Course 212: Hilary Term 2001
Part III: Normed Vector Spaces and Functional Analysis

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Contents

9 Normed Vector Spaces

A set X is a *vector space* over some field $\mathbb{F}$ if

- given any $x, y \in X$ and $\lambda \in \mathbb{F}$, there are well-defined elements $x + y$ and λx of X,
- X is an Abelian group with respect to the operation $+$ of addition,
- the identities

$$\lambda(x + y) = \lambda x + \lambda y, \qquad (\lambda + \mu)x = \lambda x + \mu x,$$

$$(\lambda\mu)x = \lambda(\mu x), \qquad 1x = x$$

  are satisfied for all $x, y \in X$ and $\lambda, \mu \in \mathbb{F}$.

Elements of the field $\mathbb{F}$ are referred to as *scalars*. We consider here only *real vector spaces* and *complex vector spaces*: these are vector spaces over the fields of real numbers and complex numbers respectively.

Definition A *norm* $\|.\|$ on a real or complex vector space X is a function, associating to each element x of X a corresponding real number $\|x\|$, such that the following conditions are satisfied:—

(i) $\|x\| \geq 0$ for all $x \in X$,

(ii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$,

(iii) $\|\lambda x\| = |\lambda| \, \|x\|$ for all $x \in X$ and for all scalars λ,

(iv) $\|x\| = 0$ if and only if $x = 0$.

A *normed vector space* $(X, \|.\|)$ consists of a a real or complex vector space X, together with a norm $\|.\|$ on X.

Note that any normed complex vector space can also be regarded as a normed real vector space.

Example The field $\mathbb{R}$ is a one-dimensional normed vector space over itself: the norm $|t|$ of $t \in \mathbb{R}$ is the absolute value of t.

Example The field $\mathbb{C}$ is a one-dimensional normed vector space over itself: the norm $|z|$ of $z \in \mathbb{C}$ is the modulus of z. The field $\mathbb{C}$ is also a two-dimensional normed vector space over $\mathbb{R}$.

Example Let $\|.\|_1$, $\|.\|_2$ and $\|.\|_\infty$ be the real-valued functions on $\mathbb{C}^n$ defined by

$$\begin{aligned}
\|\mathbf{z}\|_1 &= \sum_{j=1}^{n} |z_j|, \\
\|\mathbf{z}\|_2 &= \left(\sum_{j=1}^{n} |z_j|^2 \right)^{\frac{1}{2}}, \\
\|\mathbf{z}\|_\infty &= \max(|z_1|, |z_2|, \ldots, |z_n|),
\end{aligned}$$

for each $\mathbf{z} \in \mathbb{C}^n$, where $\mathbf{z} = (z_1, z_2, \ldots, z_n)$. Then $\|.\|_1$, $\|.\|_2$ and $\|.\|_\infty$ are norms on $\mathbb{C}^n$. In particular, if we regard $\mathbb{C}^n$ as a $2n$-dimensional real vector space naturally isomorphic to $\mathbb{R}^{2n}$ (via the isomorphism

$$(z_1, z_2, \ldots, z_n) \mapsto (x_1, y_1, x_2, y_2, \ldots, x_n, y_n),$$

where x_j and y_j are the real and imaginary parts of z_j for $j = 1, 2, \ldots, n$) then $\|.\|_2$ represents the Euclidean norm on this space. The inequality $\|\mathbf{z}+\mathbf{w}\|_2 \leq \|\mathbf{z}\|_2 + \|\mathbf{w}\|_2$ satisfied for all $\mathbf{z}, \mathbf{w} \in \mathbb{C}^n$ is therefore just the standard Triangle Inequality for the Euclidean norm.

Example The space $\mathbb{R}^n$ is also an n-dimensional real normed vector space with respect to the norms $\|.\|_1$, $\|.\|_2$ and $\|.\|_\infty$ defined above. Note that $\|.\|_2$ is the standard Euclidean norm on $\mathbb{R}^n$.

Example Let

$$\begin{aligned}
\ell_1 &= \{(z_1, z_2, z_3, \ldots) \in \mathbb{C}^\infty : |z_1| + |z_2| + |z_3| + \cdots \text{ converges}\}, \\
\ell_2 &= \{(z_1, z_2, z_3, \ldots) \in \mathbb{C}^\infty : |z_1|^2 + |z_2|^2 + |z_3|^2 + \cdots \text{ converges}\}, \\
\ell_\infty &= \{(z_1, z_2, z_3, \ldots) \in \mathbb{C}^\infty : \text{the sequence } |z_1|, |z_2|, |z_3|, \ldots \text{ is bounded}\}.
\end{aligned}$$

where $\mathbb{C}^\infty$ denotes the set of all sequences $(z_1, z_2, z_3, \ldots)$ of complex numbers. Then ℓ_1, ℓ_2 and ℓ_∞ are infinite-dimensional normed vector spaces, with norms $\|.\|_1$, $\|.\|_2$ and $\|.\|_\infty$ respectively, where

$$\begin{aligned}
\|\mathbf{z}\|_1 &= \sum_{j=1}^{+\infty} |z_j|, \\
\|\mathbf{z}\|_2 &= \left(\sum_{j=1}^{+\infty} |z_j|^2 \right)^{\frac{1}{2}}, \\
\|\mathbf{z}\|_\infty &= \sup\{|z_1|, |z_2|, |z_3|, \ldots\}.
\end{aligned}$$

(For example, to show that $\|\mathbf{z}+\mathbf{w}\|_2 \leq \|\mathbf{z}\|_2 + \|\mathbf{w}\|_2$ for all $\mathbf{z}, \mathbf{w} \in \ell_2$, we note that

$$\left(\sum_{j=1}^{n} |z_j + w_j|^2\right)^{\frac{1}{2}} \leq \left(\sum_{j=1}^{n} |z_j|^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n} |w_j|^2\right)^{\frac{1}{2}} \leq \|\mathbf{z}\|_2 + \|\mathbf{w}\|_2$$

for all natural numbers n, by the Triangle Inequality in $\mathbb{C}^n$. Taking limits as $n \to +\infty$, we deduce that $\|\mathbf{z}+\mathbf{w}\|_2 \leq \|\mathbf{z}\|_2 + \|\mathbf{w}\|_2$, as required.)

