Course 212: Hilary Term 2001
Part II: Topological Spaces

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Contents

4 Topological Spaces

The theory of topological spaces provides a setting for the notions of continuity and convergence which is more general than that provided by the theory of metric spaces. In the theory of metric spaces one can find necessary and sufficient conditions for convergence and continuity that do not refer explicitly to the distance function on a metric space but instead are expressed in terms of open sets. Thus a sequence of points in a metric space X converges to a point p of X if and only if every open set which contains the point p also contains all but finitely many members of the sequence. Also a function $f\colon X \to Y$ between metric spaces X and Y is continuous if and only if the preimage $f^{-1}(V)$ of every open set V in Y is an open set in X. It follows from this that we can generalize the notions of convergence and continuity by introducing the concept of a *topological space*: a topological space consists of a set together with a collection of subsets termed *open sets* that satisfy appropriate axioms. The axioms for open sets in a topological space are satisfied by the open sets in any metric space.

4.1 Topological Spaces: Definitions and Examples

Definition A *topological space* X consists of a set X together with a collection of subsets, referred to as *open sets*, such that the following conditions are satisfied:—

(i) the empty set $\emptyset$ and the whole set X are open sets,

(ii) the union of any collection of open sets is itself an open set,

(iii) the intersection of any *finite* collection of open sets is itself an open set.

The collection consisting of all the open sets in a topological space X is referred to as a *topology* on the set X.

Remark If it is necessary to specify explicitly the topology on a topological space then one denotes by (X, τ) the topological space whose underlying set is X and whose topology is τ. However if no confusion will arise then it is customary to denote this topological space simply by X.

Any metric space may be regarded as a topological space. Indeed let X be a metric space with distance function d. We recall that a subset V of X is an *open set* if and only if, given any point v of V, there exists some $\delta > 0$ such that $\{x \in X : d(x, v) < \delta\} \subset V$. The empty set $\emptyset$ and the whole space X are open sets. Also any union of open sets in a metric space is an open set, and any finite intersection of open sets in a metric space is an open set. Thus the topological space axioms are satisfied by the collection of open sets in any metric space. We refer to this collection of open sets as the *topology* generated by the distance function d on X.

Any subset X of n-dimensional Euclidean space $\mathbb{R}^n$ is a topological space: a subset V of X is open in X if and only if, given any point $\mathbf{v}$ of V, there exists some $\delta > 0$ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{v}| < \delta\} \subset V.$$

In particular $\mathbb{R}^n$ is itself a topological space whose topology is generated by the Euclidean distance function on $\mathbb{R}^n$. This topology on $\mathbb{R}^n$ is referred to as the *usual topology* on $\mathbb{R}^n$. One defines the usual topologies on $\mathbb{R}$ and $\mathbb{C}$ in an analogous fashion.

Example Given any set X, one can define a topology on X where every subset of X is an open set. This topology is referred to as the *discrete topology* on X.

Example Given any set X, one can define a topology on X in which the only open sets are the empty set $\emptyset$ and the whole set X.

Definition Let X be a topological space. A subset F of X is said to be a *closed set* if and only if its complement $X \setminus F$ is an open set.

We recall that the complement of the union of some collection of subsets of some set X is the intersection of the complements of those sets, and the complement of the intersection of some collection of subsets of X is the union of the complements of those sets. The following result therefore follows directly from the definition of a topological space.

Proposition 4.1 *Let X be a topological space. Then the collection of closed sets of X has the following properties:—*

(i) *the empty set $\emptyset$ and the whole set X are closed sets,*

(ii) *the intersection of any collection of closed sets is itself a closed set,*

(iii) *the union of any finite collection of closed sets is itself a closed set.*

4.2 Hausdorff Spaces

Definition A topological space X is said to be a *Hausdorff space* if and only if it satisfies the following *Hausdorff Axiom*:

- if x and y are distinct points of X then there exist open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Lemma 4.2 *All metric spaces are Hausdorff spaces.*

Proof Let X be a metric space with distance function d, and let x and y be points of X, where $x \neq y$. Let $\varepsilon = \frac{1}{2} d(x, y)$. Then the open balls $B_X(x, \varepsilon)$ and $B_X(y, \varepsilon)$ of radius ε centred on the points x and y are open sets. If $B_X(x, \varepsilon) \cap B_X(y, \varepsilon)$ were non-empty then there would exist $z \in X$ satisfying $d(x, z) < \varepsilon$ and $d(z, y) < \varepsilon$. But this is impossible, since it would then follow from the Triangle Inequality that $d(x, y) < 2\varepsilon$, contrary to the choice of ε. Thus $x \in B_X(x, \varepsilon)$, $y \in B_X(y, \varepsilon)$, $B_X(x, \varepsilon) \cap B_X(y, \varepsilon) = \emptyset$. This shows that the metric space X is a Hausdorff space. ∎

We now give an example of a topological space which is not a Hausdorff space.

Example The *Zariski topology* on the set $\mathbb{R}$ of real numbers is defined as follows: a subset U of $\mathbb{R}$ is open (with respect to the Zariski topology) if and only if either $U = \emptyset$ or else $\mathbb{R} \setminus U$ is finite. It is a straightforward exercise to verify that the topological space axioms are satisfied, so that the set $\mathbb{R}$ of real numbers is a topological space with respect to this Zariski topology. Now the intersection of any two non-empty open sets in this topology is always non-empty. (Indeed if U and V are non-empty open sets then $U = \mathbb{R} \setminus F_1$ and $V = \mathbb{R} \setminus F_2$, where F_1 and F_2 are finite sets of real numbers. But then $U \cap V = \mathbb{R} \setminus (F_1 \cup F_2)$, which is non-empty, since $F_1 \cup F_2$ is finite and $\mathbb{R}$ is infinite.) It follows immediately from this that $\mathbb{R}$, with the Zariski topology, is not a Hausdorff space.

4.3 Subspace Topologies

Let X be a topological space with topology τ, and let A be a subset of X. Let τ_A be the collection of all subsets of A that are of the form $V \cap A$ for $V \in \tau$. Then τ_A is a topology on the set A. (It is a straightforward exercise to verify that the topological space axioms are satisfied.) The topology τ_A on A is referred to as the *subspace topology* on A.

Any subset of a Hausdorff space is itself a Hausdorff space (with respect to the subspace topology).

Lemma 4.3 *Let X be a metric space with distance function d, and let A be a subset of X. A subset W of A is open with respect to the subspace topology on A if and only if, given any point w of W, there exists some $\delta > 0$ such that*

$$\{a \in A : d(a, w) < \delta\} \subset W.$$

Thus the subspace topology on A coincides with the topology on A obtained on regarding A as a metric space (with respect to the distance function d).

Proof Suppose that W is open with respect to the subspace topology on A. Then there exists some open set U in X such that $W = U \cap A$. Let w be a point of W. Then there exists some $\delta > 0$ such that

$$\{x \in X : d(x, w) < \delta\} \subset U.$$

But then

$$\{a \in A : d(a, w) < \delta\} \subset U \cap A = W.$$

Conversely, suppose that W is a subset of A with the property that, for any $w \in W$, there exists some $\delta_w > 0$ such that

$$\{a \in A : d(a, w) < \delta_w\} \subset W.$$

Define U to be the union of the open balls $B_X(w, \delta_w)$ as w ranges over all points of W, where

$$B_X(w, \delta_w) = \{x \in X : d(x, w) < \delta_w\}.$$

The set U is an open set in X, since each open ball $B_X(w, \delta_w)$ is an open set in X, and any union of open sets is itself an open set. Moreover

$$B_X(w, \delta_w) \cap A = \{a \in A : d(a, w) < \delta_w\} \subset W$$

for any $w \in W$. Therefore $U \cap A \subset W$. However $W \subset U \cap A$, since, $W \subset A$ and $\{w\} \subset B_X(w, \delta_w) \subset U$ for any $w \in W$. Thus $W = U \cap A$, where U is an open set in X. We deduce that W is open with respect to the subspace topology on A. ∎

Example Let X be any subset of n-dimensional Euclidean space $\mathbb{R}^n$. Then the subspace topology on X coincides with the topology on X generated by the Euclidean distance function on X. We refer to this topology as the *usual topology* on X.

Let X be a topological space, and let A be a subset of X. One can readily verify the following:—

- a subset B of A is closed in A (relative to the subspace topology on A) if and only if $B = A \cap F$ for some closed subset F of X;
- if A is itself open in X then a subset B of A is open in A if and only if it is open in X;
- if A is itself closed in X then a subset B of A is closed in A if and only if it is closed in X.

4.4 Continuous Functions between Topological Spaces

Definition A function $f: X \to Y$ from a topological space X to a topological space Y is said to be *continuous* if $f^{-1}(V)$ is an open set in X for every open set V in Y, where

$$f^{-1}(V) \equiv \{x \in X : f(x) \in V\}.$$

A continuous function from X to Y is often referred to as a *map* from X to Y.

Lemma 4.4 *Let X, Y and Z be topological spaces, and let $f: X \to Y$ and $g: Y \to Z$ be continuous functions. Then the composition $g \circ f: X \to Z$ of the functions f and g is continuous.*

Proof Let V be an open set in Z. Then $g^{-1}(V)$ is open in Y (since g is continuous), and hence $f^{-1}(g^{-1}(V))$ is open in X (since f is continuous). But $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$. Thus the composition function $g \circ f$ is continuous. ∎

Lemma 4.5 *Let X and Y be topological spaces, and let $f: X \to Y$ be a function from X to Y. The function f is continuous if and only if $f^{-1}(G)$ is closed in X for every closed subset G of Y.*

Proof If G is any subset of Y then $X \setminus f^{-1}(G) = f^{-1}(Y \setminus G)$ (i.e., the complement of the preimage of G is the preimage of the complement of G). The result therefore follows immediately from the definitions of continuity and closed sets. ∎

We now show that, if a topological space X is the union of a finite collection of closed sets, and if a function from X to some topological space is continuous on each of these closed sets, then that function is continuous on X.

Lemma 4.6 *Let X and Y be topological spaces, let $f\colon X \to Y$ be a function from X to Y, and let $X = A_1 \cup A_2 \cup \cdots \cup A_k$, where $A_1, A_2, \ldots, A_k$ are closed sets in X. Suppose that the restriction of f to the closed set A_i is continuous for $i = 1, 2, \ldots, k$. Then $f\colon X \to Y$ is continuous.*

Proof Let V be an open set in Y. We must show that $f^{-1}(V)$ is open in X. Now the preimage of the open set V under the restriction $f|A_i$ of f to A_i is $f^{-1}(V) \cap A_i$. It follows from the continuity of $f|A_i$ that $f^{-1}(V) \cap A_i$ is relatively open in A_i for each i, and hence there exist open sets $U_1, U_2, \ldots, U_k$ in X such that $f^{-1}(V) \cap A_i = U_i \cap A_i$ for $i = 1, 2, \ldots, k$. Let $W_i = U_i \cup (X \setminus A_i)$ for $i = 1, 2, \ldots, k$. Then W_i is an open set in X (as it is the union of the open sets U_i and $X \setminus A_i$), and $W_i \cap A_i = U_i \cap A_i = f^{-1}(V) \cap A_i$ for each i. We claim that $f^{-1}(V) = W_1 \cap W_2 \cap \cdots \cap W_k$.

