Course 212: Academic Year 1991-2
Section 6: Normed Vector Spaces

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Contents

6 Normed Vector Spaces

6.1 Norms on Real and Complex Vector Spaces

A set X is a *vector space* over some field $\mathbb{F}$ if

- given any $x, y \in X$ and $\lambda \in \mathbb{F}$, there are well-defined elements $x + y$ and λx of X,
- X is an Abelian group with respect to the operation $+$ of addition,
- the identities

$$\lambda(x + y) = \lambda x + \lambda y, \qquad (\lambda + \mu)x = \lambda x + \mu x,$$

$$(\lambda\mu)x = \lambda(\mu x), \qquad 1x = x$$

  are satisfied for all $x, y \in X$ and $\lambda, \mu \in \mathbb{F}$.

Elements of the field $\mathbb{F}$ are referred to as *scalars*. We consider here only *real vector spaces* and *complex vector spaces*: these are vector spaces over the fields of real numbers and complex numbers respectively.

Definition A *norm* $\|.\|$ on a real or complex vector space X is a function, associating to each element x of X a corresponding real number $\|x\|$, such that the following conditions are satisfied:—

(i) $\|x\| \geq 0$ for all $x \in X$,

(ii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$,

(iii) $\|\lambda x\| = |\lambda| \, \|x\|$ for all $x \in X$ and for all scalars λ.

(iv) $\|x\| = 0$ if and only if $x = 0$.

A *normed vector space* $(X, \|.\|)$ consists of a a real or complex vector space X, together with a norm $\|.\|$ on X.

Note that any normed complex vector space can also be regarded as a normed real vector space.

If $x_1, x_2, \ldots, x_m$ are elements of a normed vector space X then

$$\left\| \sum_{k=1}^{m} x_k \right\| \leq \sum_{k=1}^{m} \|x_k\|,$$

where $\|.\|$ denotes the norm on X. (This follows directly using induction on m.)

Example The field $\mathbb{R}$ is a one-dimensional normed vector space over itself: the norm $|t|$ of $t \in \mathbb{R}$ is the absolute value of t.

Example The field $\mathbb{C}$ is a one-dimensional normed vector space over itself: the norm $|z|$ of $z \in \mathbb{R}$ is the modulus of z. The field $\mathbb{C}$ is also a two-dimensional normed vector space over $\mathbb{R}$.

Example Let $\|.\|_1$, $\|.\|_2$ and $\|.\|_\infty$ be the real-valued functions on $\mathbb{C}^n$ defined by

$$\begin{aligned} \|\mathbf{z}\|_1 &= \sum_{j=1}^n |z_j|, \\ \|\mathbf{z}\|_2 &= \left(\sum_{j=1}^n |z_j|^2 \right)^{\frac{1}{2}}, \\ \|\mathbf{z}\|_\infty &= \max(|z_1|, |z_2|, \ldots, |z_n|), \end{aligned}$$

for each $\mathbf{z} \in \mathbb{C}^n$, where $\mathbf{z} = (z_1, z_2, \ldots, z_n)$. Then $\|.\|_1$, $\|.\|_2$ and $\|.\|_\infty$ are norms on $\mathbb{C}^n$. In particular, if we regard $\mathbb{C}^n$ as a $2n$-dimensional real vector space naturally isomorphic to $\mathbb{R}^{2n}$ (via the isomorphism

$$(z_1, z_2, \ldots, z_n) \mapsto (x_1, y_1, x_2, y_2, \ldots, x_n, y_n),$$

where x_j and y_j are the real and imaginary parts of z_j for $j = 1, 2, \ldots, n$) then $\|.\|_2$ represents the Euclidean norm on this space. The inequality $\|\mathbf{z}+\mathbf{w}\|_2 \leq \|\mathbf{z}\|_2 + \|\mathbf{w}\|_2$ satisfied for all $\mathbf{z}, \mathbf{w} \in \mathbb{C}^n$ is therefore just the standard Triangle Inequality for the Euclidean norm.

Example The space $\mathbb{R}^n$ is also an n-dimensional real normed vector space with respect to the norms $\|.\|_1$, $\|.\|_2$ and $\|.\|_\infty$ defined above. Note that $\|.\|_2$ is the standard Euclidean norm on $\mathbb{R}^n$.

