition 11.3 that the restriction of the exponential map to the strip

$$\{z \in \mathbb{C} : \alpha + 2\pi n < \operatorname{Im} z < \alpha + 2\pi(n+1)\}.$$

induces a one-to-one correspondence between open subsets of this strip and open subsets of $\mathbb{C} \setminus L_\alpha$. In particular, the open set V_n is mapped homeomorphically onto U for each integer n. Thus the open set U is evenly covered by the exponential map, as required. ∎

Lemma 11.4 *The set $\mathbb{C} \setminus \{0\}$ is not evenly covered by the exponential map.*

Proof Suppose that $\mathbb{C} \setminus \{0\}$ were evenly covered by the exponential map. Then, since the preimage of $\mathbb{C} \setminus \{0\}$ is the whole of the complex plane $\mathbb{C}$, we could express the complex plane as a disjoint union of open sets, each homeomorphic to $\mathbb{C} \setminus \{0\}$. But then each of these open sets would also be closed, since its complement would be the union of the other open sets. But this would contradict the connectedness of $\mathbb{C}$. The result follows. ∎

Lemma 11.5 *Let U be an open set in $\mathbb{C} \setminus \{0\}$ and let $F: U \to \mathbb{C}$ be a continuous map satisfying $\exp(F(z)) = z$ for all $z \in U$. Then $F(U)$ is an open set in $\mathbb{C}$.*

Proof Let $E_\alpha = \{z \in \mathbb{C} : \alpha < \operatorname{Im} z < \alpha + 2\pi\}$ for some real number α. Then

$$\exp(E_\alpha \cap F(U)) = \{z \in U : F(z) \in E_\alpha\} = F^{-1}(E_\alpha).$$

Thus $\exp(E_\alpha \cap F(U))$ is the preimage of the open set E_α under the continuous map $F: U \to \mathbb{C}$, and hence $\exp(E_\alpha \cap F(U))$ is open in U. But any subset of U that is open in U (relative to the subspace topology on U) must also be open in $\mathbb{C}$, since U is itself open in $\mathbb{C}$. Thus $\exp(E_\alpha \cap F(U))$ is an open subset of $\mathbb{C}$, and moreover $\exp(E_\alpha \cap F(U)) \subset \mathbb{C} \setminus L_\alpha$, where $L_\alpha = \{te^{i\alpha} : t \geq 0\}$. But the exponential map induces a one-to-one correspondence between the open sets contained in E_α and $\mathbb{C} \setminus L_\alpha$, since it maps the open set E_α homeomorphically onto the open set $\mathbb{C} \setminus L_\alpha$ (Proposition 11.2). It follows that $E_\alpha \cap F(U)$ is open in $\mathbb{C}$. But then $F(U)$ is the union of the open sets $E_\alpha \cap F(U)$ for all real numbers α, and therefore $F(U)$ is itself an open set, as required. ∎

Theorem 11.6 *An open set U in $\mathbb{C} \setminus \{0\}$ is evenly covered by the exponential map if and only if there exists a continuous map $F: U \to \mathbb{C}$ such that $\exp(F(z)) = z$ for all $z \in U$.*

Proof First suppose that U is evenly covered by the exponential map. Then the preimage $\exp^{-1}(U)$ of U is a disjoint union of open sets, each of which is mapped homeomorphically onto U by the exponential map. Let $\tilde{U}$ be one of these open sets, and let $F: U \to \tilde{U}$ be the inverse of the restriction $\exp|\tilde{U}: \tilde{U} \to U$ of the exponential map to $\tilde{U}$. Then F is continuous, since $\exp|\tilde{U}$ is a homeomorphism, and $\exp(F(z)) = z$ for all $z \in U$.

Conversely suppose that U is open, and that there exists a continuous map $F: U \to \mathbb{C}$ with the property that $\exp(F(z)) = z$ for all $z \in U$. We must show that U is evenly covered by the exponential map. Now $F(U)$ is open in $\mathbb{C}$, by Lemma 11.5, and the continuous function $F: U \to F(U)$ is the inverse of the restriction $\exp|F(U): F(U) \to U$ of the exponential map to $F(U)$. Thus $F(U)$ is mapped homeomorphically onto U by the exponential map. For each integer n, let $V_n = \{z \in \mathbb{C} : z - 2\pi i n \in F(U)\}$. Then each set V_n is open in $\mathbb{C}$. Now $\exp(z) = \exp(z - 2\pi i n)$ for each integer n. It follows that each set V_n is mapped homeomorphically onto U by the exponential map. Moreover if $z + 2\pi i m = w + 2\pi i n$, where m and n are integers and $z, w \in F(U)$, then $\exp(z) = \exp(w)$, hence $z = w$ (since the exponential map is injective on $F(U)$), and thus $m = n$. We deduce that the sets V_n are disjoint. If $z \in \mathbb{C}$ satisfies $\exp(z) \in U$ then $\exp(z) = \exp(w)$ for some $w \in F(U)$. But then $z = w + 2\pi i n$ for some integer n, and therefore $z \in V_n$. We conclude that the preimage $\exp^{-1}(U)$ of U is the disjoint union of the open sets V_n, and each of these open sets is mapped homeomorphically onto U by the exponential map. Thus U is evenly covered by the exponential map, as required. ∎