Let $W = W_1 \cap W_2 \cap \cdots \cap W_k$. Then $f^{-1}(V) \subset W$, since $f^{-1}(V) \subset W_i$ for each i. Also

$$W = \bigcup_{i=1}^{k} (W \cap A_i) \subset \bigcup_{i=1}^{k} (W_i \cap A_i) = \bigcup_{i=1}^{k} (f^{-1}(V) \cap A_i) \subset f^{-1}(V),$$

since $X = A_1 \cup A_2 \cup \cdots \cup A_k$ and $W_i \cap A_i = f^{-1}(V) \cap A_i$ for each i. Therefore $f^{-1}(V) = W$. But W is open in X, since it is the intersection of a finite collection of open sets. We have thus shown that $f^{-1}(V)$ is open in X for any open set V in Y. Thus $f\colon X \to Y$ is continuous, as required. ∎

Alternative Proof A function $f\colon X \to Y$ is continuous if and only if $f^{-1}(G)$ is closed in X for every closed set G in Y (Lemma 4.5). Let G be an closed set in Y. Then $f^{-1}(G) \cap A_i$ is relatively closed in A_i for $i = 1, 2, \ldots, k$, since the restriction of f to A_i is continuous for each i. But A_i is closed in X, and therefore a subset of A_i is relatively closed in A_i if and only if it is closed in X. Therefore $f^{-1}(G) \cap A_i$ is closed in X for $i = 1, 2, \ldots, k$. Now $f^{-1}(G)$ is the union of the sets $f^{-1}(G) \cap A_i$ for $i = 1, 2, \ldots, k$. It follows that $f^{-1}(G)$, being a finite union of closed sets, is itself closed in X. It now follows from Lemma 4.5 that $f\colon X \to Y$ is continuous. ∎

Example Let Y be a topological space, and let $\alpha\colon [0, 1] \to Y$ and $\beta\colon [0, 1] \to Y$ be continuous functions defined on the interval $[0, 1]$, where $\alpha(1) = \beta(0)$. Let $\gamma\colon [0, 1] \to Y$ be defined by

$$\gamma(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2}; \\ \beta(2t - 1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Now $\gamma|[0, \frac{1}{2}] = \alpha \circ \rho$ where $\rho\colon [0, \frac{1}{2}] \to [0, 1]$ is the continuous function defined by $\rho(t) = 2t$ for all $t \in [0, \frac{1}{2}]$. Thus $\gamma|[0, \frac{1}{2}]$ is continuous, being a composition of two continuous functions. Similarly $\gamma|[\frac{1}{2}, 1]$ is continuous. The subintervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ are closed in $[0, 1]$, and $[0, 1]$ is the union of these two subintervals. It follows from Lemma 4.6 that $\gamma\colon [0, 1] \to Y$ is continuous.

4.5 Homeomorphisms

Definition Let X and Y be topological spaces. A function $h\colon X \to Y$ is said to be a *homeomorphism* if and only if the following conditions are satisfied:

- the function $h\colon X \to Y$ is both injective and surjective (so that the function $h\colon X \to Y$ has a well-defined inverse $h^{-1}\colon Y \to X$),
- the function $h\colon X \to Y$ and its inverse $h^{-1}\colon Y \to X$ are both continuous.

Two topological spaces X and Y are said to be *homeomorphic* if there exists a homeomorphism $h\colon X \to Y$ from X to Y.

If $h\colon X \to Y$ is a homeomorphism between topological spaces X and Y then h induces a one-to-one correspondence between the open sets of X and the open sets of Y. Thus the topological spaces X and Y can be regarded as being identical as topological spaces.

4.6 Sequences and Convergence

Definition A sequence $x_1, x_2, x_3, \ldots$ of points in a topological space X is said to *converge* to a point p of X if, given any open set U containing the point p, there exists some natural number N such that $x_j \in U$ for all $j \geq N$. If the sequence (x_j) converges to p then we refer to p as a *limit* of the sequence.

This definition of convergence generalizes the definition of convergence for a sequence of points in a metric space.

It can happen that a sequence of points in a topological space can have more than one limit. For example, consider the set $\mathbb{R}$ of real numbers with the Zariski topology. (The open sets of $\mathbb{R}$ in the Zariski topology are the empty set and those subsets of $\mathbb{R}$ whose complements are finite.) Let $x_1, x_2, x_3, \ldots$ be the sequence in $\mathbb{R}$ defined by $x_j = j$ for all natural numbers j. One can readily check that this sequence converges to every real number p (with respect to the Zariski topology on $\mathbb{R}$).

Lemma 4.7 *A sequence $x_1, x_2, x_3, \ldots$ of points in a Hausdorff space X converges to at most one limit.*

Proof Suppose that p and q were limits of the sequence (x_j), where $p \neq q$. Then there would exist open sets U and V such that $p \in U$, $q \in V$ and $U \cap V = \emptyset$, since X is a Hausdorff space. But then there would exist natural numbers N_1 and N_2 such that $x_j \in U$ for all j satisfying $j \geq N_1$ and $x_j \in V$ for all j satisfying $j \geq N_2$. But then $x_j \in U \cap V$ for all j satisfying $j \geq N_1$ and $j \geq N_2$, which is impossible, since $U \cap V = \emptyset$. This contradiction shows that the sequence (x_j) has at most one limit. ∎

Lemma 4.8 *Let X be a topological space, and let F be a closed set in X. Let $(x_j : j \in \mathbb{N})$ be a sequence of points in F. Suppose that the sequence (x_j) converges to some point p of X. Then $p \in F$.*

Proof Suppose that p were a point belonging to the complement $X \setminus F$ of F. Now $X \setminus F$ is open (since F is closed). Therefore there would exist some natural number N such that $x_j \in X \setminus F$ for all values of j satisfying $j \geq N$, contradicting the fact that $x_j \in F$ for all j. This contradiction shows that p must belong to F, as required. ∎

Lemma 4.9 *Let $f: X \to Y$ be a continuous function between topological spaces X and Y, and let $x_1, x_2, x_3, \ldots$ be a sequence of points in X which converges to some point p of X. Then the sequence $f(x_1), f(x_2), f(x_3), \ldots$ converges to $f(p)$.*

Proof Let V be an open set in Y which contains the point $f(p)$. Then $f^{-1}(V)$ is an open set in X which contains the point p. It follows that there exists some natural number N such that $x_j \in f^{-1}(V)$ whenever $j \geq N$. But then $f(x_j) \in V$ whenever $j \geq N$. We deduce that the sequence $f(x_1), f(x_2), f(x_3), \ldots$ converges to $f(p)$, as required. ∎

4.7 Neighbourhoods, Closures and Interiors

Definition Let X be a topological space, and let x be a point of X. Let N be a subset of X which contains the point x. Then N is said to be a *neighbourhood* of the point x if and only if there exists an open set U for which $x \in U$ and $U \subset N$.

One can readily verify that this definition of neighbourhoods in topological spaces is consistent with that for neighbourhoods in metric spaces.

Lemma 4.10 *Let X be a topological space. A subset V of X is open in X if and only if V is a neighbourhood of each point belonging to V.*

Proof It follows directly from the definition of neighbourhoods that an open set V is a neighbourhood of any point belonging to V. Conversely, suppose that V is a subset of X which is a neighbourhood of each $v \in V$. Then, given any point v of V, there exists an open set U_v such that $v \in U_v$ and $U_v \subset V$. Thus V is an open set, since it is the union of the open sets U_v as v ranges over all points of V. ∎

Definition Let X be a topological space and let A be a subset of X. The *closure* $\overline{A}$ of A in X is defined to be the intersection of all of the closed subsets of X that contain A. The *interior* A^0 of A in X is defined to be the union of all of the open subsets of X that are contained in A.

Let X be a topological space and let A be a subset of X. It follows directly from the definition of $\overline{A}$ that the closure $\overline{A}$ of A is uniquely characterized by the following two properties:

(i) the closure $\overline{A}$ of A is a closed set containing A,

(ii) if F is any closed set containing A then F contains $\overline{A}$.

Similarly the interior A^0 of A is uniquely characterized by the following two properties:

(i) the interior A^0 of A is an open set contained in A,

(ii) if U is any open set contained in A then U is contained in A^0.

Moreover a point x of A belongs to the interior A^0 of A if and only if A is a neighbourhood of x.

Lemma 4.11 *Let X be a topological space, and let A be a subset of X. Suppose that a sequence $x_1, x_2, x_3, \ldots$ of points of A converges to some point p of X. Then p belongs to the closure $\overline{A}$ of A.*

Proof If F is any closed set containing A then $x_j \in F$ for all j, and therefore $p \in F$, by Lemma 4.8. Therefore $p \in \overline{A}$ by definition of $\overline{A}$. ∎

5 Product Topologies

5.1 The Cartesian Product of Subsets of Euclidean Space

Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. We can regard the Cartesian product $X \times Y$ of X and Y as a subset of $\mathbb{R}^{m+n}$. If $\mathbf{x}$ and $\mathbf{y}$ are points of X and Y respectively, with

$$\mathbf{x} = (x_1, x_2, \ldots, x_m), \quad \mathbf{y} = (y_1, y_2, \ldots, y_n),$$

then $(\mathbf{x}, \mathbf{y})$ is that point of $\mathbb{R}^{m+n}$ with Cartesian coordinates given by

$$(\mathbf{x}, \mathbf{y}) = (x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n).$$

It follows immediately from the definition of the Euclidean distance function that

$$|(\mathbf{x}, \mathbf{y}) - (\mathbf{v}, \mathbf{w})|^2 = |\mathbf{x} - \mathbf{v}|^2 + |\mathbf{y} - \mathbf{w}|^2.$$

for all points $\mathbf{x}$ and $\mathbf{v}$ of X and points $\mathbf{y}$ and $\mathbf{w}$ of Y.

Lemma 5.1 *Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. A subset U of $X \times Y$ is open in $X \times Y$ if and only if, given any point $(\mathbf{v}, \mathbf{w})$ of U, there exist positive real numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that*

$$\{(\mathbf{x}, \mathbf{y}) \in X \times Y : |\mathbf{x} - \mathbf{v}| < \delta_1 \text{ and } |\mathbf{y} - \mathbf{w}| < \delta_2\} \subset U.$$

Proof We recall that a subset U of $X \times Y$ is open in $X \times Y$ if and only if, given any point $(\mathbf{v}, \mathbf{w})$ of U, there exists a positive real number $\delta > 0$ such that

$$\{(\mathbf{x}, \mathbf{y}) \in X \times Y : |(\mathbf{x}, \mathbf{y}) - (\mathbf{v}, \mathbf{w})| < \delta\} \subset U.$$

Let $(\mathbf{v}, \mathbf{w})$ be a point of U. Suppose that there exists a positive real number $\delta > 0$ such that

$$\{(\mathbf{x}, \mathbf{y}) \in X \times Y : |(\mathbf{x}, \mathbf{y}) - (\mathbf{v}, \mathbf{w})| < \delta\} \subset U.$$

Let $\delta_1 = \delta_2 = \delta/\sqrt{2}$. Then $\delta_1^2 + \delta_2^2 = \delta^2$. Thus if $|\mathbf{x} - \mathbf{v}| < \delta_1$ and $|\mathbf{y} - \mathbf{w}| < \delta_2$, then $|(\mathbf{x}, \mathbf{y}) - (\mathbf{v}, \mathbf{w})| < \delta$, and hence $(\mathbf{x}, \mathbf{y}) \in U$.

Conversely suppose that there exist positive real numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\{(\mathbf{x}, \mathbf{y}) \in X \times Y : |\mathbf{x} - \mathbf{v}| < \delta_1 \text{ and } |\mathbf{y} - \mathbf{w}| < \delta_2\} \subset U.$$

Let δ be the minimum of δ_1 and δ_2. If $|(\mathbf{x}, \mathbf{y}) - (\mathbf{v}, \mathbf{w})| < \delta$ then $|\mathbf{x} - \mathbf{v}| < \delta_1$ and $|\mathbf{y} - \mathbf{w}| < \delta_2$, and hence $(\mathbf{x}, \mathbf{y}) \in U$.

The result follows. ∎

Lemma 5.2 *Let X, Y and Z be subsets of $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^k$ respectively. A function $f: X \times Y \to Z$ is continuous if and only if, given any point $(\mathbf{v}, \mathbf{w})$ of $X \times Y$, and given any positive real number $\varepsilon > 0$, there exist positive real numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that $|f(\mathbf{x}, \mathbf{y}) - f(\mathbf{v}, \mathbf{w})| < \varepsilon$ for all $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ satisfying $|\mathbf{x} - \mathbf{v}| < \delta_1$ and $|\mathbf{y} - \mathbf{w}| < \delta_2$.*

Proof Let $f\colon X \times Y \to Z$ be a function satisfying the above criterion. We must show that this function is continuous. Let U be an open set in Z. We show that $f^{-1}(U)$ is open in $X \times Y$.

Let $(\mathbf{v}, \mathbf{w})$ be a point of $f^{-1}(U)$. Then $f(\mathbf{v}, \mathbf{w})$ is a point of U. But U is open in Z, and therefore there exists a positive real number $\varepsilon > 0$ such that

$$\{\mathbf{z} \in Z : |\mathbf{z} - f(\mathbf{v}, \mathbf{w})| < \varepsilon\} \subset U.$$

But then there exist real numbers real numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that $|f(\mathbf{x}, \mathbf{y}) - f(\mathbf{v}, \mathbf{w})| < \varepsilon$, and hence $f(\mathbf{x}, \mathbf{y}) \in U$, for all $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ satisfying $|\mathbf{x} - \mathbf{v}| < \delta_1$ and $|\mathbf{y} - \mathbf{w}| < \delta_2$. Thus

$$\{(\mathbf{x}, \mathbf{y}) \in X \times Y : |\mathbf{x} - \mathbf{v}| < \delta_1 \text{ and } |\mathbf{y} - \mathbf{w}| < \delta_2\} \subset f^{-1}(U).$$

We conclude that $f^{-1}(U)$ is open in $X \times Y$ for each open set U in Z. Thus the function $f\colon X \times Y \to Z$ is continuous.

