Course 212: Academic Year 1991-2
Section 8: The Exponential Map

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8 The Exponential Map

8.1 The Exponential Map on the Complex Plane

The exponential map $\exp\colon \mathbb{C} \to \mathbb{C}$ is defined for all $z \in \mathbb{C}$ by the formula

$$\exp(z) = \sum_{n=0}^{+\infty} \frac{z^n}{n!}.$$

A straightforward application of the Ratio Test shows that this power series converges for all $z \in \mathbb{C}$, and converges uniformly on any disk of finite radius, centred on 0. It follows from this that the exponential function is continuous on $\mathbb{C}$. A general result on products of absolutely convergent infinite series can be used to show that

$$\exp(z + w) = \exp(z)\exp(w)$$

for all $z, w \in \mathbb{C}$. It follows immediately from this that $\exp(z) \neq 0$ for all $z \in \mathbb{C}$, and $\exp(-z) = 1/\exp(z)$, since

$$\exp(z)\exp(-z) = \exp(z - z) = \exp(0) = 1.$$

Note that, for any real number y,

$$\begin{aligned}
\exp(iy) &= \sum_{m=0}^{+\infty} \frac{(iy)^{2m}}{(2m)!} + \sum_{m=0}^{+\infty} \frac{(iy)^{2m+1}}{(2m+1)!} \\
&= \sum_{m=0}^{+\infty} \frac{(-1)^m y^{2m}}{(2m)!} + i \sum_{m=0}^{+\infty} \frac{(-1)^m y^{2m+1}}{(2m+1)!} \\
&= \cos y + i \sin y
\end{aligned}$$

(where we have split up the infinite series defining $\exp(iy)$ into separate summations over the even and the odd powers of z, and used the Taylor expansions for the sine and cosine functions derived using Taylor's Theorem). This identity is known as *de Moivre's Theorem*. It follows that

$$\exp(x + iy) = e^x(\cos y + i \sin y)$$

for all $x, y \in \mathbb{R}$. Also

$$\cos y = \frac{1}{2}(\exp(iy) + \exp(-iy)), \qquad \sin y = \frac{1}{2i}(\exp(iy) - \exp(-iy)).$$

Lemma 8.1 *Let z and w be complex numbers. Then $\exp(z) = \exp(w)$ if and only if $w = z + 2\pi i n$ for some integer n.*

Proof If $w = z + 2\pi i n$ for some integer n then

$$\exp(w) = \exp(z)\exp(2\pi i n) = \exp(z)(\cos 2\pi n + i \sin 2\pi n) = \exp(z).$$

Conversely suppose that $\exp(w) = \exp(z)$. Let $w - z = u + iv$, where $u, v \in \mathbb{R}$. Then

$$e^u(\cos v + i \sin v) = \exp(w - z) = \exp(w)\exp(z)^{-1} = 1.$$

Taking the modulus of both sides, we see that $e^u = 1$, and thus $u = 0$. Also $\cos v = 1$ and $\sin v = 0$, and therefore $v = 2\pi n$ for some integer n. The result follows. ∎

Proposition 8.2 *The exponential map* $\exp \mathbb{C} \to \mathbb{C}$ *maps the open strip*

$$\{z \in \mathbb{C} : \alpha < \operatorname{Im} z < \alpha + 2\pi\}$$

homeomorphically onto $\mathbb{C} \setminus L_\alpha$ *for each* $\alpha \in \mathbb{R}$*, where*

$$L_\alpha = \{te^{i\alpha} : t \geq 0\}.$$

Proof Without loss of generality, we may assume that $\alpha = 0$, since $\exp(z) = \exp(i\alpha)\exp(z - i\alpha)$ for all $z \in \mathbb{C}$ and $\alpha \in \mathbb{R}$. Let

$$D = \mathbb{C} \setminus \{t \in \mathbb{R} : t \geq 0\}, \qquad E = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < 2\pi\},$$

and let $F\colon D \to E$ be the function sending $re^{i\theta}$ to $\log r + i\theta$ for all real numbers r and θ satisfying $r > 0$ and $0 < \theta < 2\pi$. Then $F\colon D \to E$ is the inverse of the restriction $\exp|E\colon E \to D$ of the exponential map to E, and thus $\exp|E$ is a bijection. Therefore it only remains to show that the function $F\colon D \to E$ is continuous. Now