Corollary 11.7 *There does not exist any continuous function $F: \mathbb{C} \setminus \{0\} \to \mathbb{C}$ such that $\exp(F(z)) = z$ for all non-zero complex numbers z.*

Proof The result follows immediately from Theorem 11.6 in view of the fact that the open set $\mathbb{C} \setminus \{0\}$ is not evenly covered by the exponential map (Lemma 11.4). ∎

11.3 Riemann Surfaces

Historically, mathematicians have regarded the 'logarithm function' as an example of a 'many-valued function' on the set of all non-zero complex numbers. This 'many-valued function' associates to any non-zero complex number z the set consisting of all complex numbers w satisfying $\exp(w) = z$. Any two such complex numbers w differ by an integer multiple of $2\pi i$. A continuous function $F: U \to \mathbb{C}$, defined over an open subset U of $\mathbb{C} \setminus \{0\}$, and satisfying $\exp(F(z)) = z$ for all $z \in U$, is referred to as a (continuous) *branch* of the logarithm function. Theorem 11.6 therefore shows that a continuous branch of the logarithm function can be defined over an open subset of the punctured complex plane $\mathbb{C} \setminus \{0\}$ if and only if that open set is evenly covered by the exponential map.

Another approach to the study of 'many-valued' functions was pioneered by Riemann, in his theory of *Riemann surfaces*. Instead of regarding a function such as the logarithm function as a 'many-valued function' defined over the punctured complex plane, he regarded it as a single-valued function defined over a surface which can be projected onto the punctured complex plane. The portion of this surface covering a sufficiently small open set in the punctured plane consists of an infinite number of sheets, each of which is a copy of that open set. However if a point on the punctured plane traverses a loop that encircles zero, then the corresponding point of the Riemann surface will pass from one sheet to another. The Riemann surface will therefore resemble a spiral staircase.

We can construct a representation of such a Riemann surface as follows. Let

$$S = \{(z, w) \in \mathbb{C}^2 : z \neq 0 \text{ and } \exp(w) = z\}.$$

Define functions $p: S \to \mathbb{C}\setminus\{0\}$ and $\log: S \to \mathbb{C}$ by $p(z, w) = z$ and $\log(z, w) = w$ for all $(z, w) \in S$. Then $\exp \circ \log = p$. If U is an evenly-covered open set in $\mathbb{C} \setminus \{0\}$, and if w_0 is a complex number satisfying $\exp(w_0) \in U$, then we can find a continuous map $F: U \to \mathbb{C}$ satisfying $\exp(F(z)) = z$ for all $z \in U$ (Theorem 11.6), and moreover we can choose F so that $w_0 \in F(U)$. Then the map $\varphi: U \to S$ sending $z \in U$ to $(z, F(z))$ gives a parameterization of the surface S around the point w_0. The sheets of the Riemann surface S covering U are the open sets W_n for all integers n, where

$$W_n = \{(z, w) \in S : z \in U \text{ and } w - 2\pi i n \in F(U)\}.$$

It is easy to see that the 'logarithm function' $\log\colon S \to \mathbb{C}$ is actually a homeomorphism from S to $\mathbb{C}$. Thus the Riemann surface S is homeomorphic to the whole complex plane.

Now consider the loop $\sigma\colon [0,1] \to \mathbb{C}$ defined by $\sigma(t) = \exp(2\pi i t)$ for all $t \in [0,1]$. A point traversing this loop will encircle zero once in the anticlockwise direction as the parameter t varies from 0 to 1. It is not difficult to see that a corresponding point moving continuously on the Riemann surface S and starting at $(1,0)$ will traverse the path $\tilde{\sigma}\colon [0,1] \to S$, where $\tilde{\sigma}(t) = (\exp(2\pi i t), 2\pi i t)$ for all $t \in [0,1]$, and will travel from $(1,0)$ to $(1, 2\pi i)$. Thus the point on the Riemann surface has passed from one sheet of the Riemann surface to another as the point on the punctured plane has traversed a loop in the punctured complex plane encircling zero.

Alternatively, one can represent the Riemann surface for the logarithm function as a surface in $\mathbb{R}^3$. Indeed let Σ denote the surface in $\mathbb{R}^3$ consisting of all points (x,y,z) that satisfy the conditions

$$x^2 + y^2 \neq 0, \quad \cos 2\pi z = \frac{x}{\sqrt{x^2+y^2}} \text{ and } \sin 2\pi z = \frac{y}{\sqrt{x^2+y^2}}.$$

The function sending $(x,y,z) \in \Sigma$ to $(x + iy, \frac{1}{2}\log(x^2+y^2) + 2\pi i z)$ is a homeomorphism from Σ to S. The projection function $p\colon S \to \mathbb{C} \setminus \{0\}$ corresponds, under this homeomorphism, to the map sending $(x,y,z) \in \Sigma$ to $x + iy$, and the 'logarithm function' $\log\colon S \to \mathbb{C}$ corresponds to the map sending $(x,y,z) \in \Sigma$ to $\frac{1}{2}\log(x^2+y^2) + 2\pi i z$.