Conversely suppose that $f\colon X \times Y \to Z$ is a continuous function. Let $(\mathbf{v}, \mathbf{w})$ be a point of $X \times Y$ and let $\varepsilon > 0$ be given. Then

$$\{\mathbf{z} \in Z : |\mathbf{z} - f(\mathbf{v}, \mathbf{w})| < \varepsilon\}$$

is an open set in Z, and hence its preimage is an open set in $X \times Y$. It follows from Lemma 5.1 that there exist positive real numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\{(\mathbf{x}, \mathbf{y}) \in X \times Y : |\mathbf{x} - \mathbf{v}| < \delta_1 \text{ and } |\mathbf{y} - \mathbf{w}| < \delta_2\}$$

is contained in the preimage of $\{\mathbf{z} \in Z : |\mathbf{z} - f(\mathbf{v}, \mathbf{w})| < \varepsilon\}$. But this means that $|f(\mathbf{x}, \mathbf{y}) - f(\mathbf{v}, \mathbf{w})| < \varepsilon$ for all $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ satisfying $|\mathbf{x} - \mathbf{v}| < \delta_1$ and $|\mathbf{y} - \mathbf{w}| < \delta_2$, as required. ∎

The next result shows how one can describe the collection of open sets of $X \times Y$ in terms of the collections of open sets in X and in Y, without explicit reference to norms or distance functions. This motivates the definition of the product topology on the Cartesian product of two topological spaces.

Proposition 5.3 *Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. A subset U of $X \times Y$ is open in $X \times Y$ if and only if, given any point $(\mathbf{v}, \mathbf{w})$ of U, there exist an open set V in X and an open set W in Y such that $\mathbf{v} \in V$, $\mathbf{w} \in W$ and $V \times W \subset U$.*

Proof Let U be open in $X \times Y$. It follows from Lemma 5.1 that there exist positive real numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\{(\mathbf{x}, \mathbf{y}) \in X \times Y : |\mathbf{x} - \mathbf{v}| < \delta_1 \text{ and } |\mathbf{y} - \mathbf{w}| < \delta_2\} \subset U.$$

Let

$$V = \{\mathbf{x} \in X : |\mathbf{x} - \mathbf{v}| < \delta_1\} \quad \text{and} \quad W = \{\mathbf{y} \in Y : |\mathbf{y} - \mathbf{w}| < \delta_2\}.$$

Then V is open in X, W is open in Y, $\mathbf{v} \in V$, $\mathbf{w} \in W$ and $V \times W \subset U$.

Conversely suppose that U is a subset of $X \times Y$ and that, given any point $(\mathbf{v}, \mathbf{w})$ of U, there exist an open set V in X and an open set W in Y such that $\mathbf{v} \in V$, $\mathbf{w} \in W$ and $V \times W \subset U$. Then, given any point $(\mathbf{v}, \mathbf{w})$ of U, there exist positive real numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{v}| < \delta_1\} \subset V$$

and

$$\{\mathbf{y} \in Y : |\mathbf{y} - \mathbf{w}| < \delta_2\} \subset W.$$

But then

$$\{(\mathbf{x}, \mathbf{y}) \in X \times Y : |\mathbf{x} - \mathbf{v}| < \delta_1 \text{ and } |\mathbf{y} - \mathbf{w}| < \delta_2\} \subset V \times W \subset U.$$

It follows from Lemma 5.1 that U is open in $X \times Y$, as required. ∎

5.2 Product Topologies

The *Cartesian product* $X_1 \times X_2 \times \cdots \times X_n$ of sets $X_1, X_2, \ldots, X_n$ is defined to be the set of all ordered n-tuples $(x_1, x_2, \ldots, x_n)$, where $x_i \in X_i$ for $i = 1, 2, \ldots, n$.

The sets $\mathbb{R}^2$ and $\mathbb{R}^3$ are the Cartesian products $\mathbb{R} \times \mathbb{R}$ and $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ respectively.

Cartesian products of sets are employed as the domains of functions of several variables. For example, if X, Y and Z are sets, and if an element $f(x, y)$ of Z is determined for each choice of an element x of X and an element y of Y, then we have a function $f\colon X \times Y \to Z$ whose domain is the Cartesian product $X \times Y$ of X and Y: this function sends the ordered pair (x, y) to $f(x, y)$ for all $x \in X$ and $y \in Y$.

Now suppose that X, Y and Z are topological spaces. We wish to define a notion of continuity for functions $f\colon X \times Y \to Z$ from $X \times Y$ to Z. In order to do this, we show that the topologies of X and Y together induce in a natural way a topology on $X \times Y$; this topology is referred to as the *product topology* on $X \times Y$.

First we observe that if V is a subset of X and if W is a subset of Y then $V \times W$ is a subset of $X \times Y$: an element of $V \times W$ is an ordered pair (v, w) with $v \in V$ and $w \in W$, and such an ordered pair belongs to $X \times Y$.

Definition Let X and Y be topological spaces. A subset U of $X \times Y$ is said to be *open* in $X \times Y$ (with respect to the product topology) if, given any point (x, y) of U, there exist an open set V in X and an open set W in Y such that $x \in V$, $y \in W$ and $V \times W \subset U$. The empty set is regarded as an open set in $X \times Y$.

Lemma 5.4 *Let X and Y be topological spaces. Then the collection of open sets in $X \times Y$ is a topology on $X \times Y$.*

Proof The definition of open sets ensures that the empty set and the whole set $X \times Y$ are open in $X \times Y$. We must prove that any union or finite intersection of open sets in $X \times Y$ is an open set.

Let E be the union of a collection of open sets in $X \times Y$, and let (x, y) be a point of E. Then $(x, y) \in D$ for some open set D in the collection. It follows from this that there exists an open set V in X and an open set W in Y such that $x \in V$, $y \in W$ and $V \times W \subset D$. But then $V \times W \subset E$. It follows that E is open in $X \times Y$.

Let $U = U_1 \cap U_2 \cap \cdots \cap U_m$, where $U_1, U_2, \ldots, U_m$ are open sets in $X \times Y$, and let (x, y) be a point of U. Then there exist open sets V_k in X and open sets W_k in Y for $k = 1, 2, \ldots, m$ such that $x \in V_k$, $y \in W_k$ and $V_k \times W_k \subset U_k$ for $k = 1, 2, \ldots, m$. Let

$$V = V_1 \cap V_2 \cap \cdots \cap V_m, \qquad W = W_1 \cap W_2 \cap \cdots \cap W_m.$$

Then $x \in V$ and $y \in W$. Also $V \times W \subset V_k \times W_k \subset U_k$ for $k = 1, 2, \ldots, m$, hence $V \times W \subset U$. It follows that U is open in $X \times Y$, as required. ∎

Let X and Y be topological spaces. The collection of open sets in $X \times Y$ defined as described above is referred to as the *product topology* on $X \times Y$. The definition of the product topology can easily be generalized to Cartesian products of any finite number of topological spaces.

Definition Let $X_1, X_2, \ldots, X_n$ be topological spaces. A subset U of the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ is said to be *open* (with respect to the product topology) if, given any point p of U, there exist open sets V_i in X_i for $i = 1, 2, \ldots, n$ such that $\{p\} \subset V_1 \times V_2 \times \cdots \times V_n \subset U$.

Lemma 5.5 *Let $X_1, X_2, \ldots, X_n$ be topological spaces. Then the collection of open sets in $X_1 \times X_2 \times \cdots \times X_n$ is a topology on $X_1 \times X_2 \times \cdots \times X_n$.*

Proof Let $X = X_1 \times X_2 \times \cdots \times X_n$. The definition of open sets ensures that the empty set and the whole set X are open in X. We must prove that any union or finite intersection of open sets in X is an open set.

Let E be a union of a collection of open sets in X and let p be a point of E. Then Then $p \in D$ for some open set D in the collection. It follows from this that there exist open sets V_i in X_i for $i = 1, 2, \ldots, n$ such that

$$\{p\} \subset V_1 \times V_2 \times \cdots \times V_n \subset D \subset E.$$

Thus E is open in X.

Let $U = U_1 \cap U_2 \cap \cdots \cap U_m$, where $U_1, U_2, \ldots, U_m$ are open sets in X, and let p be a point of U. Then there exist open sets V_{ki} in X_i for $k = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, n$ such that $\{p\} \subset V_{k1} \times V_{k2} \times \cdots \times V_{kn} \subset U_k$ for $k = 1, 2, \ldots, m$. Let $V_i = V_{1i} \cap V_{2i} \cap \cdots \cap V_{mi}$ for $i = 1, 2, \ldots, n$. Then

$$\{p\} \subset V_1 \times V_2 \times \cdots \times V_n \subset V_{k1} \times V_{k2} \times \cdots \times V_{kn} \subset U_k$$

for $k = 1, 2, \ldots, m$, and hence $\{p\} \subset V_1 \times V_2 \times \cdots \times V_n \subset U$. It follows that U is open in X, as required. ∎

Lemma 5.6 *Let $X_1, X_2, \ldots, X_n$ and Z be topological spaces. Then a function $f: X_1 \times X_2 \times \cdots \times X_n \to Z$ is continuous if and only if, given any point p of $X_1 \times X_2 \times \cdots \times X_n$, and given any open set U in Z containing $f(p)$, there exist open sets V_i in X_i for $i = 1, 2, \ldots, n$ such that $p \in V_1 \times V_2 \cdots \times V_n$ and $f(V_1 \times V_2 \times \cdots \times V_n) \subset U$.*

Proof Let V_i be an open set in X_i for $i = 1, 2, \ldots, n$, and let U be an open set in Z. Then $V_1 \times V_2 \times \cdots \times V_n \subset f^{-1}(U)$ if and only if $f(V_1 \times V_2 \times \cdots \times V_n) \subset U$. It follows that $f^{-1}(U)$ is open in the product topology on $X_1 \times X_2 \times \cdots \times X_n$ if and only if, given any point p of $X_1 \times X_2 \times \cdots \times X_n$ satisfying $f(p) \in U$, there exist open sets V_i in X_i for $i = 1, 2, \ldots, n$ such that $f(V_1 \times V_2 \times \cdots \times V_n) \subset U$. The required result now follows from the definition of continuity. ∎

Let $X_1, X_2, \ldots, X_n$ be topological spaces, and let V_i be an open set in X_i for $i = 1, 2, \ldots, n$. It follows directly from the definition of the product topology that $V_1 \times V_2 \times \cdots \times V_n$ is open in $X_1 \times X_2 \times \cdots \times X_n$.

Theorem 5.7 *Let $X = X_1 \times X_2 \times \cdots \times X_n$, where $X_1, X_2, \ldots, X_n$ are topological spaces and X is given the product topology, and for each i, let $p_i: X \to X_i$ denote the projection function which sends $(x_1, x_2, \ldots, x_n) \in X$ to x_i. Then the functions $p_1, p_2, \ldots, p_n$ are continuous. Moreover a function $f: Z \to X$ mapping a topological space Z into X is continuous if and only if $p_i \circ f: Z \to X_i$ is continuous for $i = 1, 2, \ldots, n$.*

Proof Let V be an open set in X_i. Then

$$p_i^{-1}(V) = X_1 \times \cdots \times X_{i-1} \times V \times X_{i+1} \times \cdots \times X_n,$$

and therefore $p_i^{-1}(V)$ is open in X. Thus $p_i: X \to X_i$ is continuous for all i.

Let $f: Z \to X$ be continuous. Then, for each i, $p_i \circ f: Z \to X_i$ is a composition of continuous functions, and is thus itself continuous.

Conversely suppose that $f: Z \to X$ is a function with the property that $p_i \circ f$ is continuous for all i. Let U be an open set in X. We must show that $f^{-1}(U)$ is open in Z.