If $x_1, x_2, \ldots, x_m$ are elements of a normed vector space X then

$$\left\|\sum_{k=1}^{m} x_k\right\| \leq \sum_{k=1}^{m} \|x_k\|,$$

where $\|.\|$ denotes the norm on X. (This can be verified by induction on m, using the inequality $\|x+y\| \leq \|x\| + \|y\|$.)

A norm $\|.\|$ on a vector space X induces a corresponding distance function on X: the distance $d(x,y)$ between elements x and y of X is defined by $d(x,y) = \|x-y\|$. This distance function satisfies the metric space axioms. Thus any vector space with a given norm can be regarded as a metric space. A norm on a vector space X therefore generates a topology on X: a subset U of X is an open set if and only if, given any point u of U, there exists some $\delta > 0$ such that

$$\{x \in X : \|x-u\| < \delta\} \subset U.$$

The function $x \mapsto \|x\|$ is a continuous function from X to $\mathbb{R}$, since

$$\|x\| - \|y\| = \|(x-y)+y\| - \|y\| \leq (\|x-y\| + \|y\|) - \|y\| = \|x-y\|,$$

and $\|y\| - \|x\| \leq \|x-y\|$, and therefore $|\|x\| - \|y\|| \leq \|x-y\|$.

The Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ of vector spaces $X_1, X_2, \ldots, X_n$ can itself be regarded as a vector space: if $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are points of $X_1 \times X_2 \times \cdots \times X_n$, and if λ is any scalar, then

$$\begin{aligned} (x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n), \\ \lambda(x_1, x_2, \ldots, x_n) &= (\lambda x_1, \lambda x_2, \ldots, \lambda x_n). \end{aligned}$$

Lemma 9.1 *Let $X_1, X_2, \ldots, X_n$ be normed vector spaces, and let $\|.\|_{\max}$ be the norm on $X_1 \times X_2 \times \cdots \times X_n$ defined by*

$$\|(x_1, x_2, \ldots, x_n)\|_{\max} = \max(\|x_1\|_1, \|x_2\|_2, \ldots, \|x_n\|_n),$$

where $\|.\|_i$ is the norm on X_i for $i = 1, 2, \ldots, n$. Then the topology on $X_1 \times X_2 \times \cdots \times X_n$ generated by the norm $\|.\|_{\max}$ is the product topology on $X_1 \times X_2 \times \cdots \times X_n$.

Proof It is a straightforward exercise to verify that $\|.\|_{\max}$ is indeed a norm on X, where $X = X_1 \times X_2 \times \cdots \times X_n$.

Let U be a subset of X. Suppose that U is open with respect to the product topology. Let $\mathbf{u}$ be any point of U, given by $\mathbf{u} = (u_1, u_2, \ldots, u_n)$. We must show that there exists some $\delta > 0$ such that

$$\{\mathbf{x} \in X : \|\mathbf{x} - \mathbf{u}\|_{\max} < \delta\} \subset U.$$

Now it follows from the definition of the product topology that there exist open sets $V_1, V_2, \ldots, V_n$ in $X_1, X_2, \ldots, X_n$ such that $u_i \in V_i$ for all i and $V_1 \times V_2 \times \cdots \times V_n \subset U$. We can then take δ to be the minimum of $\delta_1, \delta_2, \ldots, \delta_n$, where $\delta_1, \delta_2, \ldots, \delta_n$ are chosen such that

$$\{x_i \in X_i : \|x_i - u_i\| < \delta_i\} \subset V_i$$

for $i = 1, 2, \ldots, n$.

Conversely suppose that U is open with respect to the topology generated by the norm $\|.\|_{\max}$. Let $\mathbf{u}$ be any point of U. Then there exists $\delta > 0$ such that

$$\{\mathbf{x} \in X : \|\mathbf{x} - \mathbf{u}\|_{\max} < \delta\} \subset U.$$

Let $V_i = \{x_i \in X_i : \|x_i - u_i\| < \delta\}$ for $i = 1, 2, \ldots, n$. Then, for each i, V_i is an open set in X_i, $u_i \in V_i$, and $V_1 \times V_2 \times \cdots \times V_n \subset U$. We deduce that U is also open with respect to the product topology, as required. ∎

Proposition 9.2 *Let X be a normed vector space over the field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then the function from $X \times X$ to X sending $(x, y) \in X \times X$ to $x + y$ is continuous. Also the function from $\mathbb{F} \times X$ to X sending $(\lambda, x) \in \mathbb{F} \times X$ to λx is continuous.*

Proof Let $(u, v) \in X \times X$, and let $\varepsilon > 0$ be given. Let $\delta = \frac{1}{2}\varepsilon$. If $(x, y) \in X \times X$ satisfies $\|(x, y) - (u, v)\|_{\max} < \delta$, then $\|x - u\| < \delta$ and $\|y - v\| < \delta$, and hence

$$\|(x + y) - (u + v)\| \leq \|x - u\| + \|y - v\| < \varepsilon.$$

This shows that the function $(x, y) \mapsto x + y$ is continuous at $(u, v) \in X \times X$.

Next let $(\mu, u) \in \mathbb{F} \times X$, and let $\varepsilon > 0$ be given. Let

$$\delta = \text{minimum}\left(\frac{\varepsilon}{2(\|u\| + 1)}, \frac{\varepsilon}{2(|\mu| + 1)}, 1\right).$$

Now $\lambda x - \mu u = \lambda(x - u) + (\lambda - \mu)u$ for all $\lambda \in \mathbb{F}$ and $x \in X$. Thus if $(\lambda, x) \in \mathbb{F} \times X$ satisfies $\|(\lambda, x) - (\mu, u)\|_{\max} < \delta$, then

$$|\lambda - \mu| < \frac{\varepsilon}{2(\|u\| + 1)}, \qquad \|x - u\| < \frac{\varepsilon}{2(|\mu| + 1)}, \qquad |\lambda| < |\mu| + 1,$$

and hence

$$\|\lambda x - \mu u\| \leq |\lambda| \, \|x - u\| + |\lambda - \mu| \, \|u\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that the function $(\lambda, x) \mapsto \lambda x$ is continuous at $(\mu, u) \in \mathbb{F} \times X$, as required. ∎