Analogous techniques can be used to define Riemann surfaces for other 'multi-valued functions', such as $z \mapsto \sqrt{z^2 - 1}$. The introduction by Riemann of this way of viewing 'many-valued functions' led to the discovery of very deep and powerful theorems in the theory of complex functions whose statement and proof involves geometric and topological concepts and techniques.

11.4 Path Lifting and the Monodromy Theorem

Let X be a topological space, and let $f\colon X \to \mathbb{C} \setminus \{0\}$ be a continuous map. We shall study the problem of determining whether or not there exists a continuous map $\tilde{f}\colon X \to \mathbb{C}$ satisfying $\exp \circ \tilde{f} = f$. A problem of this sort is referred to as a *lifting problem*. We have already seen that such a map $\tilde{f}$ does not exist when $X = \mathbb{C} \setminus \{0\}$ and $f\colon X \to \mathbb{C} \setminus \{0\}$ is the identity map. On the other hand, we shall prove the existence of the required lift $\tilde{f}$ of f when X is either the interval $[0,1]$ or the square $[0,1] \times [0,1]$. First we prove a uniqueness result concerning such lifts.

Proposition 11.8 *Let X be a connected topological space, and let $g: X \to \mathbb{C}$ and $h: X \to \mathbb{C}$ be continuous maps. Suppose that $\exp \circ g = \exp \circ h$ and that $g(x_0) = h(x_0)$ for some $x_0 \in X$. Then $g = h$.*

Proof Let $X_0 = \{x \in X : g(x) = h(x)\}$. Note that X_0 is non-empty, by hypothesis. We show that X_0 is both open and closed in X.

Let x be a point of X. There exists an open neighbourhood U of $\exp(g(x))$ in $\mathbb{C} \setminus \{0\}$ which is evenly covered by the exponential map. Then $\exp^{-1}(U)$ is a disjoint union of open sets, each of which is mapped homeomorphically onto U by the exponential map. One of these open sets contains $g(x)$; let this set be denoted by $\tilde{U}$. Also one of these open sets contains $h(x)$; let this open set be denoted by $\tilde{V}$. Note that $\tilde{U} = \tilde{V}$ if $x \in X_0$ (so that $g(x) = h(x)$), and $\tilde{U} \cap \tilde{V} = \emptyset$ if $x \in X \setminus X_0$ (so that $g(x) \neq h(x)$). Let $N_x = g^{-1}(\tilde{U}) \cap h^{-1}(\tilde{V})$. Then N_x is an open set in X containing x.

Consider the case when $x \in X_0$. In this case $\tilde{V} = \tilde{U}$, so that $g(N_x) \subset \tilde{U}$ and $h(N_x) \subset \tilde{U}$. But $\exp \circ g = \exp \circ h$, and the restriction $\exp|\tilde{U}$ of the exponential map to $\tilde{U}$ maps $\tilde{U}$ homeomorphically onto U. Therefore $g|N_x = h|N_x$, and thus $N_x \subset X_0$. We have thus shown that, for each $x \in X_0$, there exists an open set N_x such that $x \in N_x$ and $N_x \subset X_0$. We conclude that X_0 is open.

Next consider the case when $x \in X \setminus X_0$. In this case $\tilde{U} \cap \tilde{V} = \emptyset$. But $g(N_x) \subset \tilde{U}$ and $h(N_x) \subset \tilde{V}$. Therefore $g(x') \neq h(x')$ for all $x' \in N_x$, so that $N_x \subset X \setminus X_0$. We have thus shown that, for each $x \in X \setminus X_0$, there exists an open set N_x such that $x \in N_x$ and $N_x \subset X \setminus X_0$. We conclude that $X \setminus X_0$ is open, and thus X_0 is closed.

The subset X_0 of X is both open and closed. Also X_0 is non-empty, since there exists some point x_0 of X for which $g(x_0) = h(x_0)$. It follows from the connectedness of X that $X_0 = X$. Therefore $g = h$. ∎

Lemma 11.9 *Let X be a topological space, let A be a connected subset of X, and let $f: X \to \mathbb{C} \setminus \{0\}$ and $g: A \to \mathbb{C}$ be continuous maps with the property that $\exp \circ g = f|A$. Suppose that $f(X) \subset U$, where U is an open subset of $\mathbb{C} \setminus \{0\}$ that is evenly covered by the exponential map. Then there exists a continuous map $\tilde{f}: X \to \mathbb{C}$ such that $\tilde{f}|A = g$ and $\exp \circ \tilde{f} = f$.*

Proof Choose $a_0 \in A$. Now U is evenly covered by the exponential map. Therefore $\exp^{-1}(U)$ is a disjoint union of open sets, each of which is mapped homeomorphically onto U by the exponential map. One of these open sets contains $g(a_0)$; let this set be denoted by $\tilde{U}$. Let $F: U \to \tilde{U}$ be the inverse of the homeomorphism $\exp|\tilde{U}: \tilde{U} \to U$, and let $\tilde{f} = F \circ f$. Then $\exp \circ \tilde{f} = f$. Also $\exp \circ \tilde{f}|A = \exp \circ g$ and $\tilde{f}(a_0) = g(a_0)$. It follows from Proposition 11.8 that $\tilde{f}|A = g$, since A is connected. Thus $\tilde{f}: X \to \mathbb{C}$ is the required map. ∎