Let z be a point of $f^{-1}(U)$, and let $f(z) = (u_1, u_2, \ldots, u_n)$. Now U is open in X, and therefore there exist open sets $V_1, V_2, \ldots, V_n$ in $X_1, X_2, \ldots, X_n$ respectively such that $u_i \in V_i$ for all i and $V_1 \times V_2 \times \cdots \times V_n \subset U$. Let

$$N_z = f_1^{-1}(V_1) \cap f_2^{-1}(V_2) \cap \cdots \cap f_n^{-1}(V_n),$$

where $f_i = p_i \circ f$ for $i = 1, 2, \ldots, n$. Now $f_i^{-1}(V_i)$ is an open subset of Z for $i = 1, 2, \ldots, n$, since V_i is open in X_i and $f_i: Z \to X_i$ is continuous. Thus N_z, being a finite intersection of open sets, is itself open in Z. Moreover

$$f(N_z) \subset V_1 \times V_2 \times \cdots \times V_n \subset U,$$

so that $N_z \subset f^{-1}(U)$. It follows that $f^{-1}(U)$ is the union of the open sets N_z as z ranges over all points of $f^{-1}(U)$. Therefore $f^{-1}(U)$ is open in Z. This shows that $f: Z \to X$ is continuous, as required. ∎

Proposition 5.8 *The usual topology on $\mathbb{R}^n$ coincides with the product topology on $\mathbb{R}^n$ obtained on regarding $\mathbb{R}^n$ as the Cartesian product $\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ of n copies of the real line $\mathbb{R}$.*

Proof We must show that a subset U of $\mathbb{R}^n$ is open with respect to the usual topology if and only if it is open with respect to the product topology.

Let U be a subset of $\mathbb{R}^n$ that is open with respect to the usual topology, and let $\mathbf{u} \in U$. Then there exists some $\delta > 0$ such that $B(\mathbf{u}, \delta) \subset U$, where

$$B(\mathbf{u}, \delta) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{u}| < \delta\}.$$

Let $I_1, I_2, \ldots, I_n$ be the open intervals in $\mathbb{R}$ defined by

$$I_i = \{t \in \mathbb{R} : u_i - \frac{\delta}{\sqrt{n}} < t < u_i + \frac{\delta}{\sqrt{n}}\} \qquad (i = 1, 2, \ldots, n),$$

Then $I_1, I_2, \ldots, I_n$ are open sets in $\mathbb{R}$. Moreover

$$\{\mathbf{u}\} \subset I_1 \times I_2 \times \cdots \times I_n \subset B(\mathbf{u}, \delta) \subset U,$$

since

$$|\mathbf{x} - \mathbf{u}|^2 = \sum_{i=1}^{n} (x_i - u_i)^2 < n \left(\frac{\delta}{\sqrt{n}} \right)^2 = \delta^2$$

for all $\mathbf{x} \in I_1 \times I_2 \times \cdots \times I_n$. This shows that any subset U of $\mathbb{R}^n$ that is open with respect to the usual topology on $\mathbb{R}^n$ is also open with respect to the product topology on $\mathbb{R}^n$.

Conversely suppose that U is a subset of $\mathbb{R}^n$ that is open with respect to the product topology on $\mathbb{R}^n$, and let $\mathbf{u} \in U$. Then there exist open sets $V_1, V_2, \ldots, V_n$ in $\mathbb{R}$ containing $u_1, u_2, \ldots, u_n$ respectively such that $V_1 \times V_2 \times \cdots \times V_n \subset U$. Now we can find $\delta_1, \delta_2, \ldots, \delta_n$ such that $\delta_i > 0$ and $(u_i - \delta_i, u_i + \delta_i) \subset V_i$ for all i. Let $\delta > 0$ be the minimum of $\delta_1, \delta_2, \ldots, \ldots, \delta_n$. Then

$$B(\mathbf{u}, \delta) \subset V_1 \times V_2 \times \cdots V_n \subset U,$$

for if $\mathbf{x} \in B(\mathbf{u}, \delta)$ then $|x_i - u_i| < \delta_i$ for $i = 1, 2, \ldots, n$. This shows that any subset U of $\mathbb{R}^n$ that is open with respect to the product topology on $\mathbb{R}^n$ is also open with respect to the usual topology on $\mathbb{R}^n$. ∎

The following result is now an immediate corollary of Proposition 5.8 and Theorem 5.7.

Corollary 5.9 *Let X be a topological space and let $f: X \to \mathbb{R}^n$ be a function from X to $\mathbb{R}^n$. Let us write*

$$f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$$

for all $x \in X$, where the components $f_1, f_2, \ldots, f_n$ of f are functions from X to $\mathbb{R}$. The function f is continuous if and only if its components $f_1, f_2, \ldots, f_n$ are all continuous.

Let $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ be continuous real-valued functions on some topological space X. We claim that $f + g$, $f - g$ and $f.g$ are continuous. Now it is a straightforward exercise to verify that the sum and product functions $s: \mathbb{R}^2 \to \mathbb{R}$ and $p: \mathbb{R}^2 \to \mathbb{R}$ defined by $s(x, y) = x + y$ and $p(x, y) = xy$ are continuous, and $f + g = s \circ h$ and $f.g = p \circ h$, where $h: X \to \mathbb{R}^2$ is defined by $h(x) = (f(x), g(x))$. Moreover it follows from Corollary 5.9 that the function h is continuous, and compositions of continuous functions are continuous. Therefore $f + g$ and $f.g$ are continuous, as claimed. Also $-g$ is continuous, and $f - g = f + (-g)$, and therefore $f - g$ is continuous. If in addition the continuous function g is non-zero everywhere on X then $1/g$ is continuous (since $1/g$ is the composition of g with the reciprocal function $t \mapsto 1/t$), and therefore f/g is continuous.

Lemma 5.10 *The Cartesian product $X_1 \times X_2 \times \ldots, X_n$ of Hausdorff spaces $X_1, X_2, \ldots, X_n$ is Hausdorff.*

Proof Let $X = X_1 \times X_2 \times \ldots, X_n$, and let u and v be distinct points of X, where $u = (x_1, x_2, \ldots, x_n)$ and $v = (y_1, y_2, \ldots, y_n)$. Then $x_i \neq y_i$ for some integer i between 1 and n. But then there exist open sets U and V in X_i such that $x_i \in U$, $y_i \in V$ and $U \cap V = \emptyset$ (since X_i is a Hausdorff space). Let $p_i: X \to X_i$ denote the projection function. Then $p_i^{-1}(U)$ and $p_i^{-1}(V)$ are open sets in X, since p_i is continuous. Moreover $u \in p_i^{-1}(U)$, $v \in p_i^{-1}(V)$, and $p_i^{-1}(U) \cap p_i^{-1}(V) = \emptyset$. Thus X is Hausdorff, as required. ∎

6 Identification Maps and Quotient Topologies

6.1 Cut and Paste Constructions

Suppose we start out with a square of paper. If we join together two opposite edges of this square we obtain a cylinder. The boundary of the cylinder consists of two circles. If we join together the two boundary circles we obtain a torus (which corresponds to the surface of a doughnut).

Let the square be represented by the set $[0,1] \times [0,1]$ consisting of all ordered pairs (s,t) where s and t are real numbers between 0 and 1. There is an equivalence relation on the square $[0,1] \times [0,1]$, where points (s,t) and (u,v) of the square are related if and only if at least one of the following conditions is satisfied:

- $s = u$ and $t = v$;
- $s = 0$, $u = 1$ and $t = v$;
- $s = 1$, $u = 0$ and $t = v$;
- $t = 0$, $v = 1$ and $s = u$;
- $t = 1$, $v = 0$ and $s = u$;
- (s,t) and (u,v) both belong to $\{(0,0), (0,1), (1,0), (1,1)\}$.

Note that if $0 < s < 1$ and $0 < t < 1$ then the equivalence class of the point (s,t) is the set $\{(s,t)\}$ consisting of that point. If $s = 0$ or 1 and if $0 < t < 1$ then the equivalence class of (s,t) is the set $\{(0,t), (1,t)\}$. Similarly if $t = 0$ or 1 and if $0 < s < 1$ then the equivalence class of (s,t) is the set $\{(s,0), (s,1)\}$. The equivalence class of each corner of the square is the set $\{(0,0), (1,0), (0,1), (1,1)\}$ consisting of all four corners. Thus each equivalence class contains either one point in the interior of the square, or two points on opposite edges of the square, or four points at the four corners of the square. Let T^2 denote the set of these equivalence classes. We have a map $q\colon [0,1] \times [0,1] \to T^2$ which sends each point (s,t) of the square to its equivalence class. Each element of the set T^2 is the image of one, two or four points of the square. The elements of T^2 represent points on the torus obtained from the square by first joining together two opposite sides of the square to form a cylinder and then joining together the boundary circles of this cylinder as described above. We say that the torus T^2 is obtained from the square $[0,1] \times [0,1]$ by *identifying* the points $(0,t)$ and $(1,t)$ for all $t \in [0,1]$ and identifying the points $(s,0)$ and $(s,1)$ for all $s \in [0,1]$.

The topology on the square $[0,1] \times [0,1]$ induces a corresponding topology on the set T^2, where a subset U of T^2 is open in T^2 if and only if $q^{-1}(U)$ is open in the square $[0,1] \times [0,1]$. (The fact that these open sets in T^2 constitute a topology on the set T^2 is a consequence of Lemma 6.1.) The function $q\colon [0,1] \times [0,1] \to T^2$ is then a continuous surjection. We say that the topological space T^2 is the *identification space* obtained from the square $[0,1] \times [0,1]$ by identifying points on the sides to the square as described above. The continuous map q from the square to the torus is an example of an *identification map*, and the topology on the torus T^2 is referred to as the *quotient topology* on T^2 induced by the identification map $q\colon [0,1] \times [0,1] \to T^2$.

Another well-known identification space obtained from the square is the *Klein bottle* (*Kleinsche Flasche*). The Klein bottle K^2 is obtained from the square $[0,1] \times [0,1]$ by identifying $(0,t)$ with $(1,1-t)$ for all $t \in [0,1]$ and identifying $(s,0)$ with $(s,1)$ for all $s \in [0,1]$. These identifications correspond to an equivalence relation on the square, where points (s,t) and (u,v) of the square are equivalent if and only if one of the following conditions is satisfied:

- $s = u$ and $t = v$;
- $s = 0$, $u = 1$ and $t = 1 - v$;
- $s = 1$, $u = 0$ and $t = 1 - v$;
- $t = 0$, $v = 1$ and $s = u$;
- $t = 1$, $v = 0$ and $s = u$;
- (s,t) and (u,v) both belong to $\{(0,0), (0,1), (1,0), (1,1)\}$.

The corresponding set of equivalence classes is the Klein bottle K^2. Thus each point of the Klein bottle K^2 represents an equivalence class consisting of either one point in the interior of the square, or two points $(0,t)$ and $(1,1-t)$ with $0 < t < 1$ on opposite edges of the square, or two points $(s,0)$ and $(s,1)$ with $0 < s < 1$ on opposite edges of the square, or the four corners of the square. There is a surjection $r\colon [0,1] \times [0,1] \to K^2$ from the square to the Klein bottle that sends each point of the square to its equivalence class. The identifications used to construct the Klein bottle ensure that $r(0,t) = r(1,1-t)$ for all $t \in [0,1]$ and $r(s,0) = r(s,1)$ for all $s \in [0,1]$. One can construct a quotient topology on the Klein bottle K^2, where a subset U of K^2 is open in K^2 if and only if its preimage $r^{-1}(U)$ is open in the square $[0,1] \times [0,1]$.

6.2 Identification Maps and Quotient Topologies

Definition Let X and Y be topological spaces and let $q\colon X \to Y$ be a function from X to Y. The function q is said to be an *identification map* if and only if the following conditions are satisfied:

- the function $q\colon X \to Y$ is surjective,
- a subset U of Y is open in Y if and only if $q^{-1}(U)$ is open in X.

It follows directly from the definition that any identification map is continuous. Moreover, in order to show that a continuous surjection $q\colon X \to Y$ is an identification map, it suffices to prove that if V is a subset of Y with the property that $q^{-1}(V)$ is open in X then V is open in Y.

Example Let S^1 denote the unit circle $\{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$ in $\mathbb{R}^2$, and let $q\colon [0,1] \to S^1$ be the continuous map defined by $q(t) = (\cos 2\pi t, \sin 2\pi t)$ for all $t \in [0,1]$. We show that $q\colon [0,1] \to S^1$ is an identification map. This map is continuous and surjective. It remains to show that if V is a subset of S^1 with the property that $q^{-1}(V)$ is open in $[0,1]$ then V is open in S^1.

Note that $|q(s) - q(t)| = 2|\sin \pi(s-t)|$ for all $s,t \in [0,1]$ satisfying $|s-t| \leq \frac{1}{2}$. Let V be a subset of S^1 with the property that $q^{-1}(V)$ is open in $[0,1]$, and let $\mathbf{v}$ be an element of V. We show that there exists $\varepsilon > 0$ such that all points $\mathbf{u}$ of S^1 satisfying $|\mathbf{u} - \mathbf{v}| < \varepsilon$ belong to V. We consider separately the cases when $\mathbf{v} = (1,0)$ and when $\mathbf{v} \neq (1,0)$.