Corollary 9.3 *Let X be a normed vector space over the field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let (x_n) and (y_n) be convergent sequences in X, and let (λ_n) be a convergent sequence in $\mathbb{F}$. Then the sequences $(x_n + y_n)$ and $(\lambda_n x_n)$ are convergent in X, and*

$$\begin{aligned} \lim_{n \to +\infty} (x_n + y_n) &= \lim_{n \to +\infty} x_n + \lim_{n \to +\infty} y_n, \\ \lim_{n \to +\infty} (\lambda_n x_n) &= \left(\lim_{n \to +\infty} \lambda_n \right) \left(\lim_{n \to +\infty} x_n \right). \end{aligned}$$

Proof Let $x = \lim_{n \to +\infty} x_n$, $y = \lim_{n \to +\infty} y_n$ and $\lambda = \lim_{n \to +\infty} \lambda_n$. Using Lemma 9.1, together with the definition of convergence in metric spaces, it follows easily that the sequences (x_n, y_n) and (λ_n, x_n) converge to (x, y) and (λ, x) in $X \times X$ and $\mathbb{F} \times X$ respectively. The convergence of $(x_n + y_n)$ and $\lambda_n x_n$ to $x + y$ and λx respectively now follows from Proposition 9.2. ∎

Let X be a normed vector space, and let $x_1, x_2, x_3, \ldots$ be elements of X. The infinite series $\sum_{n=1}^{+\infty} x_n$ is said to *converge* to some element s of X if, given any $\varepsilon > 0$, there exists some natural number N such that

$$\|s - \sum_{n=1}^{m} x_n\| < \varepsilon$$

for all $m \geq N$ (where $\|.\|$ denotes the norm on X).

We say that a normed vector space X is *complete* if and only if every Cauchy sequence in X is convergent. (A sequence $x_1, x_2, x_3, \ldots$ is a *Cauchy sequence* if and only if, given any $\varepsilon > 0$, there exists some natural number N such that $\|x_j - x_k\| < \varepsilon$ for all j and k satisfying $j \geq N$ and $k \geq N$.) A complete normed vector space is referred to as a *Banach space.* (The basic theory of such spaces was extensively developed by the famous Polish mathematician Stefan Banach and his co-workers.)

Lemma 9.4 *Let X be a Banach space, and let $x_1, x_2, x_3, \ldots$ be elements of X. Suppose that $\sum_{n=1}^{+\infty} \|x_n\|$ is convergent. Then $\sum_{n=1}^{+\infty} x_n$ is convergent, and*

$$\left\| \sum_{n=1}^{+\infty} x_n \right\| \leq \sum_{n=1}^{+\infty} \|x_n\|.$$

Proof For each natural number n, let

$$s_n = x_1 + x_2 + \cdots + x_n.$$

Let $\varepsilon > 0$ be given. We can find N such that $\sum_{n=N}^{+\infty} \|x_n\| < \varepsilon$, since $\sum_{n=1}^{+\infty} \|x_n\|$ is convergent. Let $s_n = x_1 + x_2 + \cdots + x_n$. If $j \geq N$, $k \geq N$ and $j < k$ then

$$\|s_k - s_j\| = \left\| \sum_{n=j+1}^{k} x_n \right\| \leq \sum_{n=j+1}^{k} \|x_n\| \leq \sum_{n=N}^{+\infty} \|x_n\| < \varepsilon.$$

Thus $s_1, s_2, s_3, \ldots$ is a Cauchy sequence in X, and therefore converges to some element s of X, since X is complete. But then $s = \sum_{j=1}^{+\infty} x_j$. Moreover, on choosing m large enough to ensure that $\|s - s_m\| < \varepsilon$, we deduce that

$$\|s\| \leq \left\| \sum_{n=1}^{m} x_n \right\| + \left\| s - \sum_{n=1}^{m} x_n \right\| \leq \sum_{n=1}^{m} \|x_n\| + \left\| s - \sum_{n=1}^{m} x_n \right\| < \sum_{n=1}^{+\infty} \|x_n\| + \varepsilon.$$

Since this inequality holds for all $\varepsilon > 0$, we conclude that

$$\|s\| \leq \sum_{n=1}^{+\infty} \|x_n\|,$$

as required. ∎

9.1 Bounded Linear Transformations

Let X and Y be real or complex vector spaces. A function $T\colon X \to Y$ is said to be a *linear transformation* if $T(x + y) = Tx + Ty$ and $T(\lambda x) = \lambda Tx$ for all elements x and y of X and scalars λ. A linear transformation mapping X into itself is referred to as a *linear operator* on X.

Definition Let X and Y be normed vector spaces. A linear transformation $T\colon X \to Y$ is said to be *bounded* if there exists some non-negative real number C with the property that $\|Tx\| \leq C\|x\|$ for all $x \in X$. If T is bounded, then the smallest non-negative real number C with this property is referred to as the *operator norm* of T, and is denoted by $\|T\|$.

Lemma 9.5 *Let X and Y be normed vector spaces, and let $S\colon X \to Y$ and $T\colon X \to Y$ be bounded linear transformations. Then $S + T$ and λS are bounded linear transformations for all scalars λ, and*

$$\|S + T\| \leq \|S\| + \|T\|, \qquad \|\lambda S\| = |\lambda|\|S\|.$$

Moreover $\|S\| = 0$ if and only if $S = 0$. Thus the vector space $B(X, Y)$ of bounded linear transformations from X to Y is a normed vector space (with respect to the operator norm).

Proof $\|(S+T)x\| \leq \|Sx\|+\|Tx\| \leq (\|S\|+\|T\|)\|x\|$ for all $x \in X$. Therefore $S+T$ is bounded, and $\|S+T\| \leq \|S\|+\|T\|$. Using the fact that $\|(\lambda S)x\| = |\lambda|\,\|Sx\|$ for all $x \in X$, we see that λS is bounded, and $\|\lambda S\| = |\lambda|\,\|S\|$. If $S = 0$ then $\|S\| = 0$. Conversely if $\|S\| = 0$ then $\|Sx\| \leq \|S\|\,\|x\| = 0$ for all $x \in X$, and hence $S = 0$. The result follows. ∎

Lemma 9.6 *Let X, Y and Z be normed vector spaces, and let $S: X \to Y$ and $T: Y \to Z$ be bounded linear transformations. Then the composition TS of S and T is also bounded, and $\|TS\| \leq \|T\|\,\|S\|$.*

Proof $\|TSx\| \leq \|T\|\,\|Sx\| \leq \|T\|\,\|S\|\,\|x\|$ for all $x \in X$. The result follows. ∎

Proposition 9.7 *Let X and Y be normed vector spaces, and let $T: X \to Y$ be a linear transformation from X to Y. Then the following conditions are equivalent:—*

(i) *$T: X \to Y$ is continuous,*

(ii) *$T: X \to Y$ is continuous at 0,*

(iii) *$T: X \to Y$ is bounded.*

Proof Obviously (i) implies (ii). We show that (ii) implies (iii) and (iii) implies (i). The equivalence of the three conditions then follows immediately.