Theorem 11.10 (Path Lifting Theorem) *Let* $\gamma\colon [0,1] \to \mathbb{C} \setminus \{0\}$ *be a continuous path in* $\mathbb{C} \setminus \{0\}$*, and let* z *be a complex number satisfying* $\exp(z) = \gamma(0)$*. Then there exists a unique continuous path* $\tilde{\gamma}\colon [0,1] \to \mathbb{C}$ *such that* $\tilde{\gamma}(0) = z$ *and* $\exp \circ \tilde{\gamma} = \gamma$*.*

Proof Let $\mathcal{U}$ be the collection consisting of the sets $\mathbb{C} \setminus L_\alpha$ for $0 \leq \alpha < 2\pi$, where $L_\alpha = \{te^{i\alpha} : t \geq 0\}$. Then $\mathcal{U}$ is an open cover of $\mathbb{C} \setminus \{0\}$, and each of the open sets belonging to $\mathcal{U}$ is evenly covered by the exponential map. Now the collection of sets of the form $\gamma^{-1}(U)$ with $U \in \mathcal{U}$ is an open cover of the interval $[0,1]$. But $[0,1]$ is compact, by the Heine-Borel Theorem. It follows from the Lebesgue Lemma that there exists some $\delta > 0$ such that every subinterval of length less than δ is mapped by γ into one of the open sets belonging to $\mathcal{U}$. Partition the interval $[0,1]$ into subintervals $[t_{i-1}, t_i]$, where $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$, and where the length of each subinterval is less than δ. Now it follows from Lemma 11.9 that once $\tilde{\gamma}(t_{i-1})$ has been determined, we can extend $\tilde{\gamma}$ continuously over the ith subinterval $[t_{i-1}, t_i]$. Thus by extending $\tilde{\gamma}$ successively over $[t_0, t_1]$, $[t_1, t_2]$,..., $[t_{n-1}, t_n]$, we can lift the path $\gamma\colon [0,1] \to \mathbb{C} \setminus \{0\}$ to a path $\tilde{\gamma}\colon [0,1] \to \mathbb{C}$ starting at z. The uniqueness of $\tilde{\gamma}$ follows from Proposition 11.8. ∎

Alternative Proof Let J be the set of all real numbers s in the interval $[0,1]$ with the property that there exists some (necessarily unique) continuous map $\tilde{\gamma}_s\colon [0,s] \to \mathbb{C}$ satisfying $\tilde{\gamma}_s(0) = z$ and $\exp(\tilde{\gamma}_s(t)) = \gamma(t)$ for all $t \in [0,s]$. Clearly $0 \in J$, and thus J is a non-empty subset of $[0,1]$. We must prove that $1 \in J$.

Let $\tau = \sup J$, and let U be an open subset of $\mathbb{C} \setminus \{0\}$ which contains the point $\gamma(\tau)$ and is evenly covered by the exponential map. Choose $\delta > 0$ such that $\gamma(t) \in U$ for all $t \in [0,1]$ satisfying $|t - \tau| < \delta$, and choose $s \in J$ satisfying $\tau - \delta < s \leq \tau$. Then there exists a continuous function $F\colon U \to \mathbb{C}$ such that $\exp(F(z)) = z$ for all $z \in U$ (Theorem 11.6), and F can be chosen so that $\tilde{\gamma}_s(s) \in F(U)$. Given any $u \in [0,1]$ satisfying $s \leq u < \tau + \delta$, we define a continuous lift $\tilde{\gamma}_u\colon [0,u] \to \mathbb{C}$ of $\gamma|[0,u]$ by the formula

$$\tilde{\gamma}_u(t) = \begin{cases} \tilde{\gamma}_s(t) & \text{if } 0 \leq t \leq s, \\ F(\gamma(t)) & \text{if } s \leq t \leq u. \end{cases}$$

It follows that $u \in J$. In particular $\tau \in J$. If $\tau < 1$ then there would exist $u \in [0,1]$ satisfying $\tau < u < \tau + \delta$. But then $u \in J$, which is impossible, since $\tau = \sup J$. Thus $\tau = 1$, and so $1 \in J$, as required. ∎

Theorem 11.11 (The Monodromy Theorem) *Let* $H\colon [0,1] \times [0,1] \to \mathbb{C} \setminus \{0\}$ *be a continuous map, and let* z *be a complex number satisfying* $\exp(z) = H(0,0)$*. Then there exists a unique continuous map* $\tilde{H}\colon [0,1] \times [0,1] \to \mathbb{C}$ *such that* $\tilde{H}(0,0) = z$ *and* $\exp \circ \tilde{H} = H$*.*