$$F(x+iy) = \begin{cases} \frac{1}{2}\log(x^2+y^2) + i \arccos\left(\dfrac{x}{\sqrt{x^2+y^2}}\right) & \text{if } y > 0, \\ \frac{1}{2}\log(x^2+y^2) + 2\pi i - i \arccos\left(\dfrac{x}{\sqrt{x^2+y^2}}\right) & \text{if } y < 0, \\ \frac{1}{2}\log(x^2+y^2) + \pi i - i \arcsin\left(\dfrac{y}{\sqrt{x^2+y^2}}\right) & \text{if } x < 0, \end{cases}$$

where $\arcsin\colon [-1, 1] \to [-\pi/2, -\pi/2]$ and $\arccos\colon [-1, 1] \to [0, \pi]$ are the inverses of the restrictions of the sine and cosine functions to the intervals $[-\pi/2, \pi/2]$ and $[0, \pi]$ respectively. Also the functions log, arcsin and arccos are continuous, since the inverse of any strictly increasing or strictly decreasing function defined over some interval in $\mathbb{R}$ is itself continuous. We conclude that the restrictions of $F\colon D \to E$ to each of the open sets

$$\{z \in \mathbb{C} : \operatorname{Im} z > 0\}, \qquad \{z \in \mathbb{C} : \operatorname{Im} z < 0\}, \qquad \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$$

is continuous, and the domain D of F is the union of these three open sets. It follows that $F\colon D \to E$ is itself continuous. Thus $\exp|E\colon E \to D$ is a homeomorphism, as required. ∎

Definition We say that an open U in $\mathbb{C} \setminus \{0\}$ is *evenly covered* by the exponential map if the preimage $\exp^{-1}(U)$ of U is a disjoint union of open sets in $\mathbb{C}$, each of which is mapped homeomorphically onto U by the exponential map.

Corollary 8.3 *Let U be an open set in $\mathbb{C} \setminus \{0\}$. Suppose that $U \cap L_\alpha = \emptyset$ for some $\alpha \in \mathbb{R}$, where*

$$L_\alpha = \{te^{i\alpha} : t \geq 0\}.$$

Then U is evenly covered by the exponential map.

Proof For each integer n let

$$V_n = \exp^{-1}(U) \cap \{z \in \mathbb{C} : \alpha + 2\pi n < \operatorname{Im} z < \alpha + 2\pi(n+1)\}.$$

Then each V_n is an open set in $\mathbb{C}$ (since it is an intersection of open sets). Moreover $\exp^{-1}(U)$ is the union of the sets V_n, since $U \cap L_\alpha = \emptyset$ and therefore no point z of $\exp^{-1}(U)$ satisfies $\operatorname{Im} z = \alpha + 2\pi n$ for any integer n. Now it follows from Proposition 8.3 that the restriction of the exponential map to the strip

$$\{z \in \mathbb{C} : \alpha + 2\pi n < \operatorname{Im} z < \alpha + 2\pi(n+1)\}.$$

induces a one-to-one correspondence between open subsets of this strip and open subsets of $\mathbb{C} \setminus L_\alpha$. In particular, the open set V_n is mapped homeomorphically onto U for each integer n. Thus the open set U is evenly covered by the exponential map, as required. ∎

Lemma 8.4 *The set $\mathbb{C} \setminus \{0\}$ is not evenly covered by the exponential map.*

Proof Suppose that $\mathbb{C} \setminus \{0\}$ were evenly covered by the exponential map. Then, since the preimage of $\mathbb{C} \setminus \{0\}$ is the whole of the complex plane $\mathbb{C}$, we could express the complex plane as a disjoint union of open sets, each homeomorphic to $\mathbb{C} \setminus \{0\}$. But then each of these open sets would also be closed, since its complement would be the union of the other open sets. But this would contradict the connectedness of $\mathbb{C}$. The result follows. ∎

Lemma 8.5 *Let U be an open set in $\mathbb{C} \setminus \{0\}$ and let $F: U \to \mathbb{C}$ be a continuous map satisfying $\exp(F(z)) = z$ for all $z \in U$. Then $F(U)$ is an open set in $\mathbb{C}$.*