Suppose that $T: X \to Y$ is continuous at 0. Then there exists $\delta > 0$ such that $\|Tx\| < 1$ for all $x \in X$ satisfying $\|x\| < \delta$. Let C be any positive real number satisfying $C > 1/\delta$. If x is any non-zero element of X then $\|\lambda x\| < \delta$, where $\lambda = 1/(C\|x\|)$, and hence

$$\|Tx\| = C\|x\|\,\|\lambda Tx\| = C\|x\|\,\|T(\lambda x)\| < C\|x\|.$$

Thus $\|Tx\| \leq C\|x\|$ for all $x \in X$, and hence $T: X \to Y$ is bounded. Thus (ii) implies (iii).

Finally suppose that $T: X \to Y$ is bounded. Let x be a point of X, and let $\varepsilon > 0$ be given. Choose $\delta > 0$ satisfying $\|T\|\delta < \varepsilon$. If $x' \in X$ satisfies $\|x' - x\| < \delta$ then

$$\|Tx' - Tx\| = \|T(x' - x)\| \leq \|T\|\,\|x' - x\| < \|T\|\delta < \varepsilon.$$

Thus $T: X \to Y$ is continuous. Thus (iii) implies (i), as required. ∎

Proposition 9.8 *Let X be a normed vector space and let Y be a Banach space. Then the space $B(X,Y)$ of bounded linear transformations from X to Y is also a Banach space.*

Proof We have already shown that $B(X,Y)$ is a normed vector space (see Lemma 9.5). Thus it only remains to show that $B(X,Y)$ is complete.

Let $S_1, S_2, S_3, \ldots$ be a Cauchy sequence in $B(X,Y)$. Let $x \in X$. We claim that $S_1x, S_2x, S_3x, \ldots$ is a Cauchy sequence in Y. This result is trivial if $x = 0$. If $x \neq 0$, and if $\varepsilon > 0$ is given then there exists some natural number N such that $\|S_j - S_k\| < \varepsilon/\|x\|$ whenever $j \geq N$ and $k \geq N$. But then $\|S_jx - S_kx\| \leq \|S_j - S_k\|\,\|x\| < \varepsilon$ whenever $j \geq N$ and $k \geq N$. This shows that $S_1x, S_2x, S_3x, \ldots$ is indeed a Cauchy sequence. It therefore converges to some element of Y, since Y is a Banach space.

Let the function $S\colon X \to Y$ be defined by $Sx = \lim_{n\to+\infty} S_nx$. Then

$$S(x+y) = \lim_{n\to+\infty}(S_nx + S_ny) = \lim_{n\to+\infty} S_nx + \lim_{n\to+\infty} S_ny = Sx + Sy,$$

(see Corollary 9.3), and

$$S(\lambda x) = \lim_{n\to+\infty} S_n(\lambda x) = \lambda \lim_{n\to+\infty} S_nx = \lambda Sx,$$

Thus $S\colon X \to Y$ is a linear transformation.

Next we show that $S_n \to S$ in $B(X,Y)$ as $n \to +\infty$. Let $\varepsilon > 0$ be given. Then there exists some natural number N such that $\|S_j - S_n\| < \frac{1}{2}\varepsilon$ whenever $j \geq N$ and $n \geq N$, since the sequence $S_1, S_2, S_3, \ldots$ is a Cauchy sequence in $B(X,Y)$. But then $\|S_jx - S_nx\| \leq \frac{1}{2}\varepsilon\|x\|$ for all $j \geq N$ and $n \geq N$, and thus

$$\begin{aligned}
\|Sx - S_nx\| &= \left\|\lim_{j\to+\infty}(S_jx - S_nx)\right\| \leq \lim_{j\to+\infty}\|S_jx - S_nx\| \\
&\leq \lim_{j\to+\infty}\|S_j - S_n\|\,\|x\| \leq \tfrac{1}{2}\varepsilon\|x\|
\end{aligned}$$

for all $n \geq N$ (since the norm is a continuous function on Y). But then

$$\|Sx\| \leq \|S_nx\| + \|Sx - S_nx\| \leq \left(\|S_n\| + \tfrac{1}{2}\varepsilon\right)\|x\|$$

for any $n \geq N$, showing that $S\colon X \to Y$ is a bounded linear transformation, and $\|S - S_n\| \leq \frac{1}{2}\varepsilon < \varepsilon$ for all $n \geq N$, showing that $S_n \to S$ in $B(X,Y)$ as $n \to +\infty$. Thus the Cauchy sequence $S_1, S_2, S_3, \ldots$ is convergent in $B(X,Y)$, as required. ∎

Corollary 9.9 *Let X and Y be Banach spaces, and let $T_1, T_2, T_3, \ldots$ be bounded linear transformations from X to Y. Suppose that $\sum_{n=0}^{+\infty} \|T_n\|$ is convergent. Then $\sum_{n=0}^{+\infty} T_n$ is convergent, and*

$$\left\| \sum_{n=0}^{+\infty} T_n \right\| \leq \sum_{n=0}^{+\infty} \|T_n\|.$$

Proof The space $B(X, Y)$ of bounded linear maps from X to Y is a Banach space by Proposition 9.8. The result therefore follows immediately on applying Lemma 9.4. ∎

Example Let T be a bounded linear operator on a Banach space X (i.e., a bounded linear transformation from X to itself). The infinite series

$$\sum_{n=0}^{+\infty} \frac{\|T\|^n}{n!}$$

converges to $\exp(\|T\|)$. It follows immediately from Lemma 9.6 (using induction on n) that $\|T^n\| \leq \|T\|^n$ for all $n \geq 0$ (where T^0 is the identity operator on X). It therefore follows from Corollary 9.9 that there is a well-defined bounded linear operator $\exp T$ on X, defined by

$$\exp T = \sum_{n=0}^{+\infty} \frac{1}{n!} T^n$$

(where T^0 is the identity operator I on X).