Proof Again, let $\mathcal{U}$ be the open cover of $\mathbb{C} \setminus \{0\}$ consisting of the open sets $\mathbb{C} \setminus L_\alpha$ for $0 \leq \alpha < 2\pi$, where $L_\alpha = \{te^{i\alpha} : t \geq 0\}$. Then the collection of sets of the form $H^{-1}(U)$ with $U \in \mathcal{U}$ is an open cover of the unit square $[0,1] \times [0,1]$. But the unit square is compact. An application of the Lebesgue Lemma (as in the proof of Theorem 11.10) shows that there exists some $\delta > 0$ with the property that any square contained in $[0,1] \times [0,1]$ whose sides have length less than δ is mapped by H into some open set in $\mathbb{C} \setminus \{0\}$ which is evenly covered by the exponential map. It follows from Lemma 11.9 that if the lift $\tilde{H}$ of H has already been determined over a corner, or along one side, or along two adjacent sides of a square whose sides have length less than δ, then $\tilde{H}$ can be extended over the whole of that square. Thus if we subdivide the square $[0,1] \times [0,1]$ into smaller squares whose sides have length less than δ then we can extend the map g to a lift $\tilde{H}$ of H by successively extending $\tilde{H}$ in turn over each of the smaller squares. Indeed suppose that the unit square has been subdivided into squares $S_{j,k}$ for $j, k = 1, 2, \ldots, n$, where $1/n < \delta$ and

$$S_{j,k} = \{(x,y) \in [0,1] \times [0,1] : \frac{j-1}{n} \leq x \leq \frac{j}{n} \text{ and } \frac{k-1}{n} \leq y \leq \frac{k}{n}\}.$$

Then we can extend the map $\tilde{H}$ successively over the squares

$$S_{1,1}, S_{1,2}, \ldots, S_{1,n}, S_{2,1}, S_{2,2}, \ldots, S_{2,n}, S_{3,1}, \ldots, S_{n-1,n} S_{n,1}, S_{n,2}, \ldots, S_{n,n}.$$

The uniqueness of $\tilde{H}$ follows from Proposition 11.8. ∎

12 Winding Numbers

12.1 Winding Numbers of Closed Curves

Let $\gamma\colon [0,1] \to \mathbb{C}$ be a continuous closed curve in the complex plane which is defined on some closed interval $[0,1]$ (so that $\gamma(0) = \gamma(1)$), and let w be a complex number which does not belong to the image of the closed curve γ. It then follows from the Path Lifting Theorem (Theorem 11.10) that there exists a continuous path $\tilde{\gamma}\colon [0,1] \to \mathbb{C}$ in $\mathbb{C}$ such that $\gamma(t) - w = \exp(\tilde{\gamma}(t))$ for all $t \in [0,1]$. Let us define

$$n(\gamma, w) = \frac{\tilde{\gamma}(1) - \tilde{\gamma}(0)}{2\pi i}.$$

Now $\exp(\tilde{\gamma}(1)) = \gamma(1) - w = \gamma(0) - w = \exp(\tilde{\gamma}(0))$ (since γ is a closed curve). It follows from this that $n(\gamma, w)$ is an integer. This integer is known as the *winding number* of the closed curve γ about w.

Lemma 12.1 *The value of the winding number $n(\gamma, w)$ does not depend on the choice of the lift $\tilde{\gamma}$ of the curve γ.*

Proof Let $\sigma\colon [0,1] \to \mathbb{C}$ be a continuous curve in $\mathbb{C}$ with the property that $\exp(\sigma(t)) = \gamma(t) - w = \exp(\tilde{\gamma}(t))$ for all $t \in [0,1]$. Then

$$\frac{\sigma(t) - \tilde{\gamma}(t)}{2\pi i}$$

is an integer for all $t \in [0,1]$. But the map sending $t \in [0,1]$ to $\sigma(t) - \tilde{\gamma}(t)$ is continuous on $[0,1]$. This map must therefore be a constant map, since the interval $[0,1]$ is connected. Thus there exists some integer m with the property that $\sigma(t) = \tilde{\gamma}(t) + 2\pi i m$ for all $t \in [0,1]$. But then

$$\sigma(1) - \sigma(0) = \tilde{\gamma}(1) - \tilde{\gamma}(0).$$

This proves that the value of the winding number $n(\gamma, w)$ of the closed curve γ about w is indeed independent of the choice of the lift $\tilde{\gamma}$ of γ. ∎

A continuous curve is said to be *piecewise* C^1 if it is made up of a finite number of continuously differentiable segments. We now show how the winding number of a piecewise C^1 closed curve in the complex plane can be expressed as a contour integral.

Proposition 12.2 *Let $\gamma\colon [0,1] \to \mathbb{C}$ be a piecewise C^1 closed curve in the complex plane, and let w be a point of $\mathbb{C}$ that does not lie on the curve γ. Then the winding number $n(\gamma, w)$ of γ about w is given by*

$$n(\gamma, w) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - w}.$$

Proof By definition

$$n(\gamma, w) = \frac{\sigma(1) - \sigma(0)}{2\pi i},$$

where $\sigma\colon [0,1] \to \mathbb{C}$ is a path in $\mathbb{C}$ such that $\gamma(t) - w = \exp(\sigma(t))$ for all $t \in [0,1]$. Taking derivatives, we see that

$$\gamma'(t) = \exp(\sigma(t))\sigma'(t) = (\gamma(t) - w)\sigma'(t).$$

Thus

$$\begin{aligned} n(\gamma, w) &= \frac{\sigma(1) - \sigma(0)}{2\pi i} = \frac{1}{2\pi i} \int_0^1 \sigma'(t)\, dt \\ &= \frac{1}{2\pi i} \int_0^1 \frac{\gamma'(t)\, dt}{\gamma(t) - w} = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - w}. \end{aligned}$$ ∎

One of the most important properties of winding numbers of closed curves in the complex plane is their invariance under continuous deformations of the closed curve.