Proof Let $E_\alpha = \{z \in \mathbb{C} : \alpha < \operatorname{Im} z < \alpha + 2\pi\}$ for some real number α. Then

$$\exp(E_\alpha \cap F(U)) = \{z \in U : F(z) \in E_\alpha\} = F^{-1}(E_\alpha).$$

Thus $\exp(E_\alpha \cap F(U))$ is the preimage of the open set E_α under the continuous map $F: U \to \mathbb{C}$, and hence $\exp(E_\alpha \cap F(U))$ is open in U. But any subset of U that is open in U (relative to the subspace topology on U) must also be open in $\mathbb{C}$, since U is itself open in $\mathbb{C}$. Thus $\exp(E_\alpha \cap F(U))$ is an open subset of $\mathbb{C}$, and moreover $\exp(E_\alpha \cap F(U)) \subset \mathbb{C} \setminus L_\alpha$, where $L_\alpha = \{te^{i\alpha} : t \geq 0\}$. But the exponential map induces a one-to-one correspondence between the open sets contained in E_α and $\mathbb{C} \setminus L_\alpha$, since it maps the open set E_α homeomorphically onto the open set $\mathbb{C} \setminus L_\alpha$ (Proposition 8.2). It follows that $E_\alpha \cap F(U)$ is open in $\mathbb{C}$. But then $F(U)$ is the union of the open sets $E_\alpha \cap F(U)$ for all real numbers α, and therefore $F(U)$ is itself an open set, as required. ∎

Theorem 8.6 *An open set U in $\mathbb{C} \setminus \{0\}$ is evenly covered by the exponential map if and only if there exists a continuous map $F: U \to \mathbb{C}$ such that $\exp(F(z)) = z$ for all $z \in U$.*

Proof First suppose that U is evenly covered by the exponential map. Then the preimage $\exp^{-1}(U)$ of U is a disjoint union of open sets, each of which is mapped homeomorphically onto U by the exponential map. Let $\tilde{U}$ be one of these open sets, and let $F: U \to \tilde{U}$ be the inverse of the restriction $\exp|\tilde{U}: \tilde{U} \to U$ of the exponential map to $\tilde{U}$. Then F is continuous, since $\exp|\tilde{U}$ is a homeomorphism, and $\exp(F(z)) = z$ for all $z \in U$.

Conversely suppose that U is open, and that there exists a continuous map $F: U \to \mathbb{C}$ with the property that $\exp(F(z)) = z$ for all $z \in U$. We must show that U is evenly covered by the exponential map. Now $F(U)$ is open in $\mathbb{C}$, by Lemma 8.5, and the continuous function $F: U \to F(U)$ is the inverse of the restriction $\exp|F(U): F(U) \to U$ of the exponential map to $F(U)$. Thus $F(U)$ is mapped homeomorphically onto U by the exponential map. For each integer n, let

$$V_n = \{z \in \mathbb{C} : z - 2\pi i n \in F(U)\}.$$

Then each set V_n is open in $\mathbb{C}$. Now $\exp(z) = \exp(z - 2\pi i n)$ for each integer n. It follows that each set V_n is mapped homeomorphically onto U by the exponential map. Moreover if $z + 2\pi i m = w + 2\pi i n$, where m and n are integers and $z, w \in F(U)$, then $\exp(z) = \exp(w)$, hence $z = w$ (since the exponential map is injective on $F(U)$), and thus $m = n$. We deduce that the sets V_n are disjoint. If $z \in \mathbb{C}$ satisfies $\exp(z) \in U$ then $\exp(z) = \exp(w)$ for some $w \in F(U)$. But then $z = w + 2\pi i n$ for some integer n, and therefore $z \in V_n$. We conclude that the preimage $\exp^{-1}(U)$ of U is the disjoint union of the open sets V_n, and each of these open sets is mapped homeomorphically onto U by the exponential map. Thus U is evenly covered by the exponential map, as required. ∎

Corollary 8.7 *There does not exist any continuous function* $F: \mathbb{C} \setminus \{0\} \to \mathbb{C}$ *such that* $\exp(F(z)) = z$ *for all non-zero complex numbers* z.