Suppose that $\mathbf{v} = (1,0)$. Then $(1,0) \in V$, and hence $0 \in q^{-1}(V)$ and $1 \in q^{-1}(V)$. But $q^{-1}(V)$ is open in $[0,1]$. It follows that there exists a real number δ satisfying $0 < \delta < \frac{1}{2}$ such that $[0,\delta) \subset q^{-1}(V)$ and $(1-\delta,1] \in q^{-1}(V)$. Let $\varepsilon = 2\sin \pi\delta$. Now if $-\pi \leq \theta \leq \pi$ then the Euclidean distance between the points $(1,0)$ and $(\cos\theta, \sin\theta)$ is $2\sin\frac{1}{2}|\theta|$. Moreover, this distance increases monotonically as $|\theta|$ increases from 0 to π. Thus any point on the unit circle S^1 whose distance from $(1,0)$ is less than ε must be of the form $(\cos\theta, \sin\theta)$, where $|\theta| < 2\pi\delta$. Thus if $\mathbf{u} \in S^1$ satisfies $|\mathbf{u} - \mathbf{v}| < \varepsilon$ then $\mathbf{u} = q(s)$ for some $s \in [0,1]$ satisfying either $0 \leq s < \delta$ or $1 - \delta < s \leq 1$. But then $s \in q^{-1}(V)$, and hence $\mathbf{u} \in V$.

Next suppose that $\mathbf{v} \neq (1,0)$. Then $\mathbf{v} = q(t)$ for some real number t satisfying $0 < t < 1$. But $q^{-1}(V)$ is open in $[0,1]$, and $t \in q^{-1}(V)$. It follows that $(t-\delta, t+\delta) \subset q^{-1}(V)$ for some real number δ satisfying $\delta > 0$. Let $\varepsilon = 2\sin\pi\delta$. If $\mathbf{u} \in S^1$ satisfies $|\mathbf{u} - \mathbf{v}| < \varepsilon$ then $\mathbf{u} = q(s)$ for some $s \in (t-\delta, t+\delta)$. But then $s \in q^{-1}(V)$, and hence $\mathbf{u} \in V$.

We have thus shown that if V is a subset of S^1 with the property that $q^{-1}(V)$ is open in $[0,1]$ then there exists $\varepsilon > 0$ such that $\mathbf{u} \in V$ for all elements $\mathbf{u}$ of S^1 satisfying $|\mathbf{u} - \mathbf{v}| < \varepsilon$. It follows from this that V is open in S^1. Thus the continuous surjection $q\colon [0,1] \to S^1$ is an identification map.

Lemma 6.1 *Let X be a topological space, let Y be a set, and let $q\colon X \to Y$ be a surjection. Then there is a unique topology on Y for which the function $q\colon X \to Y$ is an identification map.*

Proof Let τ be the collection consisting of all subsets U of Y for which $q^{-1}(U)$ is open in X. Now $q^{-1}(\emptyset) = \emptyset$, and $q^{-1}(Y) = X$, so that $\emptyset \in \tau$ and $Y \in \tau$. If $\{V_\alpha : \alpha \in A\}$ is any collection of subsets of Y indexed by a set A, then it is a straightforward exercise to verify that

$$\bigcup_{\alpha\in A} q^{-1}(V_\alpha) = q^{-1}\left(\bigcup_{\alpha\in A} V_\alpha\right), \qquad \bigcap_{\alpha\in A} q^{-1}(V_\alpha) = q^{-1}\left(\bigcap_{\alpha\in A} V_\alpha\right)$$

(i.e., given any collection of subsets of Y, the union of the preimages of the sets is the preimage of the union of those sets, and the intersection of the preimages of the sets is the preimage of the intersection of those sets). It follows easily from this that unions and finite intersections of sets belonging to τ must themselves belong to τ. Thus τ is a topology on Y, and the function $q\colon X \to Y$ is an identification map with respect to the topology τ. Clearly τ is the unique topology on Y for which the function $q\colon X \to Y$ is an identification map. ∎

Let X be a topological space, let Y be a set, and let $q\colon X \to Y$ be a surjection. The unique topology on Y for which the function q is an identification map is referred to as the *quotient topology* (or *identification topology*) on Y.

Let $\sim$ be an equivalence relation on a topological space X. If Y is the corresponding set of equivalence classes of elements of X then there is a surjection $q: X \to Y$ that sends each element of X to its equivalence class. Lemma 6.1 ensures that there is a well-defined quotient topology on Y, where a subset U of Y is open in Y if and only if $q^{-1}(U)$ is open in X. (Appropriate equivalence relations on the square yield the torus and the Klein bottle, as discussed above.)

Lemma 6.2 *Let X and Y be topological spaces and let $q: X \to Y$ be an identification map. Let Z be a topological space, and let $f: Y \to Z$ be a function from Y to Z. Then the function f is continuous if and only if the composition function $f \circ q: X \to Z$ is continuous.*

Proof Suppose that f is continuous. Then the composition function $f \circ q$ is a composition of continuous functions and hence is itself continuous.

Conversely suppose that $f \circ q$ is continuous. Let U be an open set in Z. Then $q^{-1}(f^{-1}(U))$ is open in X (since $f \circ q$ is continuous), and hence $f^{-1}(U)$ is open in Y (since the function q is an identification map). Therefore the function f is continuous, as required. ∎

Example Let S^1 be the unit circle in $\mathbb{R}^2$, and let $q: [0,1] \to S^1$ be the map that sends $t \in [0,1]$ to $(\cos 2\pi t, \sin 2\pi t)$. Then $q: [0,1] \to S^1$ is an identification map, and therefore a function $f: S^1 \to Z$ from S^1 to some topological space Z is continuous if and only if $f \circ q: [0,1] \to Z$ is continuous.

Example The Klein bottle K^2 is the identification space obtained from the square $[0,1] \times [0,1]$ by identifying $(0,t)$ with $(1, 1-t)$ for all $t \in [0,1]$ and identifying $(s,0)$ with $(s,1)$ for all $s \in [0,1]$. Let $q: [0,1] \times [0,1] \to K^2$ be the identification map determined by these identifications. Let Z be a topological space. A function $g: [0,1] \times [0,1] \to Z$ mapping the square into Z which satisfies $g(0,t) = g(1, 1-t)$ for all $t \in [0,1]$ and $g(s,0) = g(s,1)$ for all $s \in [0,1]$, determines a corresponding function $f: K^2 \to Z$, where $g = f \circ q$. It follows from Lemma 6.2 that the function $f: K^2 \to Z$ is continuous if and only if $g: [0,1] \times [0,1] \to Z$ is continuous.

Example Let S^n be the n-sphere, consisting of all points $\mathbf{x}$ in $\mathbb{R}^{n+1}$ satisfying $|\mathbf{x}| = 1$. Let $\mathbb{R}P^n$ be the set of all lines in $\mathbb{R}^{n+1}$ passing through the origin (i.e., $\mathbb{R}P^n$ is the set of all one-dimensional vector subspaces of $\mathbb{R}^{n+1}$). Let $q: S^n \to \mathbb{R}P^n$ denote the function which sends a point $\mathbf{x}$ of S^n to the element of $\mathbb{R}P^n$ represented by the line in $\mathbb{R}^{n+1}$ that passes through both $\mathbf{x}$ and the origin. Note that each element of $\mathbb{R}P^n$ is the image (under q) of exactly two antipodal points $\mathbf{x}$ and $-\mathbf{x}$ of S^n. The function q induces a corresponding quotient topology on $\mathbb{R}P^n$ such that $q: S^n \to \mathbb{R}P^n$ is an identification map. The set $\mathbb{R}P^n$, with this topology, is referred to as *real projective n-space*. In particular

$\mathbb{R}P^2$ is referred to as the *real projective plane*. It follows from Lemma 6.2 that a function $f: \mathbb{R}P^n \to Z$ from $\mathbb{R}P^n$ to any topological space Z is continuous if and only if the composition function $f \circ q: S^n \to Z$ is continuous.

7 Compactness

7.1 Compact Topological Spaces

Let X be a topological space, and let A be a subset of X. A collection of subsets of X in X is said to *cover* A if and only if every point of A belongs to at least one of these subsets. In particular, an *open cover* of X is collection of open sets in X that covers X.

If $\mathcal{U}$ and $\mathcal{V}$ are open covers of some topological space X then $\mathcal{V}$ is said to be a *subcover* of $\mathcal{U}$ if and only if every open set belonging to $\mathcal{V}$ also belongs to $\mathcal{U}$.

Definition A topological space X is said to be *compact* if and only if every open cover of X possesses a finite subcover.

Lemma 7.1 *Let X be a topological space. A subset A of X is compact (with respect to the subspace topology on A) if and only if, given any collection $\mathcal{U}$ of open sets in X covering A, there exists a finite collection $V_1, V_2, \ldots, V_r$ of open sets belonging to $\mathcal{U}$ such that $A \subset V_1 \cup V_2 \cup \cdots \cup V_r$.*

Proof A subset B of A is open in A (with respect to the subspace topology on A) if and only if $B = A \cap V$ for some open set V in X. The desired result therefore follows directly from the definition of compactness. ∎

We now show that any closed bounded interval in the real line is compact. This result is known as the *Heine-Borel Theorem*. The proof of this theorem uses the *least upper bound principle* which states that, given any non-empty set S of real numbers which is bounded above, there exists a *least upper bound* (or *supremum*) $\sup S$ for the set S.

Theorem 7.2 (Heine-Borel) *Let a and b be real numbers satisfying $a < b$. Then the closed bounded interval $[a, b]$ is a compact subset of $\mathbb{R}$.*

Proof Let $\mathcal{U}$ be a collection of open sets in $\mathbb{R}$ with the property that each point of the interval $[a, b]$ belongs to at least one of these open sets. We must show that $[a, b]$ is covered by finitely many of these open sets.

Let S be the set of all $\tau \in [a, b]$ with the property that $[a, \tau]$ is covered by some finite collection of open sets belonging to $\mathcal{U}$, and let $s = \sup S$. Now

$s \in W$ for some open set W belonging to $\mathcal{U}$. Moreover W is open in $\mathbb{R}$, and therefore there exists some $\delta > 0$ such that $(s - \delta, s + \delta) \subset W$. Moreover $s - \delta$ is not an upper bound for the set S, hence there exists some $\tau \in S$ satisfying $\tau > s - \delta$. It follows from the definition of S that $[a, \tau]$ is covered by some finite collection $V_1, V_2, \ldots, V_r$ of open sets belonging to $\mathcal{U}$.

Let $t \in [a, b]$ satisfy $\tau \leq t < s + \delta$. Then

$$[a, t] \subset [a, \tau] \cup (s - \delta, s + \delta) \subset V_1 \cup V_2 \cup \cdots \cup V_r \cup W,$$

and thus $t \in S$. In particular $s \in S$, and moreover $s = b$, since otherwise s would not be an upper bound of the set S. Thus $b \in S$, and therefore $[a, b]$ is covered by a finite collection of open sets belonging to $\mathcal{U}$, as required. ∎

Lemma 7.3 *Let A be a closed subset of some compact topological space X. Then A is compact.*

Proof Let $\mathcal{U}$ be any collection of open sets in X covering A. On adjoining the open set $X \setminus A$ to $\mathcal{U}$, we obtain an open cover of X. This open cover of X possesses a finite subcover, since X is compact. Moreover A is covered by the open sets in the collection $\mathcal{U}$ that belong to this finite subcover. It follows from Lemma 7.1 that A is compact, as required. ∎

Lemma 7.4 *Let $f \colon X \to Y$ be a continuous function between topological spaces X and Y, and let A be a compact subset of X. Then $f(A)$ is a compact subset of Y.*

Proof Let $\mathcal{V}$ be a collection of open sets in Y which covers $f(A)$. Then A is covered by the collection of all open sets of the form $f^{-1}(V)$ for some $V \in \mathcal{V}$. It follows from the compactness of A that there exists a finite collection $V_1, V_2, \ldots, V_k$ of open sets belonging to $\mathcal{V}$ such that

$$A \subset f^{-1}(V_1) \cup f^{-1}(V_2) \cup \cdots \cup f^{-1}(V_k).$$

But then $f(A) \subset V_1 \cup V_2 \cup \cdots \cup V_k$. This shows that $f(A)$ is compact. ∎

Lemma 7.5 *Let $f \colon X \to \mathbb{R}$ be a continuous real-valued function on a compact topological space X. Then f is bounded above and below on X.*