Proposition 9.10 *Let T be a bounded linear operator on a Banach space X. Suppose that $\|T\| < 1$. Then the operator $I - T$ has a bounded inverse $(I - T)^{-1}$ (where I denotes the identity operator on X). Moreover*

$$(I - T)^{-1} = I + T + T^2 + T^3 + \cdots.$$

Proof $\|T^n\| \leq \|T\|^n$ for all n, and the geometric series

$$1 + \|T\| + \|T\|^2 + \|T\|^3 + \cdots$$

is convergent (since $\|T\| < 1$). It follows from Corollary 9.9 that the infinite series

$$I + T + T^2 + T^3 + \cdots$$

converges to some bounded linear operator S on X. Now

$$\begin{aligned}(I-T)S &= \lim_{n\to+\infty}(I-T)(I+T+T^2+\cdots+T^n) = \lim_{n\to+\infty}(I-T^{n+1}) \\ &= I - \lim_{n\to+\infty} T^{n+1} = I,\end{aligned}$$

since $\|T\|^{n+1} \to 0$ and therefore $T^{n+1} \to 0$ as $n \to +\infty$. Similarly $S(I-T) = I$. This shows that $I - T$ is invertible, with inverse S, as required. ∎

9.2 The Equivalence of Norms on a Finite-Dimensional Vector Space

Let $\|.\|$ and $\|.\|_*$ be norms on a real or complex vector space X. The norms $\|.\|$ and $\|.\|_*$ are said to be *equivalent* if and only if there exist constants c and C, where $0 < c \leq C$, such that

$$c\|x\| \leq \|x\|_* \leq C\|x\|$$

for all $x \in X$.

Lemma 9.11 *Two norms $\|.\|$ and $\|.\|_*$ on a real or complex vector space X are equivalent if and only if they induce the same topology on X.*

Proof Suppose that the norms $\|.\|$ and $\|.\|_*$ induce the same topology on X. Then there exists some $\delta > 0$ such that

$$\{x \in X : \|x\| < \delta\} \subset \{x \in X : \|x\|_* < 1\},$$

since the set $\{x \in X : \|x\|_* < 1\}$ is open with respect to the topology on X induced by both $\|.\|_*$ and $\|.\|$. Let C be any positive real number satisfying $C\delta > 1$. Then

$$\left\|\frac{1}{C\|x\|}x\right\| = \frac{1}{C} < \delta,$$

and hence

$$\|x\|_* = C\|x\| \left\|\frac{1}{C\|x\|}x\right\|_* < C\|x\|.$$

for all non-zero elements x of X, and thus $\|x\|_* \leq C\|x\|$ for all $x \in X$. On interchanging the roles of the two norms, we deduce also that there exists a positive real number c such that $\|x\| \leq (1/c)\|x\|_*$ for all $x \in X$. But then $c\|x\| \leq \|x\|_* \leq C\|x\|$ for all $x \in X$. We conclude that the norms $\|.\|$ and $\|.\|_*$ are equivalent.

Conversely suppose that the norms $\|.\|$ and $\|.\|_*$ are equivalent. Then there exist constants c and C, where $0 < c \leq C$, such that $c\|x\| \leq \|x\|_* \leq C\|x\|$ for all $x \in X$. Let U be a subset of X that is open with respect to the topology on X induced by the norm $\|.\|_*$, and let $u \in U$. Then there exists some $\delta > 0$ such that

$$\{x \in X : \|x - u\|_* < C\delta\} \subset U.$$

But then

$$\{x \in X : \|x - u\| < \delta\} \subset \{x \in X : \|x - u\|_* < C\delta\} \subset U,$$

showing that U is open with respect to the topology induced by the norm $\|.\|$. Similarly any subset of X that is open with respect to the topology induced by the norm $\|.\|$ must also be open with respect to the topology induced by $\|.\|_*$. Thus equivalent norms induce the same topology on X. ∎

It follows immediately from Lemma 9.11 that if $\|.\|$, $\|.\|_*$ and $\|.\|_\sharp$ are norms on a real (or complex) vector space X, if the norms $\|.\|$ and $\|.\|_*$ are equivalent, and if the norms $\|.\|_*$ and $\|.\|_\sharp$ are equivalent, then the norms $\|.\|$ and $\|.\|_\sharp$ are also equivalent. This fact can easily be verified directly from the definition of equivalence of norms.

We recall that the usual topology on $\mathbb{R}^n$ is that generated by the Euclidean norm on $\mathbb{R}^n$.

Lemma 9.12 *Let $\|.\|$ be a norm on $\mathbb{R}^n$. Then the function $\mathbf{x} \mapsto \|\mathbf{x}\|$ is continuous with respect to the usual topology on on $\mathbb{R}^n$.*

Proof Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ denote the basis of $\mathbb{R}^n$ given by

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \cdots, \quad \mathbf{e}_n = (0, 0, 0, \ldots, 1).$$

Let $\mathbf{x}$ and $\mathbf{y}$ be points of $\mathbb{R}^n$, given by

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \qquad \mathbf{y} = (y_1, y_2, \ldots, y_n).$$

Using Schwarz' Inequality, we see that

$$\begin{aligned} \|\mathbf{x} - \mathbf{y}\| &= \left\| \sum_{j=1}^n (x_j - y_j)\mathbf{e}_j \right\| \leq \sum_{j=1}^n |x_j - y_j| \, \|\mathbf{e}_j\| \\ &\leq \left(\sum_{j=1}^n (x_j - y_j)^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^n \|\mathbf{e}_j\|^2 \right)^{\frac{1}{2}} = C\|\mathbf{x} - \mathbf{y}\|_2, \end{aligned}$$

where

$$C^2 = \|\mathbf{e}_1\|^2 + \|\mathbf{e}_2\|^2 + \cdots + \|\mathbf{e}_n\|^2$$

and $\|\mathbf{x} - \mathbf{y}\|_2$ denotes the Euclidean norm of $\mathbf{x} - \mathbf{y}$, defined by

$$\|\mathbf{x} - \mathbf{y}\|_2 = \left(\sum_{j=1}^{n} (x_j - y_j)^2 \right)^{\frac{1}{2}}.$$