Proposition 12.3 *Let w be a complex number and, for each $\tau \in [0,1]$, let $\gamma_\tau \colon [0,1] \to \mathbb{C}$ be a closed curve in $\mathbb{C}$ which does not pass through w. Suppose that the map sending $(t,\tau) \in [0,1] \times [0,1]$ to $\gamma_\tau(t)$ is a continuous map from $[0,1] \times [0,1]$ to $\mathbb{C}$. Then $n(\gamma_\tau, w) = n(\gamma_0, w)$ for all $\tau \in [0,1]$. In particular, $n(\gamma_1, w) = n(\gamma_0, w)$.*

Proof Let $H \colon [0,1] \times [0,1] \to \mathbb{C} \setminus \{0\}$ be defined by $H(t,\tau) = \gamma_\tau(t) - w$. It follows from the Monodromy Theorem (Theorem 11.11) that there exists a continuous map $\tilde{H} \colon [0,1] \times [0,1] \to \mathbb{C}$ such that $H = \exp \circ \tilde{H}$. But then

$$\tilde{H}(1,\tau) - \tilde{H}(0,\tau) = 2\pi i n(\gamma_\tau, w)$$

for all $\tau \in [0,1]$, and therefore the function $\tau \mapsto n(\gamma_\tau, w)$ is a continuous function on the interval $[0,1]$ taking values in the set $\mathbb{Z}$ of integers. But such a function must be constant on $[0,1]$, since the interval $[0,1]$ is connected. Thus $n(\gamma_0, w) = n(\gamma_1, w)$, as required. ∎

Corollary 12.4 *Let $\gamma_0 \colon [0,1] \to \mathbb{C}$ and $\gamma_1 \colon [0,1] \to \mathbb{C}$ be continuous closed curves in $\mathbb{C}$, and let w be a complex number which does not lie on the images of the closed curves γ_0 and γ_1. Suppose that $|\gamma_1(t) - \gamma_0(t)| < |w - \gamma_0(t)|$ for all $t \in [0,1]$. Then $n(\gamma_0, w) = n(\gamma_1, w)$.*

Proof Let $\gamma_\tau(t) = (1-\tau)\gamma_0(t) + \tau\gamma_1(t)$ for all $t \in [0,1]$ and $\tau \in [0,1]$. Then

$$|\gamma_\tau(t) - \gamma_0(t)| = \tau|\gamma_1(t) - \gamma_0(t)| < |w - \gamma_0(t)|,$$

for all $t \in [0,1]$ and $\tau \in [0,1]$, and thus the closed curve γ_τ does not pass through w. The result therefore follows from Proposition 12.3. ∎

Corollary 12.5 *Let $\gamma \colon [0,1] \to \mathbb{C}$ be a continuous closed curve in $\mathbb{C}$, and let $\sigma \colon [0,1] \to \mathbb{C}$ be a continuous path in $\mathbb{C}$ whose image does not intersect the image of γ. Then $n(\gamma, \sigma(0)) = n(\gamma, \sigma(1))$. Thus the function $w \mapsto n(\gamma, w)$ is constant over each path-component of the set $\mathbb{C} \setminus \gamma([0,1])$.*

Proof For each $\tau \in [0,1]$, let $\gamma_\tau \colon [0,1] \to \mathbb{C}$ be the closed curve given by $\gamma_\tau(t) = \gamma(t) - \sigma(\tau)$. Then the closed curves γ_τ do not pass through 0 (since the curves γ and σ do not intersect), and the map from $[0,1] \times [0,1]$ to $\mathbb{C}$ sending (t,τ) to $\gamma_\tau(t)$ is continuous. It follows from Proposition 12.3 that

$$n(\gamma, \sigma(0)) = n(\gamma_0, 0) = n(\gamma_1, 0) = n(\gamma, \sigma(1)),$$

as required. ∎

12.2 The Fundamental Theorem of Algebra

Theorem 12.6 (The Fundamental Theorem of Algebra) *Let $P: \mathbb{C} \to \mathbb{C}$ be a non-constant polynomial with complex coefficients. Then there exists some complex number z_0 such that $P(z_0) = 0$.*

Proof The result is trivial if $P(0) = 0$. Thus it suffices to prove the result when $P(0) \neq 0$.

For any $r \geq 0$, let the closed curve σ_r denote the circle about zero of radius r, traversed once in the anticlockwise direction, given by $\sigma_r(t) = r\exp(2\pi i t)$ for all $t \in [0, 1]$. Consider the winding number $n(P \circ \sigma_r, 0)$ of $P \circ \sigma_r$ about zero. We claim that this winding number is equal to m for large values of r, where m is the degree of the polynomial P.