Proof The open set $\mathbb{C} \setminus \{0\}$ is not evenly covered by the exponential map (Lemma 8.4). The result therefore follows immediately from Theorem 8.6. ∎

Historically, mathematicians have regarded the 'logarithm function' as an example of a 'many-valued function' on the set of all non-zero complex numbers. This 'many-valued function' associates to any non-zero complex number z the set consisting of all complex numbers w satisfying $\exp(w) = z$. Any two such complex numbers w differ by an integer multiple of $2\pi i$. A continuous function $F: U \to \mathbb{C}$, defined over an open subset U of $\mathbb{C} \setminus \{0\}$, and satisfying $\exp(F(z)) = z$ for all $z \in U$, is referred to as a (continuous) *branch* of the logarithm function. Theorem 8.6 therefore shows that a continuous branch of the logarithm function can be defined over an open subset of the punctured complex plane $\mathbb{C} \setminus \{0\}$ if and only if that open set is evenly covered by the exponential map.

Another approach to the study of 'many-valued' functions was pioneered by Riemann, in his theory of *Riemann surfaces*. Instead of regarding a function such as the logarithm function as a 'many-valued function' defined over the punctured complex plane, he regarded it as a single-valued function defined over a surface which can be projected onto the punctured complex plane. The portion of this surface covering a sufficiently small open set in the punctured plane consists of an infinite number of sheets, each of which is a copy of that open set. However if a point on the punctured plane traverses a loop that encircles zero, then the corresponding point of the Riemann surface will pass from one sheet to another. The Riemann surface will therefore resemble a spiral staircase.

We can construct a representation of such a Riemann surface as follows. Let

$$S = \{(z, w) \in \mathbb{C}^2 : z \neq 0 \text{ and } \exp(w) = z\}.$$

Define functions $p: S \to \mathbb{C} \setminus \{0\}$ and $\log: S \to \mathbb{C}$ by $p(z, w) = z$ and $\log(z, w) = w$ for all $(z, w) \in S$. Then $\exp \circ \log = p$. If U is an open set in $\mathbb{C} \setminus \{0\}$, and if

w_0 is a complex number satisfying $\exp(w_0) \in U$, then we can find a continuous map $F: U \to \mathbb{C}$ satisfying $\exp(F(z)) = z$ for all $z \in U$ (Theorem 8.6), and moreover we can choose F so that $w_0 \in F(U)$. Then the map $\varphi: U \to S$ sending $z \in U$ to $(z, F(z))$ gives a parameterization of the surface S around the point w_0. The sheets of the Riemann surface S covering U are the open sets W_n for all integers n, where

$$W_n = \{(z, w) \in S : z \in U \text{ and } w - 2\pi i n \in F(U)\}.$$

It is easy to see that the 'logarithm function' $\log: S \to \mathbb{C}$ is actually a homeomorphism from S to $\mathbb{C}$. Thus the Riemann surface S is homeomorphic to the whole complex plane.

Now consider the loop $\sigma: [0, 1] \to \mathbb{C}$ defined by $\sigma(t) = \exp(2\pi i t)$ for all $t \in [0, 1]$. A point traversing this loop will encircle zero once in the anticlockwise direction as the parameter t varies from 0 to 1. It is not difficult to see that a corresponding point moving continuously on the Riemann surface S and starting at $(1, 0)$ will traverse the path $\tilde{\sigma}: [0, 1] \to S$, where

$$\tilde{\sigma}(t) = (\exp(2\pi i t), 2\pi i t)$$

for all $t \in [0, 1]$, and will travel from $(1, 0)$ to $(1, 2\pi i)$. Thus the point on the Riemann surface has passed from one sheet of the Riemann surface to another as the point on the punctured plane has traversed a loop in the punctured complex plane encircling zero.