Also $|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq \|\mathbf{x} - \mathbf{y}\|$, since

$$\|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\|, \qquad \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{x}\|.$$

We conclude therefore that

$$|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq C\|\mathbf{x} - \mathbf{y}\|_2,$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and thus the function $\mathbf{x} \mapsto \|\mathbf{x}\|$ is continuous on $\mathbb{R}^n$ (with respect to the usual topology on $\mathbb{R}^n$). ∎

Theorem 9.13 *Any two norms on $\mathbb{R}^n$ are equivalent, and induce the usual topology on $\mathbb{R}^n$.*

Proof Let $\|.\|$ be any norm on $\mathbb{R}^n$. We show that $\|.\|$ is equivalent to the Euclidean norm $\|.\|_2$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, defined by

$$S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 = 1\}.$$

Now S^{n-1} is a compact subset of $\mathbb{R}^n$, since it is both closed and bounded. Also the function $\mathbf{x} \mapsto \|\mathbf{x}\|$ is continuous (Lemma 9.12). But any continuous real-valued function on a compact topological space attains both its maximum and minimum values on that space. Therefore there exist points $\mathbf{u}$ and $\mathbf{v}$ of S^{n-1} such that $\|\mathbf{u}\| \leq \|\mathbf{x}\| \leq \|\mathbf{v}\|$ for all $\mathbf{x} \in S^{n-1}$. Set $c = \|\mathbf{u}\|$ and $C = \|\mathbf{v}\|$. Then $0 < c \leq C$ (since it follows from the definition of norms that the norm of any non-zero element of $\mathbb{R}^n$ is necessarily non-zero).

If $\mathbf{x}$ is any non-zero element of $\mathbb{R}^n$ then $\lambda\mathbf{x} \in S^{n-1}$, where $\lambda = 1/\|\mathbf{x}\|_2$. But $\|\lambda\mathbf{x}\| = |\lambda|\,\|\mathbf{x}\|$ (see the the definition of norms). Therefore $c \leq |\lambda|\,\|\mathbf{x}\| \leq C$, and hence $c\|\mathbf{x}\|_2 \leq \|\mathbf{x}\| \leq C\|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{R}^n$, showing that the norm $\|.\|$ is equivalent to the Euclidean norm $\|.\|_2$ on $\mathbb{R}^n$. Therefore any two norms on $\mathbb{R}^n$ are equivalent, and thus generate the same topology on $\mathbb{R}^n$ (Lemma 9.11). This topology must then be the usual topology on $\mathbb{R}^n$. ∎

Let X be a finite-dimensional real vector space. Then X is isomorphic to $\mathbb{R}^n$, where n is the dimension of X. It follows immediately from Theorem 9.13 and Lemma 9.11 that all norms on X are equivalent and therefore generate the same topology on X. This result does not generalize to infinite-dimensional vector spaces.

10 Introduction to Functional Analysis

Let X be a topological space. We say that a function $f\colon X \to \mathbb{R}^n$ from X to $\mathbb{R}^n$ is *bounded* if there exists some non-negative constant K such that $|f(x)| \leq K$ for all $x \in X$. If f and g are bounded continuous functions from X to $\mathbb{R}^n$, then so is $f + g$. Also λf is bounded and continuous for any real number λ. It follows from this that the space $C(X, \mathbb{R}^n)$ of bounded continuous functions from X to $\mathbb{R}^n$ is a vector space over $\mathbb{R}$. Given $f \in C(X, \mathbb{R}^n)$, we define the *supremum norm* $\|f\|$ of f by the formula

$$\|f\| = \sup_{x \in X} |f(x)|.$$

One can readily verify that $\|.\|$ is a norm on the vector space $C(X, \mathbb{R}^n)$. We shall show that $C(X, \mathbb{R}^n)$, with the supremum norm, is a Banach space (i.e., the supremum norm on $C(X, \mathbb{R}^n)$ is complete). The proof of this result will make use of the following characterization of continuity for functions whose range is $\mathbb{R}^n$.

Lemma 10.1 *A function $f\colon X \to \mathbb{R}^n$ mapping a topological space X into $\mathbb{R}^n$ is continuous if and only if it satisfies the following criterion: given any point x of X and given any $\varepsilon > 0$, there exists some open set U_x in X such that $x \in U_x$ and $|f(u) - f(x)| < \varepsilon$ for all $u \in U_x$.*

Proof Suppose that $f\colon X \to \mathbb{R}^n$ is continuous. Let $x \in X$ and $\varepsilon > 0$ be given. Let

$$U_x = \{u \in X : |f(u) - f(x)| < \varepsilon\}.$$

Then U_x is open in X, since it is the preimage under f of the open ball of radius ε about $f(x)$ in $\mathbb{R}^n$. Thus U_x is the required open set.

Conversely suppose that $f\colon X \to \mathbb{R}^n$ is a function satisfying the given criterion. We must show that f is continuous. Let V be an open set in $\mathbb{R}^n$, and let $x \in f^{-1}(V)$. Then there exists some $\varepsilon > 0$ with the property that

$$\{\mathbf{y} \in \mathbb{R}^n : |\mathbf{y} - f(x)| < \varepsilon\} \subset V.$$

Now the criterion satisfied by f ensures the existence of some open set U_x in X such that $x \in U_x$ and $|f(u) - f(x)| < \varepsilon$ for all $u \in U_x$, and moreover the choice of ε ensures that $U_x \subset f^{-1}(V)$. Therefore the preimage $f^{-1}(V)$ of the open set V is the union of the open sets U_x as x ranges over all points of $f^{-1}(V)$, and is thus itself an open set. Thus $f\colon X \to \mathbb{R}^n$ is continuous, as required. ∎

Theorem 10.2 *The normed vector space $C(X, \mathbb{R}^n)$ of all bounded continuous functions from some topological space X to $\mathbb{R}^n$, with the supremum norm, is a Banach space.*

Proof Let $f_1, f_2, f_3, \ldots$ be a Cauchy sequence in $C(X, \mathbb{R}^n)$. Then, for each $x \in X$, the sequence $f_1(x), f_2(x), f_3(x), \ldots$ is a Cauchy sequence in $\mathbb{R}^n$ (since $|f_j(x) - f_k(x)| \leq \|f_j - f_k\|$ for all natural numbers j and k), and $\mathbb{R}^n$ is a complete metric space. Thus, for each $x \in X$, the sequence $f_1(x), f_2(x), f_3(x), \ldots$ converges to some point $f(x)$ of $\mathbb{R}^n$. We must show that the limit function f defined in this way is bounded and continuous.