Let $P(z) = a_0 + a_1 z + \cdots + a_m z^m$, where $a_1, a_2, \ldots, a_m$ are complex numbers, and where $a_m \neq 0$. We write $P(z) = P_m(z) + Q(z)$, where $P_m(z) = a_m z^m$ and $Q(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1}$. Let

$$R = \frac{|a_0| + |a_1| + \cdots + |a_m|}{|a_m|}.$$

If $|z| > R$ then

$$\left| \frac{Q(z)}{P_m(z)} \right| = \frac{1}{|a_m z|} \left| \frac{a_0}{z^{m-1}} + \frac{a_1}{z^{m-2}} + \cdots + a_{m-1} \right| < 1,$$

since $R \geq 1$, and thus $|P(z) - P_m(z)| < |P_m(z)|$. It follows from Corollary 12.4 that $n(P \circ \sigma_r, 0) = n(P_m \circ \sigma_r, 0) = m$ for all $r > R$.

Given $r \geq 0$, let $\gamma_\tau = P \circ \sigma_{\tau r}$ for all $\tau \in [0, 1]$. Then $n(\gamma_0, 0) = 0$, since γ_0 is a constant curve with value $P(0)$. Thus if the polynomial P were everywhere non-zero, then it would follow from Proposition 12.3 that $n(\gamma_1, 0) = n(\gamma_0, 0) = 0$. But $n(\gamma_1, 0) = n(P \circ \sigma_r, 0) = m$ for all $r > R$, and $m > 0$. Therefore the polynomial P must have at least one zero in the complex plane. ∎

The proof of the Fundamental Theorem of Algebra given above depends on the continuity of the polynomial P, together with the fact that the winding number $n(P \circ \sigma_r, 0)$ is non-zero for sufficiently large r, where σ_r denotes the circle of radius r about zero, described once in the anticlockwise direction. We can therefore generalize the proof of the Fundamental Theorem of Algebra in order to obtain the following result (sometimes referred to as the *Kronecker Principle*).

Proposition 12.7 *Let $f\colon D \to \mathbb{C}$ be a continuous map defined on the closed unit disk D in $\mathbb{C}$, and let $w \in \mathbb{C} \setminus f(D)$. Then $n(f \circ \sigma, w) = 0$, where $\sigma\colon [0,1] \to \mathbb{C}$ is the parameterization of unit circle defined by $\sigma(t) = \exp(2\pi i t)$, and $n(f \circ \sigma, w)$ is the winding number of $f \circ \sigma$ about w.*

Proof Define $\gamma_\tau(t) = f(\tau \exp(2\pi i t))$ for all $t \in [0,1]$ and $\tau \in [0,1]$. Then none of the closed curves γ_τ passes through w, and γ_0 is the constant curve with value $f(0)$. It follows from Proposition 12.3 that

$$n(f \circ \sigma, w) = n(\gamma_1, w) = n(\gamma_0, w) = 0,$$

as required. ∎

12.3 The Brouwer Fixed Point Theorem

We now use Proposition 12.7 to show that there is no continuous 'retraction' mapping the closed unit disk onto its boundary circle.

Corollary 12.8 *There does not exist a continuous map $r\colon D \to \partial D$ with the property that $r(z) = z$ for all $z \in \partial D$, where ∂D denotes the boundary circle of the closed unit disk D.*

Proof Let $\sigma\colon [0,1] \to \mathbb{C}$ be defined by $\sigma(t) = \exp(2\pi i t)$. If a continuous map $r\colon D \to \partial D$ with the required property were to exist, then $r(z) \neq 0$ for all $z \in D$ (since $r(D) \subset \partial D$), and therefore $n(\sigma, 0) = n(r \circ \sigma, 0) = 0$, by Proposition 12.7. But $\sigma = \exp \circ \tilde{\sigma}$, where $\tilde{\sigma}(t) = 2\pi i t$ for all $t \in [0,1]$, and thus

$$n(\sigma, 0) = \frac{\tilde{\sigma}(1) - \tilde{\sigma}(0)}{2\pi i} = 1.$$

This shows that there cannot exist any continuous map r with the required property. ∎

Theorem 12.9 (The Brouwer Fixed Point Theorem in Two Dimensions) *Let $f\colon D \to D$ be a continuous map which maps the closed unit disk D into itself. Then there exists some $z_0 \in D$ such that $f(z_0) = z_0$.*

Proof Suppose that there did not exist any fixed point z_0 of $f\colon D \to D$. Then one could define a continuous map $r\colon D \to \partial D$ as follows: for each $z \in D$, let $r(z)$ be the point on the boundary ∂D of D obtained by continuing the line segment joining $f(z)$ to z beyond z until it intersects ∂D at the point $r(z)$. Then $r\colon D \to \partial D$ would be a continuous map, and moreover $r(z) = z$ for all $z \in \partial D$. But Corollary 12.8 shows that there does not exist any continuous map $r\colon D \to \partial D$ with this property. We conclude that $f\colon D \to D$ must have at least one fixed point. ∎

Remark The Brouwer Fixed Point Theorem is also valid in higher dimensions. This theorem states that any continuous map from the closed n-dimensional ball into itself must have at least one fixed point. The proof of the theorem for $n > 2$ is analogous to the proof for $n = 2$, once one has shown that there is no continuous map from the closed n-dimensional ball to its boundary which is the identity map on the boundary. However winding numbers cannot be used to prove this result, and thus more powerful topological techniques need to be employed.