Alternatively, one can represent the Riemann surface for the logarithm function as a surface in $\mathbb{R}^3$. Indeed let Σ denote the surface in $\mathbb{R}^3$ consisting of all points (x, y, z) that satisfy the conditions

$$x^2 + y^2 \neq 0, \quad \cos 2\pi z = \frac{x}{\sqrt{x^2 + y^2}} \text{ and } \sin 2\pi z = \frac{y}{\sqrt{x^2 + y^2}}.$$

The function sending $(x, y, z) \in \Sigma$ to $(x + iy, \frac{1}{2}\log(x^2 + y^2) + 2\pi i z)$ is a homeomorphism from Σ to S. The projection function $p: S \to \mathbb{C} \setminus \{0\}$ corresponds, under this homeomorphism, to the map sending $(x, y, z) \in \Sigma$ to $x + iy$, and the 'logarithm function' $\log: S \to \mathbb{C}$ corresponds to the map sending $(x, y, z) \in \Sigma$ to $\frac{1}{2}\log(x^2 + y^2) + 2\pi i z$.

Analogous techniques can be used to define Riemann surfaces for other 'multi-valued functions', such as $z \mapsto \sqrt{z^2 - 1}$. The introduction by Riemann of this way of viewing 'many-valued functions' led to the discovery of very deep and powerful theorems in the theory of complex functions whose statement and proof involves geometric and topological concepts and techniques.

8.2 Path Lifting and the Monodromy Theorem

Let X be a topological space, and let $f: X \to \mathbb{C} \setminus \{0\}$ be a continuous map. We shall study the problem of determining whether or not there exists a continuous map $\tilde{f}: X \to \mathbb{C}$ satisfying $\exp \circ \tilde{f} = f$. A problem of this sort is referred to as a *lifting problem*. We have already seen that such a map $\tilde{f}$ does not exist when $X = \mathbb{C} \setminus \{0\}$ and $f: X \to \mathbb{C} \setminus \{0\}$ is the identity map. On the other hand, we shall prove the existence of the required lift $\tilde{f}$ of f when X is either the interval $[0, 1]$ or the square $[0, 1] \times [0, 1]$. First we prove a uniqueness result concerning such lifts.

Proposition 8.8 *Let X be a connected topological space, and let $g: X \to \mathbb{C}$ and $h: X \to \mathbb{C}$ be continuous maps. Suppose that $\exp \circ g = \exp \circ h$ and that $g(x_0) = h(x_0)$ for some $x_0 \in X$. Then $g = h$.*

Proof Let $X_0 = \{x \in X : g(x) = h(x)\}$. Note that X_0 is non-empty, by hypothesis. We show that X_0 is both open and closed in X.

Let x be a point of X. There exists an open neighbourhood U of $\exp(g(x))$ in $\mathbb{C} \setminus \{0\}$ which is evenly covered by the exponential map. Then $\exp^{-1}(U)$ is a disjoint union of open sets, each of which is mapped homeomorphically onto U by the exponential map. One of these open sets contains $g(x)$; let this set be denoted by $\tilde{U}$. Also one of these open sets contains $h(x)$; let this open set be denoted by $\tilde{V}$. Note that $\tilde{U} = \tilde{V}$ if $x \in X_0$ (so that $g(x) = h(x)$), and $\tilde{U} \cap \tilde{V} = \emptyset$ if $x \in X \setminus X_0$ (so that $g(x) \neq h(x)$). Let $N_x = g^{-1}(\tilde{U}) \cap h^{-1}(\tilde{V})$. Then N_x is an open set in X containing x.

Consider the case when $x \in X_0$. In this case $\tilde{V} = \tilde{U}$, so that $g(N_x) \subset \tilde{U}$ and $h(N_x) \subset \tilde{U}$. But $\exp \circ g = \exp \circ h$, and the restriction $\exp|\tilde{U}$ of the exponential map to $\tilde{U}$ maps $\tilde{U}$ homeomorphically onto U. Therefore $g|N_x = h|N_x$, and thus $N_x \subset X_0$. We have thus shown that, for each $x \in X_0$, there exists an open set N_x such that $x \in N_x$ and $N_x \subset X_0$. We conclude that X_0 is open.

Next consider the case when $x \in X \setminus X_0$. In this case $\tilde{U} \cap \tilde{V} = \emptyset$. But $g(N_x) \subset \tilde{U}$ and $h(N_x) \subset \tilde{V}$. Therefore $g(x') \neq h(x')$ for all $x' \in N_x$, so that $N_x \subset X \setminus X_0$. We have thus shown that, for each $x \in X \setminus X_0$, there exists an open set N_x such that $x \in N_x$ and $N_x \subset X \setminus X_0$. We conclude that $X \setminus X_0$ is open, and thus X_0 is closed.