Let $\varepsilon > 0$ be given. Then there exists some natural number N with the property that $\|f_j - f_k\| < \frac{1}{3}\varepsilon$ for all $j \geq N$ and $k \geq N$, since $f_1, f_2, f_3, \ldots$ is a Cauchy sequence in $C(X, \mathbb{R}^n)$. But then, on taking the limit of the left hand side of the inequality $|f_j(x) - f_k(x)| < \frac{1}{3}\varepsilon$ as $k \to +\infty$, we deduce that $|f_j(x) - f(x)| \leq \frac{1}{3}\varepsilon$ for all $x \in X$ and $j \geq N$. In particular $|f_N(x) - f(x)| \leq \frac{1}{3}\varepsilon$ for all $x \in X$. It follows that $|f(x)| \leq \|f_N\| + \frac{1}{3}\varepsilon$ for all $x \in X$, showing that the limit function f is bounded.

Next we show that the limit function f is continuous. Let $x \in X$ and $\varepsilon > 0$ be given. Let N be chosen large enough to ensure that $|f_N(u) - f(u)| \leq \frac{1}{3}\varepsilon$ for all $u \in X$. Now f_N is continuous. It follows from Lemma 10.1 that there exists some open set U_x in X such that $x \in U_x$ and $|f_N(u) - f_N(x)| < \frac{1}{3}\varepsilon$ for all $u \in U_x$. Thus if $u \in U_x$ then

$$\begin{aligned} |f(u) - f(x)| &\leq |f(u) - f_N(u)| + |f_N(u) - f_N(x)| + |f_N(x) - f(x)| \\ &< \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon. \end{aligned}$$

It follows from Lemma 10.1 that the limit function f is continuous. Thus $f \in C(X, \mathbb{R}^n)$.

Finally we observe that $f_j \to f$ in $C(X, \mathbb{R}^n$ as $j \to +\infty$. Indeed we have already seen that, given $\varepsilon > 0$ there exists some natural number N such that $|f_j(x) - f(x)| \leq \frac{1}{3}\varepsilon$ for all $x \in X$ and for all $j \geq N$. Thus $\|f_j - f\| \leq \frac{1}{3}\varepsilon < \varepsilon$ for all $j \geq N$, showing that $f_j \to f$ in $C(X, \mathbb{R}^n)$ as $j \to +\infty$. This shows that $C(X, \mathbb{R}^n)$ is a complete metric space, as required. ∎

Corollary 10.3 *Let X be a metric space and let F be a closed subset of $\mathbb{R}^n$. Then the space $C(X, F)$ of bounded continuous functions from X to F is a complete metric space with respect to the distance function ρ, where*

$$\rho(f, g) = \|f - g\| = \sup_{x \in X} |f(x) - g(x)|$$

for all $f, g \in C(X, F)$.

Proof Let $f_1, f_2, f_3, \ldots$ be a Cauchy sequence in $C(X, F)$. Then $f_1, f_2, f_3, \ldots$ is a Cauchy sequence in $C(X, \mathbb{R}^n)$ and therefore converges in $C(X, \mathbb{R}^n)$ to some function $f\colon X \to \mathbb{R}^n$. Let x be some point of X. Then $f_j(x) \to f(x)$ as $j \to +\infty$. But then $f(x) \in F$, since $f_j(x) \in F$ for all j, and F is closed in $\mathbb{R}^n$. This shows that $f \in C(X, F)$, and thus the Cauchy sequence $f_1, f_2, f_3, \ldots$ converges in $C(X, F)$. We conclude that $C(X, F)$ is a complete metric space, as required. ∎

10.1 The Contraction Mapping Theorem and Picard's Theorem

Let X be a metric space with distance function d. A function $T\colon X \to X$ mapping X to itself is said to be a *contraction mapping* if there exists some constant λ satisfying $0 \leq \lambda < 1$ with the property that $d(T(x), T(x')) \leq \lambda d(x, x')$ for all $x, x' \in X$.

One can readily check that any contraction map $T\colon X \to X$ on a metric space (X, d) is continuous. Indeed let x be a point of X, and let $\varepsilon > 0$ be given. Then $d(T(x), T(x')) < \varepsilon$ for all points x' of X satisfying $d(x, x') < \varepsilon$.

Theorem 10.4 (Contraction Mapping Theorem) *Let X be a complete metric space, and let $T\colon X \to X$ be a contraction mapping defined on X. Then T has a unique fixed point in X (i.e., there exists a unique point x of X for which $T(x) = x$).*

Proof Let λ be chosen such that $0 \leq \lambda < 1$ and $d(T(u), T(u')) \leq \lambda d(u, u')$ for all $u, u' \in X$, where d is the distance function on X. First we show the existence of the fixed point x. Let x_0 be any point of X, and define a sequence $x_0, x_1, x_2, x_3, x_4, \ldots$ of points of X by the condition that $x_n = T(x_{n-1})$ for all natural numbers n. It follows by induction on n that $d(x_{n+1}, x_n) \leq \lambda^n d(x_1, x_0)$. Using the Triangle Inequality, we deduce that if j and k are natural numbers satisfying $k > j$ then

$$d(x_k, x_j) \leq \sum_{n=j}^{k-1} d(x_{n+1}, x_n) \leq \frac{\lambda^j - \lambda^k}{1 - \lambda} d(x_1, x_0) \leq \frac{\lambda^j}{1 - \lambda} d(x_1, x_0).$$

(Here we have used the identity

$$\lambda^j + \lambda^{j+1} + \cdots + \lambda^{k-1} = \frac{\lambda^j - \lambda^k}{1 - \lambda}.)$$

Using the fact that $0 \leq \lambda < 1$, we deduce that the sequence (x_n) is a Cauchy sequence in X. This Cauchy sequence must converge to some point x of X, since X is complete. But then we see that

$$T(x) = T\left(\lim_{n\to+\infty} x_n\right) = \lim_{n\to+\infty} T(x_n) = \lim_{n\to+\infty} x_{n+1} = x,$$

since $T\colon X \to X$ is a continuous function, and thus x is a fixed point of T.