Proof The range $f(X)$ of the function f is covered by some finite collection $I_1, I_2, \ldots, I_k$ of open intervals of the form $(-m, m)$, where $m \in \mathbb{N}$, since $f(X)$ is compact (Lemma 7.4) and $\mathbb{R}$ is covered by the collection of all intervals of this form. It follows that $f(X) \subset (-M, M)$, where $(-M, M)$ is the largest of the intervals $I_1, I_2, \ldots, I_k$. Thus the function f is bounded above and below on X, as required. ∎

Proposition 7.6 *Let $f: X \to \mathbb{R}$ be a continuous real-valued function on a compact topological space X. Then there exist points u and v of X such that $f(u) \leq f(x) \leq f(v)$ for all $x \in X$.*

Proof Let $m = \inf\{f(x) : x \in X\}$ and $M = \sup\{f(x) : x \in X\}$. There must exist $v \in X$ satisfying $f(v) = M$, for if $f(x) < M$ for all $x \in X$ then the function $x \mapsto 1/(M - f(x))$ would be a continuous real-valued function on X that was not bounded above, contradicting Lemma 7.5. Similarly there must exist $u \in X$ satisfying $f(u) = m$, since otherwise the function $x \mapsto 1/(f(x) - m)$ would be a continuous function on X that was not bounded above, again contradicting Lemma 7.5. But then $f(u) \leq f(x) \leq f(v)$ for all $x \in X$, as required. ∎

Proposition 7.7 *Let A be a compact subset of a metric space X. Then A is closed in X.*

Proof Let p be a point of X that does not belong to A, and let $f(x) = d(x, p)$, where d is the distance function on X. It follows from Proposition 7.6 that there is a point q of A such that $f(a) \geq f(q)$ for all $a \in A$, since A is compact. Now $f(q) > 0$, since $q \neq p$. Let δ satisfy $0 < \delta \leq f(q)$. Then the open ball of radius δ about the point p is contained in the complement of A, since $f(x) < f(q)$ for all points x of this open ball. It follows that the complement of A is an open set in X, and thus A itself is closed in X. ∎

Proposition 7.8 *Let X be a Hausdorff topological space, and let K be a compact subset of X. Let x be a point of $X \setminus K$. Then there exist open sets V and W in X such that $x \in V$, $K \subset W$ and $V \cap W = \emptyset$.*

Proof For each point $y \in K$ there exist open sets $V_{x,y}$ and $W_{x,y}$ such that $x \in V_{x,y}$, $y \in W_{x,y}$ and $V_{x,y} \cap W_{x,y} = \emptyset$ (since X is a Hausdorff space). But then there exists a finite set $\{y_1, y_2, \ldots, y_r\}$ of points of K such that K is contained in $W_{x,y_1} \cup W_{x,y_2} \cup \cdots \cup W_{x,y_r}$, since K is compact. Define

$$V = V_{x,y_1} \cap V_{x,y_2} \cap \cdots \cap V_{x,y_r}, \qquad W = W_{x,y_1} \cup W_{x,y_2} \cup \cdots \cup W_{x,y_r}.$$

Then V and W are open sets, $x \in V$, $K \subset W$ and $V \cap W = \emptyset$, as required. ∎

Corollary 7.9 *A compact subset of a Hausdorff topological space is closed.*

Proof Let K be a compact subset of a Hausdorff topological space X. It follows immediately from Proposition 7.8 that, for each $x \in X \setminus K$, there exists an open set V_x such that $x \in V_x$ and $V_x \cap K = \emptyset$. But then $X \setminus K$ is equal to the union of the open sets V_x as x ranges over all points of $X \setminus K$, and any set that is a union of open sets is itself an open set. We conclude that $X \setminus K$ is open, and thus K is closed. ∎

Proposition 7.10 *Let X be a Hausdorff topological space, and let K_1 and K_2 be compact subsets of X, where $K_1 \cap K_2 = \emptyset$. Then there exist open sets U_1 and U_2 such that $K_1 \subset U_1$, $K_2 \subset U_2$ and $U_1 \cap U_2 = \emptyset$.*

Proof It follows from Proposition 7.8 that, for each point x of K_1, there exist open sets V_x and W_x such that $x \in V_x$, $K_2 \subset W_x$ and $V_x \cap W_x = \emptyset$. But then there exists a finite set $\{x_1, x_2, \ldots, x_r\}$ of points of K_1 such that

$$K_1 \subset V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_r},$$

since K_1 is compact. Define

$$U_1 = V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_r}, \qquad U_2 = W_{x_1} \cap W_{x_2} \cap \cdots \cap W_{x_r}.$$

Then U_1 and U_2 are open sets, $K_1 \subset U_1$, $K_2 \subset U_2$ and $U_1 \cap U_2 = \emptyset$, as required. ∎

Lemma 7.11 *Let $f: X \to Y$ be a continuous function from a compact topological space X to a Hausdorff space Y. Then $f(K)$ is closed in Y for every closed set K in X.*

Proof If K is a closed set in X, then K is compact (Lemma 7.3), and therefore $f(K)$ is compact (Lemma 7.4). But any compact subset of a Hausdorff space is closed (Corollary 7.9). Thus $f(K)$ is closed in Y, as required. ∎

Remark If the Hausdorff space Y in Lemma 7.11 is a metric space, then Proposition 7.7 may be used in place of Corollary 7.9 in the proof of the lemma.

Theorem 7.12 *A continuous bijection $f: X \to Y$ from a compact topological space X to a Hausdorff space Y is a homeomorphism.*

Proof Let $g: Y \to X$ be the inverse of the bijection $f: X \to Y$. If U is open in X then $X \setminus U$ is closed in X, and hence $f(X \setminus U)$ is closed in Y, by Lemma 7.11. But $f(X \setminus U) = g^{-1}(X \setminus U) = Y \setminus g^{-1}(U)$. It follows that $g^{-1}(U)$ is open in Y for every open set U in X. Therefore $g: Y \to X$ is continuous, and thus $f: X \to Y$ is a homeomorphism. ∎

We recall that a function $f: X \to Y$ from a topological space X to a topological space Y is said to be an *identification map* if it is surjective and satisfies the following condition: a subset U of Y is open in Y if and only if $f^{-1}(U)$ is open in X.

Proposition 7.13 *A continuous surjection $f: X \to Y$ from a compact topological space X to a Hausdorff space Y is an identification map.*

Proof Let U be a subset of Y. We claim that $Y \setminus U = f(K)$, where $K = X \setminus f^{-1}(U)$. Clearly $f(K) \subset Y \setminus U$. Also, given any $y \in Y \setminus U$, there exists $x \in X$ satisfying $y = f(x)$, since $f: X \to Y$ is surjective. Moreover $x \in K$, since $f(x) \notin U$. Thus $Y \setminus U \subset f(K)$, and hence $Y \setminus U = f(K)$, as claimed.

We must show that the set U is open in Y if and only if $f^{-1}(U)$ is open in X. First suppose that $f^{-1}(U)$ is open in X. Then K is closed in X, and hence $f(K)$ is closed in Y, by Lemma 7.11. It follows that U is open in Y. Conversely if U is open in Y then $f^{-1}(Y)$ is open in X, since $f: X \to Y$ is continuous. Thus the surjection $f: X \to Y$ is an identification map. ∎

Example Let S^1 be the unit circle in $\mathbb{R}^2$, defined by $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, and let $q: [0, 1] \to S^1$ be defined by $q(t) = (\cos 2\pi t, \sin 2\pi t)$ for all $t \in [0, 1]$. It has been shown that the map q is an identification map. This also follows directly from the fact that $q: [0, 1] \to S^1$ is a continuous surjection from the compact space $[0, 1]$ to the Hausdorff space S^1.

We shall show that a finite Cartesian product of compact spaces is compact. To prove this, we apply the following result, known as the *Tube Lemma*.

Lemma 7.14 *Let X and Y be topological spaces, let K be a compact subset of Y, and U be an open set in $X \times Y$. Let $V = \{x \in X : \{x\} \times K \subset U\}$. Then V is an open set in X.*

Proof Let $x \in V$. For each $y \in K$ there exist open subsets D_y and E_y of X and Y respectively such that $(x, y) \in D_y \times E_y$ and $D_y \times E_y \subset U$. Now there exists a finite set $\{y_1, y_2, \ldots, y_k\}$ of points of K such that $K \subset E_{y_1} \cup E_{y_2} \cup \cdots \cup E_{y_k}$, since K is compact. Set $N_x = D_{y_1} \cap D_{y_2} \cap \cdots \cap D_{y_k}$. Then N_x is an open set in X. Moreover

$$N_x \times K \subset \bigcup_{i=1}^{k} (N_x \times E_{y_i}) \subset \bigcup_{i=1}^{k} (D_{y_i} \times E_{y_i}) \subset U,$$

so that $N_x \subset V$. It follows that V is the union of the open sets N_x for all $x \in V$. Thus V is itself an open set in X, as required. ∎

Theorem 7.15 *A Cartesian product of a finite number of compact spaces is itself compact.*

Proof It suffices to prove that the product of two compact topological spaces X and Y is compact, since the general result then follows easily by induction on the number of compact spaces in the product.

Let $\mathcal{U}$ be an open cover of $X \times Y$. We must show that this open cover possesses a finite subcover.

Let x be a point of X. The set $\{x\} \times Y$ is a compact subset of $X \times Y$, since it is the image of the compact space Y under the continuous map from Y to $X \times Y$ which sends $y \in Y$ to (x, y), and the image of any compact set under a continuous map is itself compact (Lemma 7.4). Therefore there exists a finite collection $U_1, U_2, \ldots, U_r$ of open sets belonging to the open cover $\mathcal{U}$ such that $\{x\} \times Y$ is contained in $U_1 \cup U_2 \cup \cdots \cup U_r$. Let V_x denote the set of all points x' of X for which $\{x'\} \times Y$ is contained in $U_1 \cup U_2 \cup \cdots \cup U_r$. Then $x \in V_x$, and Lemma 7.14 ensures that V_x is an open set in X. Note that $V_x \times Y$ is covered by finitely many of the open sets belonging to the open cover $\mathcal{U}$.

Now $\{V_x : x \in X\}$ is an open cover of the space X. It follows from the compactness of X that there exists a finite set $\{x_1, x_2, \ldots, x_r\}$ of points of X such that $X = V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_r}$. Now $X \times Y$ is the union of the sets $V_{x_j} \times Y$ for $j = 1, 2, \ldots, r$, and each of these sets can be covered by a finite collection of open sets belonging to the open cover $\mathcal{U}$. On combining these finite collections, we obtain a finite collection of open sets belonging to $\mathcal{U}$ which covers $X \times Y$. This shows that $X \times Y$ is compact. ∎

Theorem 7.16 *Let K be a subset of $\mathbb{R}^n$. Then K is compact if and only if K is both closed and bounded.*

Proof Suppose that K is compact. Then K is closed, since $\mathbb{R}^n$ is Hausdorff, and a compact subset of a Hausdorff space is closed (by Corollary 7.9). For each natural number m, let B_m be the open ball of radius m about the origin, given by $B_m = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < m\}$. Then $\{B_m : m \in \mathbb{N}\}$ is an open cover of $\mathbb{R}^n$. It follows from the compactness of K that there exist natural numbers $m_1, m_2, \ldots, m_k$ such that $K \subset B_{m_1} \cup B_{m_2} \cup \cdots \cup B_{m_k}$. But then $K \subset B_M$, where M is the maximum of $m_1, m_2, \ldots, m_k$, and thus K is bounded.

Conversely suppose that K is both closed and bounded. Then there exists some real number L such that K is contained within the closed cube C given by

$$C = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : -L \leq x_j \leq L \text{ for } j = 1, 2, \ldots, n\}.$$

Now the closed interval $[-L, L]$ is compact by the Heine-Borel Theorem (Theorem 7.2), and C is the Cartesian product of n copies of the compact set $[-L, L]$. It follows from Theorem 7.15 that C is compact. But K is a closed subset of C, and a closed subset of a compact topological space is itself compact, by Lemma 7.3. Thus K is compact, as required. ∎

7.2 Compact Metric Spaces

We recall that a metric or topological space is said to be *compact* if every open cover of the space has a finite subcover. We shall obtain some equivalent characterizations of compactness for *metric spaces* (Theorem 7.22); these characterizations do not generalize to arbitrary topological spaces.