The subset X_0 of X is both open and closed. Also X_0 is non-empty, since there exists some point x_0 of X for which $g(x_0) = h(x_0)$. It follows from the connectedness of X that $X_0 = X$. Therefore $g = h$. ∎

Lemma 8.9 *Let X be a topological space, let A be a connected subset of X, and let $f: X \to \mathbb{C} \setminus \{0\}$ and $g: A \to \mathbb{C}$ be continuous maps with the property* *that* $\exp \circ g = f|A$. *Suppose that* $f(X) \subset U$*, where* U *is an open subset of* $\mathbb{C} \setminus \{0\}$ *that is evenly covered by the exponential map. Then there exists a continuous map* $\tilde{f}: X \to \mathbb{C}$ *such that* $\tilde{f}|A = g$ *and* $\exp \circ \tilde{f} = f$.

Proof Choose $a_0 \in A$. Now U is evenly covered by the exponential map. Therefore $\exp^{-1}(U)$ is a disjoint union of open sets, each of which is mapped homeomorphically onto U by the exponential map. One of these open sets contains $g(a_0)$; let this set be denoted by $\tilde{U}$. Let $F: U \to \tilde{U}$ be the inverse of the homeomorphism $\exp |\tilde{U}: \tilde{U} \to U$, and let $\tilde{f} = F \circ f$. Then $\exp \circ \tilde{f} = f$. Also $\exp \circ \tilde{f}|A = \exp \circ g$ and $\tilde{f}(a_0) = g(a_0)$. It follows from Proposition 8.8 that $\tilde{f}|A = g$, since A is connected. Thus $\tilde{f}: X \to \mathbb{C}$ is the required map. ∎

Theorem 8.10 (Path Lifting Theorem) *Let* $\gamma: [0,1] \to \mathbb{C} \setminus \{0\}$ *be a continuous path in* $\mathbb{C} \setminus \{0\}$*, and let* z *be a complex number satisfying* $\exp(z) = \gamma(0)$. *Then there exists a unique continuous path* $\tilde{\gamma}: [0,1] \to \mathbb{C}$ *such that* $\tilde{\gamma}(0) = z$ *and* $\exp \circ \tilde{\gamma} = \gamma$.

Proof Let $\mathcal{U}$ be the collection consisting of the sets $\mathbb{C} \setminus L_\alpha$ for $0 \leq \alpha < 2\pi$, where $L_\alpha = \{te^{i\alpha} : t \geq 0\}$. Then $\mathcal{U}$ is an open cover of $\mathbb{C} \setminus \{0\}$, and each of the open sets belonging to $\mathcal{U}$ is evenly covered by the exponential map. Now the collection of sets of the form $\gamma^{-1}(U)$ with $U \in \mathcal{U}$ is an open cover of the interval $[0,1]$. But $[0,1]$ is compact, by the Heine-Borel Theorem (Theorem 4.2). It follows from the Lebesgue Lemma (Lemma 5.10) that there exists some $\delta > 0$ such that every subinterval of length less than δ is mapped by γ into one of the open sets belonging to $\mathcal{U}$. Partition the interval $[0,1]$ into subintervals $[t_{i-1}, t_i]$, where $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$, and where the length of each subinterval is less than δ. Now it follows from Lemma 8.9 that once $\tilde{\gamma}(t_{i-1})$ has been determined, we can extend $\tilde{\gamma}$ continuously over the ith subinterval $[t_{i-1}, t_i]$. Thus by extending $\tilde{\gamma}$ successively over $[t_0, t_1]$, $[t_1, t_2]$,. . ., $[t_{n-1}, t_n]$, we can lift the path $\gamma: [0,1] \to \mathbb{C} \setminus \{0\}$ to a path $\tilde{\gamma}: [0,1] \to \mathbb{C}$ starting at z. The uniqueness of $\tilde{\gamma}$ follows from Proposition 8.8. ∎

Alternative Proof Let J be the set of all real numbers s in the interval $[0,1]$ with the property that there exists some (necessarily unique) continuous map $\tilde{\gamma}_s: [0,s] \to \mathbb{C}$ satisfying $\tilde{\gamma}_s(0) = z$ and $\exp(\tilde{\gamma}_s(t)) = \gamma(t)$ for all $t \in [0,s]$. Clearly $0 \in J$, and thus J is a non-empty subset of $[0,1]$. We must prove that $1 \in J$.

