Problem Set I

1. Let $X$ and $Y$ be metric spaces with distance functions $d_X$ and $d_Y$ respectively, let $f: X \to Y$ be a function from $X$ to $Y$, and let $p$ be a point of $X$. Suppose that, given any sequence $x_1, x_2, x_3, \ldots$ of points of $X$ converging to $p$, the sequence $f(x_1), f(x_2), f(x_3), \ldots$ converges to $f(p)$. Prove that the function $f$ is continuous at $p$. [Hint: note that the function $f$ fails to be continuous at $p$ if and only if there exists $\varepsilon > 0$ such that, given any $\delta > 0$, there exists $x \in X$ satisfying $d_X(x, p) < \delta$ and $d_Y(f(x), f(p)) \geq \varepsilon$; apply this result with $\delta = 1/n$ for each natural number $n$.]

2. Let $X$ be a set, and let $d$ be the function on $X \times X$ defined by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y; \\ 0 & \text{if } x = y. \end{cases}$$

Prove that $X$ is a metric space with respect to the distance function $d$. Show also that the topology on $X$ generated by the distance function $d$ is the discrete topology on $X$ (i.e., show that every subset of $X$ is an open set).

3. Let $X$ be a topological space whose only open sets are $\emptyset$ and $X$ itself. Show that $X$ is Hausdorff if and only if $X$ consists of a single point.

4. A topological space $X$ is said to be sequentially compact if every sequence of points in $X$ has a convergent subsequence. Every closed bounded set in $\mathbb{R}^n$ is sequentially compact.

(a) Let $X$ be a sequentially compact topological space, and let $F$ be a closed subset of $X$. Prove that $F$ is sequentially compact.

(b) Let $f: X \to Y$ be a continuous map from a between topological spaces $X$ and $Y$. Suppose that $X$ is sequentially compact. Prove that the image $f(X)$ of the map $f$ is sequentially compact.

(c) Let $X$ be a sequentially compact topological space and let $f: X \to \mathbb{R}$ be a continuous function from $X$ to (where is given the usual topology). Prove that the function $f$ is bounded (i.e., there exists some constant $K$ such that $|f(x)| \leq K$ for all $x \in X$). Show also that there exist points $u$ and $v$ of $X$ such that $f(u) \leq f(x) \leq f(v)$ for all $x \in X$. [Hint: generalize corresponding results towards the end of §8 of Course 121.]
(The concept of *sequential compactness* introduced in this question should not be confused with the concept of *compactness* discussed in lectures towards the end of the Michaelmas term. It can however be proved that the concepts of *sequential compactness* and *compactness* coincide for metric spaces, though the proof is somewhat lengthy.)
Course 212—Academic year 1991-92
Problem Set II
To be handed in by FRIDAY 29th NOVEMBER

1. (a) Let \( f : X \to Y \) be a continuous map between topological spaces \( X \) and \( Y \). Suppose that \( X \) is path-connected. Prove that the image \( f(X) \) of the map \( f \) is also path-connected.

(b) Let \( X \) and \( Y \) be path-connected topological spaces. Prove that the Cartesian product \( X \times Y \) of \( X \) and \( Y \) is path-connected.

2. Determine the connected components of the following subsets of \( \mathbb{R}^2 \):
   
   (i) \( \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \);
   
   (ii) \( \{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = 1\} \);
   
   (iii) \( \{(x, y) \in \mathbb{R}^2 : y^2 = x(x^2 - 1)\} \);
   
   (iv) \( \{(x, y) \in \mathbb{R}^2 : (x - n)^2 + y^2 > \frac{1}{4} \text{ for all } n \in \mathbb{Z}\} \).

   [Fully justify your answers.]

3. Let \( X_+ \), \( X_- \) and \( X_0 \) be the subsets of \( \mathbb{R}^2 \) defined by

   \[
   X_+ = \left\{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } y = \sin \left( \frac{1}{x} \right) \right\},
   \]

   \[
   X_- = \left\{(x, y) \in \mathbb{R}^2 : x < 0 \text{ and } y = \sin \left( \frac{1}{x} \right) \right\},
   \]

   \[
   X_0 = \left\{(x, y) \in \mathbb{R}^2 : x = 0 \text{ and } -1 \leq y \leq 1 \right\},
   \]

   and let \( X = X_+ \cup X_0 \cup X_- \).

   (a) Explain why \( X_+ \), \( X_- \) and \( X_0 \) are path-connected.

   (b) Are \( X_+ \), \( X_- \) and \( X_0 \) connected? [Justify your answer.]

   (c) Show that the point \((0, 0)\) of \( X_0 \) belongs to the closures of \( X_+ \) and \( X_- \) in \( X \). [Hint: find sequences in \( X_+ \) and \( X_- \) which converge to \((0, 0)\), and use a result proved in §2 of the course concerning topological spaces.]

   (d) Explain why \((0, 0)\) belongs to the connected component of \( X \) containing the set \( X_+ \) and also to the connected component of \( X \) containing the set \( X_- \). Hence prove that the sets \( X_+ \), \( X_- \) and \( X_0 \) are contained in the same connected component of \( X \), showing
that $X$ is a connected topological space. [Hint: use results proved in §3 of the course concerning connected topological spaces.]

Let $\gamma : [0,1] \to X$ be a path in $X$, and let

$$S = \{ t \in [0,1] : \gamma(t) \in X_0 \}.$$

(e) Explain why $S$ is closed in $[0,1]$. [Hint: show that $S$ is the preimage of a closed set in $X$.]

(f) Suppose that $\gamma(s) \in X_0$ for some $s \in [0,1]$. Show that there exists some $\delta > 0$ such that $|\gamma(t) - \gamma(s)| < \frac{1}{2}$ for all $t \in [0,1]$ satisfying $|t - s| < \delta$. Then, by using the Intermediate Value Theorem, or otherwise, deduce that $\gamma(t) \in X_0$ for all $t \in [0,1]$ satisfying $|t - s| < \delta$, showing that $S$ is open in $[0,1]$.

(g) By using (e), (f) and the connectedness of the interval $[0,1]$, or otherwise, show that if $\gamma(s) \in X_0$ for some $s \in [0,1]$ then $\gamma([0,1]) \subset X_0$.

(h) Is the topological space $X$ path-connected? [Justify your answer.]

4. Let $f : [a,b] \to$ be a continuous function on the closed bounded interval $[a,b]$, where $a$ and $b$ are real numbers satisfying $a < b$.

(a) Explain why, given any $s \in [a,b]$, there exists some $\delta(s) > 0$ such that $|f(t) - f(s)| < \frac{1}{2} \varepsilon$ for all $t \in [a,b]$ satisfying $|t - s| < 2\delta(s)$.

(b) Show that there exists an finite set $\{s_1, s_2, \ldots, s_k\}$ of real numbers belonging to $[a,b]$ such that, given any $t \in [a,b]$, there exists some $s_j$ in this set such that $|t - s_j| < \delta(s_j)$. [Hint: apply the Heine-Borel Theorem.]

(c) Let $\delta$ be the minimum of $\delta(s_1), \delta(s_2), \ldots, \delta(s_k)$. Show that if $t$ and $u$ are elements of $[a,b]$ satisfying $|t - u| < \delta$ then $|f(t) - f(u)| < \varepsilon$. 

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