Proposition 7.17 *Every sequence of points in a compact metric space has a convergent subsequence.*

Proof Let X be a compact metric space, and let $x_1, x_2, x_3, \ldots$ be a sequence of points of X. We must show that this sequence has a convergent subsequence. Let F_n denote the closure of $\{x_n, x_{n+1}, x_{n+2}, \ldots\}$. We claim that the intersection of the sets $F_1, F_2, F_3, \ldots$ is non-empty. For suppose that this intersection were the empty set. Then X would be the union of the sets $V_1, V_2, V_3, \ldots$, where $V_n = X \setminus F_n$ for all n. But $V_1 \subset V_2 \subset V_3 \subset \cdots$, and each set V_n is open. It would therefore follow from the compactness of X that X would be covered by finitely many of the sets $V_1, V_2, V_3, \ldots$, and therefore $X = V_n$ for some sufficiently large n. But this is impossible, since F_n is non-empty for all natural numbers n. Thus the intersection of the sets $F_1, F_2, F_3, \ldots$ is non-empty, as claimed, and therefore there exists a point p of X which belongs to F_n for all natural numbers n.

We now obtain, by induction on n, a subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ which satisfies $d(x_{n_j}, p) < 1/j$ for all natural numbers j. Now p belongs to the closure F_1 of the set $\{x_1, x_2, x_3, \ldots\}$. Therefore there exists some natural number n_1 such that $d(x_{n_1}, p) < 1$. Suppose that x_{n_j} has been chosen so that $d(x_{n_j}, p) < 1/j$. The point p belongs to the closure F_{n_j+1} of the set $\{x_n : n > n_j\}$. Therefore there exists some natural number n_{j+1} such that $n_{j+1} > n_j$ and $d(x_{n_{j+1}}, p) < 1/(j+1)$. The subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ constructed in this manner converges to the point p, as required. ∎

We shall also prove the converse of Proposition 7.17: if X is a metric space, and if every sequence of points of X has a convergent subsequence, then X is compact (see Theorem 7.22 below).

Let X be a metric space with distance function d. A *Cauchy sequence* in X is a sequence $x_1, x_2, x_3, \ldots$ of points of X with the property that, given any $\varepsilon > 0$, there exists some natural number N such that $d(x_j, x_k) < \varepsilon$ for all j and k satisfying $j \geq N$ and $k \geq N$.

A metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges to some point of X.

Proposition 7.18 *Let X be a metric space with the property that every sequence of points of X has a convergent subsequence. Then X is complete.*

Proof Let $x_1, x_2, x_3, \ldots$ be a Cauchy sequence in X. This sequence then has a subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ which converges to some point p of X. We claim that the given Cauchy sequence also converges to p.

Let $\varepsilon > 0$ be given. Then there exists some natural number N such that $d(x_m, x_n) < \frac{1}{2}\varepsilon$ whenever $m \geq N$ and $n \geq N$, since $x_1, x_2, x_3, \ldots$ is a Cauchy sequence. Moreover n_j can be chosen large enough to ensure that $n_j \geq N$ and $d(x_{n_j}, p) < \frac{1}{2}\varepsilon$. If $n \geq N$ then

$$d(x_n, p) \leq d(x_n, x_{n_j}) + d(x_{n_j}, p) < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

This shows that the Cauchy sequence $x_1, x_2, x_3, \ldots$ converges to the point p. Thus X is complete, as required. ∎

Definition Let X be a metric space with distance function d. A subset A of X is said to be *bounded* if there exists a non-negative real number K such that $d(x, y) \leq K$ for all $x, y \in A$. The smallest real number K with this property is referred to as the *diameter* of A, and is denoted by $\operatorname{diam} A$. (Note that $\operatorname{diam} A$ is the supremum of the values of $d(x, y)$ as x and y range over all points of A.)

Let X be a metric space with distance function d, and let A be a subset of X. The *closure* $\overline{A}$ of A is the intersection of all closed sets in X that contain the set A: it can be regarded as the smallest closed set in X containing A. Let x be a point of the closure $\overline{A}$ of A. Given any $\varepsilon > 0$, there exists some point x' of A such that $d(x, x') < \varepsilon$. (Indeed the open ball in X of radius ε about the point x must intersect the set A, since otherwise the complement of this open ball would be a closed set in X containing the set A but not including the point x, which is not possible if x belongs to the closure of A.)

Lemma 7.19 *Let X be a metric space, and let A be a subset of X. Then* $\operatorname{diam} A = \operatorname{diam} \overline{A}$*, where $\overline{A}$ is the closure of A.*

Proof Clearly $\operatorname{diam} A \leq \operatorname{diam} \overline{A}$. Let x and y be points of $\overline{A}$. Then, given any $\varepsilon > 0$, there exist points x' and y' of A satisfying $d(x, x') < \varepsilon$ and $d(y, y') < \varepsilon$. It follows from the Triangle Inequality that

$$d(x, y) \leq d(x, x') + d(x', y') + d(y', y) < \operatorname{diam} A + 2\varepsilon.$$

Thus $d(x, y) < \operatorname{diam} A + 2\varepsilon$ for all $\varepsilon > 0$, and hence $d(x, y) \leq \operatorname{diam} A$. This shows that $\operatorname{diam} \overline{A} \leq \operatorname{diam} A$, as required. ∎

Definition A metric space X is said to be *totally bounded* if, given any $\varepsilon > 0$, the set X can be expressed as a finite union of subsets of X, each of which has diameter less than ε.

A subset A of a totally bounded metric space X is itself totally bounded. For if X is the union of the subsets $B_1, B_2, \ldots, B_k$, where $\operatorname{diam} B_n < \varepsilon$ for $n = 1, 2, \ldots, k$, then A is the union of $A \cap B_n$ for $n = 1, 2, \ldots, k$, and $\operatorname{diam} A \cap B_n < \varepsilon$.

Proposition 7.20 *Let X be a metric space. Suppose that every sequence of points of X has a convergent subsequence. Then X is totally bounded.*

Proof Suppose that X were not totally bounded. Then there would exist some $\varepsilon > 0$ with the property that no finite collection of subsets of X of diameter less than 3ε covers the set X. There would then exist an infinite sequence $x_1, x_2, x_3, \ldots$ of points of X with the property that $d(x_m, x_n) \geq \varepsilon$ whenever $m \neq n$. Indeed suppose that points $x_1, x_2, \ldots, x_{k-1}$ of X have already been chosen satisfying $d(x_m, x_n) \geq \varepsilon$ whenever $m < k$, $n < k$ and $m \neq n$. The diameter of each open ball $B_X(x_m, \varepsilon)$ is less than or equal to 2ε. Therefore X could not be covered by the sets $B_X(x_m, \varepsilon)$ for $m < k$, and thus there would exist a point x_k of X which does not belong to $B(x_m, \varepsilon)$ for any $m < k$. Then $d(x_m, x_k) \geq \varepsilon$ for all $m < k$. In this way we can successively choose points $x_1, x_2, x_3, \ldots$ to form an infinite sequence with the required property. However such an infinite sequence would have no convergent subsequence, which is impossible. This shows that X must be totally bounded, as required. ∎

Proposition 7.21 *Every complete totally bounded metric space is compact.*

Proof Let X be some totally bounded metric space. Suppose that there exists an open cover $\mathcal{V}$ of X which has no finite subcover. We shall prove the existence of a Cauchy sequence $x_1, x_2, x_3, \ldots$ in X which cannot converge to any point of X. (Thus if X is not compact, then X cannot be complete.)

Let $\varepsilon > 0$ be given. Then X can be covered by finitely many closed sets whose diameter is less than ε, since X is totally bounded and every subset of X has the same diameter as its closure (Lemma 7.19). At least one of these closed sets cannot be covered by a finite collection of open sets belonging to $\mathcal{V}$ (since if every one of these closed sets could be covered by a such a finite collection of open sets, then we could combine these collections to obtain a finite subcover of $\mathcal{V}$). We conclude that, given any $\varepsilon > 0$, there exists a closed subset of X of diameter less than ε which cannot be covered by any finite collection of open sets belonging to $\mathcal{V}$.

We claim that there exists a sequence $F_1, F_2, F_3, \ldots$ of closed sets in X satisfying $F_1 \supset F_2 \supset F_3 \supset \cdots$ such that each closed set F_n has the following properties: $\operatorname{diam} F_n < 1/2^n$, and no finite collection of open sets belonging to $\mathcal{V}$ covers F_n. For if F_n is a closed set with these properties then F_n is itself totally bounded, and thus the above remarks (applied with F_n in place of X) guarantee the existence of a closed subset F_{n+1} of F_n with the required properties. Thus the existence of the required sequence of closed sets follows by induction on n.

Choose $x_n \in F_n$ for each natural number n. Then $d(x_m, x_n) < 1/2^n$ for any $m > n$, since x_m and x_n belong to F_n and $\operatorname{diam} F_n < 1/2^n$. Therefore the sequence $x_1, x_2, x_3, \ldots$ is a Cauchy sequence. Suppose that this Cauchy sequence were to converge to some point p of X. Then $p \in F_n$ for each natural number n, since F_n is closed and $x_m \in F_n$ for all $m \geq n$. (If a sequence of points belonging to a closed subset of a metric or topological space is convergent then the limit of that sequence belongs to the closed set.) Moreover $p \in V$ for some open set V belonging to $\mathcal{V}$, since $\mathcal{V}$ is an open cover of X. But then there would exist $\delta > 0$ such that $B_X(p, \delta) \subset V$, where $B_X(p, \delta)$ denotes the open ball of radius δ in X centred on p. Thus if n were large enough to ensure that $1/2^n < \delta$, then $p \in F_n$ and $\operatorname{diam} F_n < \delta$, and hence $F_n \subset B_X(p, \delta) \subset V$, contradicting the fact that no finite collection of open sets belonging to $\mathcal{V}$ covers the set F_n. This contradiction shows that the Cauchy sequence $x_1, x_2, x_3, \ldots$ is not convergent.

We have thus shown that if X is a totally bounded metric space which is not compact then X is not complete. Thus every complete totally bounded metric space must be compact, as required. ∎

Theorem 7.22 *Let X be a metric space with distance function d. The following are equivalent:—*

(i) *X is compact,*

(ii) *every sequence of points of X has a convergent subsequence,*

(iii) *X is complete and totally bounded,*

Proof Propositions 7.17, 7.18 7.20 and 7.21 show that (i) implies (ii), (ii) implies (iii), and (iii) implies (i). It follows that (i), (ii) and (iii) are all equivalent to one another. ∎

Remark A subset K of $\mathbb{R}^n$ is complete if and only if it is closed in $\mathbb{R}^n$. Also it is easy to see that K is totally bounded if and only if K is a bounded subset of $\mathbb{R}^n$. Thus Theorem 7.22 is a generalization of the theorem which states that a subset K of $\mathbb{R}^n$ is compact if and only if it is both closed and bounded (Theorem 7.16).

7.3 The Lebesgue Lemma and Uniform Continuity

Lemma 7.23 (Lebesgue Lemma) *Let (X, d) be a compact metric space. Let $\mathcal{U}$ be an open cover of X. Then there exists a positive real number δ such that every subset of X whose diameter is less than δ is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$.*

Proof Every point of X is contained in at least one of the open sets belonging to the open cover $\mathcal{U}$. It follows from this that, for each point x of X, there exists some $\delta_x > 0$ such that the open ball $B(x, 2\delta_x)$ of radius $2\delta_x$ about the point x is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$. But then the collection consisting of the open balls $B(x, \delta_x)$ of radius δ_x about the points x of X forms an open cover of the compact space X. Therefore there exists a finite set $x_1, x_2, \ldots, x_r$ of points of X such that

$$B(x_1, \delta_1) \cup B(x_2, \delta_2) \cup \cdots \cup B(x_r, \delta_r) = X,$$

where $\delta_i = \delta_{x_i}$ for $i = 1, 2, \ldots, r$. Let $\delta > 0$ be given by

$$\delta = \text{minimum}(\delta_1, \delta_2, \ldots, \delta_r).$$

Suppose that A is a subset of X whose diameter is less than δ. Let u be a point of A. Then u belongs to $B(x_i, \delta_i)$ for some integer i between 1 and r. But then it follows that $A \subset B(x_i, 2\delta_i)$, since, for each point v of A,

$$d(v, x_i) \leq d(v, u) + d(u, x_i) < \delta + \delta_i \leq 2\delta_i.$$

But $B(x_i, 2\delta_i)$ is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$. Thus A is contained wholly within one of the open sets belonging to $\mathcal{U}$, as required. ∎

Let $\mathcal{U}$ be an open cover of a compact metric space X. A *Lebesgue number* for the open cover $\mathcal{U}$ is a positive real number δ such that every subset of X whose diameter is less than δ is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$. The Lebesgue Lemma thus states that there exists a Lebesgue number for every open cover of a compact metric space.