Let $\tau = \sup J$, and let U be an open subset of $\mathbb{C} \setminus \{0\}$ which contains the point $\gamma(\tau)$ and is evenly covered by the exponential map. Choose $\delta > 0$ such that $\gamma(t) \in U$ for all $t \in [0,1]$ satisfying $|t - \tau| < \delta$, and choose $s \in J$ satisfying $\tau - \delta < s \leq \tau$. Then there exists a continuous function $F: U \to \mathbb{C}$ such that $\exp(F(z)) = z$ for all $z \in U$ (Theorem 8.6), and F can be chosen so that $\tilde{\gamma}(s) \in F(U)$. Given any $u \in [0,1]$ satisfying $s \leq u < \tau + \delta$, we define a continuous lift $\tilde{\gamma}_u \colon [0,u] \to \mathbb{C}$ of $\gamma|[0,u]$ by the formula

$$\tilde{\gamma}_u(t) = \begin{cases} \tilde{\gamma}_s(t) & \text{if } 0 \leq t \leq s, \\ F(\gamma(t)) & \text{if } s \leq t \leq u. \end{cases}$$

It follows that $u \in J$. In particular $\tau \in J$. If $\tau < 1$ then there would exist $u \in [0,1]$ satisfying $\tau < u < \tau + \delta$. But then $u \in J$, which is impossible, since $\tau = \sup J$. Thus $\tau = 1$, and so $1 \in J$, as required. ∎

Theorem 8.11 (The Monodromy Theorem) *Let $H \colon [0,1] \times [0,1] \to \mathbb{C} \setminus \{0\}$ be a continuous map, and let z be a complex number satisfying $\exp(z) = H(0,0)$. Then there exists a unique continuous map $\tilde{H} \colon [0,1] \times [0,1] \to \mathbb{C}$ such that $\tilde{H}(0,0) = z$ and $\exp \circ \tilde{H} = H$.*

Proof Again, let $\mathcal{U}$ be the open cover of $\mathbb{C} \setminus \{0\}$ consisting of the open sets $\mathbb{C} \setminus L_\alpha$ for $0 \leq \alpha < 2\pi$, where $L_\alpha = \{te^{i\alpha} : t \geq 0\}$. Then the collection of sets of the form $H^{-1}(U)$ with $U \in \mathcal{U}$ is an open cover of the unit square $[0,1] \times [0,1]$. But the unit square is compact. An application of the Lebesgue Lemma (as in the proof of Theorem 8.10) shows that there exists some $\delta > 0$ with the property that any square contained in $[0,1] \times [0,1]$ whose sides have length less than δ is mapped by H into some open set in $\mathbb{C} \setminus \{0\}$ which is evenly covered by the exponential map. It follows from Lemma 8.9 that if the lift $\tilde{H}$ of H has already been determined over a corner, or along one side, or along two adjacent sides of a square whose sides have length less than δ, then $\tilde{H}$ can be extended over the whole of that square. Thus if we subdivide the square $[0,1] \times [0,1]$ into smaller squares whose sides have length less than δ then we can extend the map g to a lift $\tilde{H}$ of H by successively extending $\tilde{H}$ in turn over each of the smaller squares. Indeed suppose that the unit square has been subdivided into squares $S_{j,k}$ for $j, k = 1, 2, \ldots, n$, where $1/n < \delta$ and

$$S_{j,k} = \{(x,y) \in [0,1] \times [0,1] : \frac{j-1}{n} \leq x \leq \frac{j}{n} \text{ and } \frac{k-1}{n} \leq y \leq \frac{k}{n}\}.$$

Then we can extend the map $\tilde{H}$ successively over the squares

$$S_{1,1}, S_{1,2}, \ldots, S_{1,n}, S_{2,1}, S_{2,2}, \ldots, S_{2,n}, S_{3,1}, \ldots, S_{n-1,n} S_{n,1}, S_{n,2}, \ldots, S_{n,n}.$$

The uniqueness of $\tilde{H}$ follows from Proposition 8.8. ∎