12.4 The Borsuk-Ulam Theorem

Lemma 12.10 *Let $f\colon S^1 \to \mathbb{C} \setminus \{0\}$ be a continuous function defined on S^1, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Suppose that $f(-z) = -f(z)$ for all $z \in S^1$. Then the winding number $n(f \circ \sigma, 0)$ of $f \circ \sigma$ about 0 is odd, where $\sigma\colon [0,1] \to S^1$ is given by $\sigma(t) = \exp(2\pi i t)$.*

Proof It follows from the Path Lifting Theorem (Theorem 11.10) that there exists a continuous path $\tilde{\gamma}\colon [0,1] \to \mathbb{C}$ in $\mathbb{C}$ such that $\exp(\tilde{\gamma}(t)) = f(\sigma(t))$ for all $t \in [0,1]$. Now $f(\sigma(t + \frac{1}{2})) = -f(\sigma(t))$ for all $t \in [0, \frac{1}{2}]$, since $\sigma(t + \frac{1}{2}) = -\sigma(t)$ and $f(-z) = -f(z)$ for all $z \in \mathbb{C}$. Thus $\exp(\tilde{\gamma}(t + \frac{1}{2})) = \exp(\tilde{\gamma}(t) + \pi i)$ for all $t \in [0, \frac{1}{2}]$. It follows that $\tilde{\gamma}(t + \frac{1}{2}) = \tilde{\gamma}(t) + (2m + 1)\pi i$ for some integer m. (The value of m for which this identity is valid does not depend on t, since every continuous function from $[0, \frac{1}{2}]$ to the set of integers is necessarily constant.) Hence

$$n(f \circ \sigma, 0) = \frac{\tilde{\gamma}(1) - \tilde{\gamma}(0)}{2\pi i} = \frac{\tilde{\gamma}(1) - \tilde{\gamma}(\frac{1}{2})}{2\pi i} + \frac{\tilde{\gamma}(\frac{1}{2}) - \tilde{\gamma}(0)}{2\pi i} = 2m + 1.$$

Thus $n(f \circ \sigma, 0)$ is an odd integer, as required. ∎

We shall identify the space $\mathbb{R}^2$ with $\mathbb{C}$, identifying $(x, y) \in \mathbb{R}^2$ with the complex number $x + iy \in \mathbb{C}$ for all $x, y \in \mathbb{R}$. This is permissible, since we are interested in purely topological results concerning continuous functions defined on appropriate subsets of these spaces. Under this identification the closed unit disk D is given by

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

As usual, we define

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

Lemma 12.11 *Let $f: S^2 \to \mathbb{R}^2$ be a continuous map with the property that $f(-\mathbf{n}) = -f(\mathbf{n})$ for all $\mathbf{n} \in S^2$. Then there exists some point $\mathbf{n}_0$ of S^2 with the property that $f(\mathbf{n}_0) = 0$.*

Proof Let $\varphi: D \to S^2$ be the map defined by

$$\varphi(x, y) = (x, y, +\sqrt{1 - x^2 - y^2}).$$

(Thus the map φ maps the closed disk D homeomorphically onto the upper hemisphere in $\mathbb{R}^3$.) Let $\sigma: [0, 1] \to S^2$ be the parameterization of the equator in S^2 defined by

$$\sigma(t) = (\cos 2\pi t, \sin 2\pi t, 0)$$

for all $t \in [0, 1]$. Let $f: S^2 \to \mathbb{R}^2$ be a continuous map with the property that $f(-\mathbf{n}) = -f(\mathbf{n})$ for all $\mathbf{n} \in S^2$. If $f(\sigma(t_0)) = 0$ for some $t_0 \in [0, 1]$ then the function f has a zero at $\sigma(t_0)$. It remains to consider the case in which $f(\sigma(t)) \neq 0$ for all $t \in [0, 1]$. In that case the winding number $n(f \circ \sigma, 0)$ is an odd integer, by Lemma 12.10, and is thus non-zero. It follows from Proposition 12.7, applied to $f \circ \varphi: D \to \mathbb{R}^2$, that $0 \in f(\varphi(D))$, (since otherwise the winding number $n(f \circ \sigma, 0)$ would be zero). Thus $f(\mathbf{n}_0) = 0$ for some $\mathbf{n}_0 = \varphi(D)$, as required. ∎

Theorem 12.12 (Borsuk-Ulam) *Let $f: S^2 \to \mathbb{R}^2$ be a continuous map. Then there exists some point $\mathbf{n}$ of S^2 with the property that $f(-\mathbf{n}) = f(\mathbf{n})$.*

Proof This result follows immediately on applying Lemma 12.11 to the continuous function $g: S^2 \to \mathbb{R}^2$ defined by $g(\mathbf{n}) = f(\mathbf{n}) - f(-\mathbf{n})$. ∎

Remark It is possible to generalize the Borsuk-Ulam Theorem to n dimensions. Let S^n be the unit n-sphere centered on the origin in $\mathbb{R}^n$. The Borsuk-Ulam Theorem in n-dimensions states that if $f: S^n \to \mathbb{R}^n$ is a continuous map then there exists some point $\mathbf{x}$ of S^n with the property that $f(\mathbf{x}) = f(-\mathbf{x})$.