Let X and Y be metric spaces with distance functions d_X and d_Y respectively, and let $f\colon X \to Y$ be a function from X to Y. The function f is said to be *uniformly continuous* on X if and only if, given $\varepsilon > 0$, there exists some $\delta > 0$ such that $d_Y(f(x), f(x')) < \varepsilon$ for all points x and x' of X satisfying $d_X(x, x') < \delta$. (The value of δ should be independent of both x and x'.)

Theorem 7.24 *Let X and Y be metric spaces. Suppose that X is compact. Then every continuous function from X to Y is uniformly continuous.*

Proof Let d_X and d_Y denote the distance functions for the metric spaces X and Y respectively. Let $f\colon X \to Y$ be a continuous function from X to Y. We must show that f is uniformly continuous.

Let $\varepsilon > 0$ be given. For each $y \in Y$, define

$$V_y = \{x \in X : d_Y(f(x), y) < \tfrac{1}{2}\varepsilon\}.$$

Note that $V_y = f^{-1}\left(B_Y(y, \frac{1}{2}\varepsilon)\right)$, where $B_Y(y, \frac{1}{2}\varepsilon)$ denotes the open ball of radius $\frac{1}{2}\varepsilon$ about y in Y. Now the open ball $B_Y(y, \frac{1}{2}\varepsilon)$ is an open set in Y, and f is continuous. Therefore V_y is open in X for all $y \in Y$. Note that $x \in V_{f(x)}$ for all $x \in X$.

Now $\{V_y : y \in Y\}$ is an open cover of the compact metric space X. It follows from the Lebesgue Lemma (Lemma 7.23) that there exists some $\delta > 0$ such that every subset of X whose diameter is less than δ is a subset of some set V_y. Let x and x' be points of X satisfying $d_X(x, x') < \delta$. The diameter of the set $\{x, x'\}$ is $d_X(x, x')$, which is less than δ. Therefore there exists some $y \in Y$ such that $x \in V_y$ and $x' \in V_y$. But then $d_Y(f(x), y) < \frac{1}{2}\varepsilon$ and $d_Y(f(x'), y) < \frac{1}{2}\varepsilon$, and hence

$$d_Y(f(x), f(x')) \leq d_Y(f(x), y) + d_Y(y, f(x')) < \varepsilon.$$

This shows that $f\colon X \to Y$ is uniformly continuous, as required. ∎

Let K be a closed bounded subset of $\mathbb{R}^n$. It follows from Theorem 7.16) and Theorem 7.24 that any continuous function $f\colon K \to \mathbb{R}^k$ is uniformly continuous.

8 Connectedness

8.1 Connected Topological Spaces

Definition A topological space X is said to be *connected* if the empty set $\emptyset$ and the whole space X are the only subsets of X that are both open and closed.

Lemma 8.1 *A topological space X is connected if and only if it has the following property: if U and V are non-empty open sets in X such that $X = U \cup V$, then $U \cap V$ is non-empty.*

Proof If U is a subset of X that is both open and closed, and if $V = X \setminus U$, then U and V are both open, $U \cup V = X$ and $U \cap V = \emptyset$. Conversely if U and V are open subsets of X satisfying $U \cup V = X$ and $U \cap V = \emptyset$, then

$U = X \setminus V$, and hence U is both open and closed. Thus a topological space X is connected if and only if there do not exist non-empty open sets U and V such that $U \cup V = X$ and $U \cap V = \emptyset$. The result follows. ∎

Let $\mathbb{Z}$ be the set of integers with the usual topology (i.e., the subspace topology on $\mathbb{Z}$ induced by the usual topology on $\mathbb{R}$). Then $\{n\}$ is open for all $n \in \mathbb{Z}$, since

$$\{n\} = \mathbb{Z} \cap \{t \in \mathbb{R} : |t - n| < \tfrac{1}{2}\}.$$

It follows that every subset of $\mathbb{Z}$ is open (since it is a union of sets consisting of a single element, and any union of open sets is open). It follows that a function $f: X \to \mathbb{Z}$ on a topological space X is continuous if and only if $f^{-1}(V)$ is open in X for any subset V of $\mathbb{Z}$. We use this fact in the proof of the next theorem.

Proposition 8.2 *A topological space X is connected if and only if every continuous function $f: X \to \mathbb{Z}$ from X to the set $\mathbb{Z}$ of integers is constant.*

Proof Suppose that X is connected. Let $f: X \to \mathbb{Z}$ be a continuous function. Choose $n \in f(X)$, and let

$$U = \{x \in X : f(x) = n\}, \qquad V = \{x \in X : f(x) \neq n\}.$$

Then U and V are the preimages of the open subsets $\{n\}$ and $\mathbb{Z} \setminus \{n\}$ of $\mathbb{Z}$, and therefore both U and V are open in X. Moreover $U \cap V = \emptyset$, and $X = U \cup V$. It follows that $V = X \setminus U$, and thus U is both open and closed. Moreover U is non-empty, since $n \in f(X)$. It follows from the connectedness of X that $U = X$, so that $f: X \to \mathbb{Z}$ is constant, with value n.

Conversely suppose that every continuous function $f: X \to \mathbb{Z}$ is constant. Let S be a subset of X which is both open and closed. Let $f: X \to \mathbb{Z}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in S; \\ 0 & \text{if } x \notin S. \end{cases}$$

Now the preimage of any subset of $\mathbb{Z}$ under f is one of the open sets $\emptyset$, S, $X \setminus S$ and X. Therefore the function f is continuous. But then the function f is constant, so that either $S = \emptyset$ or $S = X$. This shows that X is connected. ∎

Lemma 8.3 *The closed interval $[a, b]$ is connected, for all real numbers a and b satisfying $a \leq b$.*

Proof Let $f\colon [a,b] \to \mathbb{Z}$ be a continuous integer-valued function on $[a,b]$. We show that f is constant on $[a,b]$. Indeed suppose that f were not constant. Then $f(\tau) \neq f(a)$ for some $\tau \in [a,b]$. But the Intermediate Value Theorem would then ensure that, given any real number c between $f(a)$ and $f(\tau)$, there would exist some $t \in [a,\tau]$ for which $f(t) = c$, and this is clearly impossible, since f is integer-valued. Thus f must be constant on $[a,b]$. We now deduce from Proposition 8.2 that $[a,b]$ is connected. ∎

Example Let $X = \{(x,y) \in \mathbb{R}^2 : x \neq 0\}$. The topological space X is not connected. Indeed if $f\colon X \to \mathbb{Z}$ is defined by

$$f(x,y) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \end{cases}$$

then f is continuous on X but is not constant.

A concept closely related to that of connectedness is *path-connectedness*. Let x_0 and x_1 be points in a topological space X. A *path* in X from x_0 to x_1 is defined to be a continuous function $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. A topological space X is said to be *path-connected* if and only if, given any two points x_0 and x_1 of X, there exists a path in X from x_0 to x_1.

Proposition 8.4 *Every path-connected topological space is connected.*

Proof Let X be a path-connected topological space, and let $f\colon X \to \mathbb{Z}$ be a continuous integer-valued function on X. If x_0 and x_1 are any two points of X then there exists a path $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. But then $f \circ \gamma\colon [0,1] \to \mathbb{Z}$ is a continuous integer-valued function on $[0,1]$. But $[0,1]$ is connected (Lemma 8.3), therefore $f \circ \gamma$ is constant (Proposition 8.2). It follows that $f(x_0) = f(x_1)$. Thus every continuous integer-valued function on X is constant. Therefore X is connected, by Proposition 8.2. ∎

The topological spaces $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{R}^n$ are all path-connected. Indeed, given any two points of one of these spaces, the straight line segment joining these two points is a continuous path from one point to the other. Also the n-sphere S^n is path-connected for all $n > 0$. We conclude that these topological spaces are connected.

Let A be a subset of a topological space X. Using Lemma 8.1 and the definition of the subspace topology, we see that A is connected if and only if the following condition is satisfied:

- if U and V are open sets in X such that $A \cap U$ and $A \cap V$ are non-empty and $A \subset U \cup V$ then $A \cap U \cap V$ is also non-empty.

Lemma 8.5 *Let X be a topological space and let A be a connected subset of X. Then the closure $\overline{A}$ of A is connected.*

Proof It follows from the definition of the closure of A that $\overline{A} \subset F$ for any closed subset F of X for which $A \subset F$. On taking F to be the complement of some open set U, we deduce that $\overline{A} \cap U = \emptyset$ for any open set U for which $A \cap U = \emptyset$. Thus if U is an open set in X and if $\overline{A} \cap U$ is non-empty then $A \cap U$ must also be non-empty.

Now let U and V be open sets in X such that $\overline{A} \cap U$ and $\overline{A} \cap V$ are non-empty and $\overline{A} \subset U \cup V$. Then $A \cap U$ and $A \cap V$ are non-empty, and $A \subset U \cup V$. But A is connected. Therefore $A \cap U \cap V$ is non-empty, and thus $\overline{A} \cap U \cap V$ is non-empty. This shows that $\overline{A}$ is connected. ∎

Lemma 8.6 *Let $f: X \to Y$ be a continuous function between topological spaces X and Y, and let A be a connected subset of X. Then $f(A)$ is connected.*

Proof Let $g: f(A) \to \mathbb{Z}$ be any continuous integer-valued function on $f(A)$. Then $g \circ f: A \to \mathbb{Z}$ is a continuous integer-valued function on A. It follows from Proposition 8.2 that $g \circ f$ is constant on A. Therefore g is constant on $f(A)$. We deduce from Proposition 8.2 that $f(A)$ is connected. ∎

Lemma 8.7 *The Cartesian product $X \times Y$ of connected topological spaces X and Y is itself connected.*

Proof Let $f: X \times Y \to \mathbb{Z}$ be a continuous integer-valued function from $X \times Y$ to Z. Choose $x_0 \in X$ and $y_0 \in Y$. The function $x \mapsto f(x, y_0)$ is continuous on X, and is thus constant. Therefore $f(x, y_0) = f(x_0, y_0)$ for all $x \in X$. Now fix x. The function $y \mapsto f(x, y)$ is continuous on Y, and is thus constant. Therefore

$$f(x, y) = f(x, y_0) = f(x_0, y_0)$$

for all $x \in X$ and $y \in Y$. We deduce from Proposition 8.2 that $X \times Y$ is connected. ∎

We deduce immediately that a finite Cartesian product of connected topological spaces is connected.

Proposition 8.8 *Let X be a topological space. For each $x \in X$, let S_x be the union of all connected subsets of X that contain x. Then*

(i) *S_x is connected,*

(ii) *S_x is closed,*

(iii) *if $x, y \in X$, then either $S_x = S_y$, or else $S_x \cap S_y = \emptyset$.*

Proof Let $f\colon S_x \to \mathbb{Z}$ be a continuous integer-valued function on S_x, for some $x \in X$. Let y be any point of S_x. Then, by definition of S_x, there exists some connected set A containing both x and y. But then f is constant on A, and thus $f(x) = f(y)$. This shows that the function f is constant on S_x. We deduce that S_x is connected. This proves (i). Moreover the closure $\overline{S_x}$ is connected, by Lemma 8.5. Therefore $\overline{S_x} \subset S_x$. This shows that S_x is closed, proving (ii).

Finally, suppose that x and y are points of X for which $S_x \cap S_y \neq \emptyset$. Let $f\colon S_x \cup S_y \to \mathbb{Z}$ be any continuous integer-valued function on $S_x \cup S_y$. Then f is constant on both S_x and S_y. Moreover the value of f on S_x must agree with that on S_y, since $S_x \cap S_y$ is non-empty. We deduce that f is constant on $S_x \cup S_y$. Thus $S_x \cup S_y$ is a connected set containing both x and y, and thus $S_x \cup S_y \subset S_x$ and $S_x \cup S_y \subset S_y$, by definition of S_x and S_y. We conclude that $S_x = S_y$. This proves (iii). ∎

Given any topological space X, the connected subsets S_x of X defined as in the statement of Proposition 8.8 are referred to as the *connected components* of X. We see from Proposition 8.8, part (iii) that the topological space X is the disjoint union of its connected components.

Example The connected components of $\{(x, y) \in \mathbb{R}^2 : x \neq 0\}$ are

$$\{(x, y) \in \mathbb{R}^2 : x > 0\} \text{ and } \{(x, y) \in \mathbb{R}^2 : x < 0\}.$$

Example The connected components of

$$\{t \in \mathbb{R} : |t - n| < \tfrac{1}{2} \text{ for some integer } n\}.$$

are the sets J_n for all $n \in \mathbb{Z}$, where $J_n = (n - \frac{1}{2}, n + \frac{1}{2})$.