If x' were another fixed point of T then we would have

$$d(x', x) = d(T(x'), T(x)) \leq \lambda d(x', x).$$

But this is impossible unless $x' = x$, since $\lambda < 1$. Thus the fixed point x of the contraction map T is unique. ∎

We use the Contraction Mapping Theorem in order to prove the following existence theorem for solutions of ordinary differential equations.

Theorem 10.5 (Picard's Theorem) *Let $F\colon U \to \mathbb{R}$ be a continuous function defined over some open set U in the plane $\mathbb{R}^2$, and let (x_0, t_0) be an element of U. Suppose that there exists some non-negative constant M such that*

$$|F(u,t) - F(v,t)| \leq M|u - v| \text{ for all } (u,t) \in U \text{ and } (v,t) \in U.$$

Then there exists a continuous function $\varphi\colon [t_0 - \delta, t_0 + \delta] \to \mathbb{R}$ defined on the interval $[t_0 - \delta, t_0 + \delta]$ for some $\delta > 0$ such that $x = \varphi(t)$ is a solution to the differential equation

$$\frac{dx(t)}{dt} = F(x(t), t)$$

with initial condition $x(t_0) = x_0$.

Proof Solving the differential equation with the initial condition $x(t_0) = x_0$ is equivalent to finding a continuous function $\varphi\colon I \to \mathbb{R}$ satisfying the integral equation

$$\varphi(t) = x_0 + \int_{t_0}^{t} F(\varphi(s), s)\, ds.$$

where I denotes the closed interval $[t_0 - \delta, t_0 + \delta]$. (Note that any continuous function φ satisfying this integral equation is automatically differentiable, since the indefinite integral of a continuous function is always differentiable.)

Let $K = |F(x_0, t_0)| + 1$. Using the continuity of the function F, together with the fact that U is open in $\mathbb{R}^2$, one can find some $\delta_0 > 0$ such that the open disk of radius δ_0 about (x_0, t_0) is contained in U and $|F(x,t)| \leq K$ for all points (x,t) in this open disk. Now choose $\delta > 0$ such that

$$\delta\sqrt{1 + K^2} < \delta_0 \text{ and } M\delta < 1.$$

Note that if $|t - t_0| \leq \delta$ and $|x - x_0| \leq K\delta$ then (x, t) belongs to the open disk of radius δ_0 about (x_0, t_0), and hence $(x, t) \in U$ and $|F(x, t)| \leq K$.

Let J denote the closed interval $[x_0 - K\delta, x_0 + K\delta]$. The space $C(I, J)$ of continuous functions from the interval I to the interval J is a complete metric space, by Corollary 10.3. Define $T: C(I, J) \to C(I, J)$ by

$$T(\varphi)(t) = x_0 + \int_{t_0}^{t} F(\varphi(s), s)\, ds.$$

We claim that T does indeed map $C(I, J)$ into itself and is a contraction mapping.

Let $\varphi: I \to J$ be an element of $C(I, J)$. Note that if $|t - t_0| \leq \delta$ then

$$|(\varphi(t), t) - (x_0, t_0)|^2 = (\varphi(t) - x_0)^2 + (t - t_0)^2 \leq \delta^2 + K^2\delta^2 < \delta_0^2,$$

hence $|F(\varphi(t), t)| \leq K$. It follows from this that

$$|T(\varphi)(t) - x_0| \leq K\delta$$

for all t satisfying $|t - t_0| < \delta$. The function $T(\varphi)$ is continuous, and is therefore a well-defined element of $C(I, J)$ for all $\varphi \in C(I, J)$.

We now show that T is a contraction mapping on $C(I, J)$. Let φ and ψ be elements of $C(I, J)$. The hypotheses of the theorem ensure that

$$|F(\varphi(t), t) - F(\psi(t), t)| \leq M|\varphi(t) - \psi(t)| \leq M\rho(\varphi, \psi)$$

for all $t \in I$, where $\rho(\varphi, \psi) = \sup_{t \in I} |\varphi(t) - \psi(t)|$. Therefore

$$\begin{aligned} |T(\varphi)(t) - T(\psi)(t)| &= \left| \int_{t_0}^{t} (F(\varphi(s), s) - F(\psi(s), s))\, ds \right| \\ &\leq M|t - t_0|\rho(\varphi, \psi) \end{aligned}$$

for all t satisfying $|t - t_0| \leq \delta$. Therefore $\rho(T(\varphi), T(\psi)) \leq M\delta\rho(\varphi, \psi)$ for all $\varphi, \psi \in C(I, J)$. But δ has been chosen such that $M\delta < 1$. This shows that $T: C(I, J) \to C(I, J)$ is a contraction mapping on $C(I, J)$. It follows from the Contraction Mapping Theorem (Theorem 10.4) that there exists a unique element φ of $C(I, J)$ satisfying $T(\varphi) = \varphi$. This function φ is the required solution to the differential equation. ∎

A straightforward but somewhat technical least upper bound argument can be used to show that if $x = \psi(t)$ is any other continuous solution to the differential equation

$$\frac{dx}{dt} = F(x, t)$$

on the interval $[t_0 - \delta, t_0 + \delta]$ satisfying the initial condition $\psi(t_0) = x_0$, then $|\psi(t) - x_0| \leq K\delta$ for all t satisfying $|t - t_0| \leq \delta$. Thus such a solution to the differential equation must belong to the space $C(I, J)$ defined in the proof of Theorem 10.5. The uniqueness of the fixed point of the contraction mapping $T \colon C(I, J) \to C(I, J)$ then shows that $\psi = \varphi$, where $\varphi \colon [t_0 - \delta, t_0 + \delta] \to \mathbb{R}$ is the solution to the differential equation whose existence was proved in Theorem 10.5. This shows that the solution to the differential equation is in fact unique on the interval $[t_0 - \delta, t_0 + \delta]$.