Example Let

$$\begin{aligned} \ell_1 &= \{(z_1, z_2, z_3, \ldots) \in \mathbb{C}^\infty : |z_1| + |z_2| + |z_3| + \cdots \text{ converges}\}, \\ \ell_2 &= \{(z_1, z_2, z_3, \ldots) \in \mathbb{C}^\infty : |z_1|^2 + |z_2|^2 + |z_3|^2 + \cdots \text{ converges}\}, \\ \ell_\infty &= \{(z_1, z_2, z_3, \ldots) \in \mathbb{C}^\infty : \text{the sequence } |z_1|, |z_2|, |z_3|, \ldots \text{ is bounded}\}. \end{aligned}$$

where $\mathbb{C}^\infty$ denotes the set of all sequences $(z_1, z_2, z_3, \ldots)$ of complex numbers. Then ℓ_1, ℓ_2 and ℓ_∞ are infinite-dimensional normed vector spaces, with norms

$\|.\|_1$, $\|.\|_2$ and $\|.\|_\infty$ respectively, where

$$\begin{aligned}\|\mathbf{z}\|_1 &= \sum_{j=1}^{+\infty} |z_j|,\\ \|\mathbf{z}\|_2 &= \left(\sum_{j=1}^{+\infty} |z_j|^2\right)^{\frac{1}{2}},\\ \|\mathbf{z}\|_\infty &= \sup\{|z_1|, |z_2|, |z_3|, \ldots\}.\end{aligned}$$

(For example, to show that $\|\mathbf{z}+\mathbf{w}\|_2 \leq \|\mathbf{z}\|_2 + \|\mathbf{w}\|_2$ for all $\mathbf{z}, \mathbf{w} \in \ell_2$, we note that

$$\left(\sum_{j=1}^{n} |z_j + w_j|^2\right)^{\frac{1}{2}} = \left(\sum_{j=1}^{n} |z_j|^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n} |w_j|^2\right)^{\frac{1}{2}} \leq \|\mathbf{z}\|_2 + \|\mathbf{w}\|_2$$

for all natural numbers n, by the Triangle Inequality in $\mathbb{C}^n$. Taking limits as $n \to +\infty$, we deduce that $\|\mathbf{z}+\mathbf{w}\|_2 \leq \|\mathbf{z}\|_2 + \|\mathbf{w}\|_2$, as required.)

If $x_1, x_2, \ldots, x_m$ are elements of a normed vector space X then

$$\left\|\sum_{n=1}^{m} x_n\right\| \leq \sum_{n=1}^{m} \|x_n\|,$$

where $\|.\|$ denotes the norm on X. (This follows directly using induction on m.)

A norm $\|.\|$ on a vector space X induces a corresponding distance function on X: the distance $d(x, y)$ between elements x and y of X is defined by $d(x, y) = \|x - y\|$. This distance function satisfies the metric space axioms. Thus any vector space with a given norm can be regarded as a metric space. A norm on a vector space X therefore generates a topology on X: a subset U of X is an open set if and only if, given any point u of U, there exists some $\delta > 0$ such that

$$\{x \in X : \|x - u\| < \delta\} \subset U.$$

The function $x \mapsto \|x\|$ is a continuous function from X to $\mathbb{R}$, since

$$\|x\| - \|y\| = \|(x - y) + y\| - \|y\| \leq (\|x - y\| + \|y\|) - \|y\| = \|x - y\|,$$

and $\|y\| - \|x\| \leq \|x - y\|$, and therefore $|\|x\| - \|y\|| \leq \|x - y\|$.

The Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ of vector spaces $X_1, X_2, \ldots, X_n$ can itself be regarded as a vector space: if $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are points of $X_1 \times X_2 \times \cdots \times X_n$, and if λ is any scalar, then

$$\begin{aligned}(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n),\\ \lambda(x_1, x_2, \ldots, x_n) &= (\lambda x_1, \lambda x_2, \ldots, \lambda x_n).\end{aligned}$$

Lemma 6.1 *Let $X_1, X_2, \ldots, X_n$ be normed vector spaces, and let $\|.\|_{\max}$ be the norm on $X_1 \times X_2 \times \cdots \times X_n$ defined by*

$$\|(x_1, x_2, \ldots, x_n)\|_{\max} = \max(\|x_1\|_1, \|x_2\|_2, \ldots, \|x_n\|_n),$$

where $\|.\|_i$ is the norm on X_i for $i = 1, 2, \ldots, n$. Then the topology on $X_1 \times X_2 \times \cdots \times X_n$ generated by the norm $\|.\|_{\max}$ is the product topology on $X_1 \times X_2 \times \cdots \times X_n$.

Proof It is a straightforward exercise to verify that $\|.\|_{\max}$ is indeed a norm on X, where $X = X_1 \times X_2 \times \cdots \times X_n$.

Let U be a subset of X. Suppose that U is open with respect to the product topology. Let $\mathbf{u}$ be any point of U, given by $\mathbf{u} = (u_1, u_2, \ldots, u_n)$. We must show that there exists some $\delta > 0$ such that

$$\{\mathbf{x} \in X : \|\mathbf{x} - \mathbf{u}\|_{\max} < \delta\} \subset U.$$

Now it follows from the definition of the product topology that there exist open sets $V_1, V_2, \ldots, V_n$ in $X_1, X_2, \ldots, X_n$ such that $u_i \in V_i$ for all i and $V_1 \times V_2 \times \cdots \times V_n \subset U$. We can then take δ to be the minimum of $\delta_1, \delta_2, \ldots, \delta_n$, where $\delta_1, \delta_2, \ldots, \delta_n$ are chosen such that

$$\{x_i \in X_i : \|x_i - u_i\| < \delta_i\} \subset V_i$$

for $i = 1, 2, \ldots, n$.

Conversely suppose that U is open with respect to the topology generated by the norm $\|.\|_{\max}$. Let $\mathbf{u}$ be any point of U. Then there exists $\delta > 0$ such that

$$\{\mathbf{x} \in X : \|\mathbf{x} - \mathbf{u}\|_{\max} < \delta\} \subset U.$$

Let $V_i = \{x_i \in X_i : \|x_i - u_i\| < \delta_i\}$ for $i = 1, 2, \ldots, n$. Then, for each i, V_i is an open set in X_i, $u_i \in V_i$, and $V_1 \times V_2 \times \cdots \times V_n \subset U$. We deduce that U is also open with respect to the product topology, as required. ∎

Proposition 6.2 *Let X be a normed vector space over the field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then the function from $X \times X$ to X sending $(x, y) \in X \times X$ to $x+y$ is continuous. Also the function from $\mathbb{F} \times X$ to X sending $(\lambda, x) \in \mathbb{F} \times X$ to λx is continuous.*

Proof Let $(u, v) \in X \times X$, and let $\varepsilon > 0$ be given. Let $\delta = \frac{1}{2}\varepsilon$. If $(x, y) \in X \times X$ satisfies $\|(x, y) - (u, v)\|_{\max} < \delta$, then $\|x - u\| < \delta$ and $\|y - v\| < \delta$, and hence

$$\|(x + y) - (u + v)\| \leq \|x - u\| + \|y - v\| < \varepsilon.$$

This shows that the function $(x, y) \mapsto x + y$ is continuous at $(u, v) \in X \times X$.

Next let $(\mu, u) \in \mathbb{F} \times X$, and let $\varepsilon > 0$ be given. Let

$$\delta = \text{minimum}\left(\frac{\varepsilon}{2(\|u\| + 1)}, \frac{\varepsilon}{2(\|\mu\| + 1)}, 1\right).$$

Now $\lambda x - \mu u = \lambda(x - u) + (\lambda - \mu)u$ for all $\lambda \in \mathbb{F}$ and $x \in X$. Thus if $(\lambda, x) \in \mathbb{F} \times X$ satisfies $\|(\lambda, x) - (\mu, u)\|_{\max} < \delta$, then

$$|\lambda - \mu\| < \frac{\varepsilon}{2(\|u\| + 1)}, \qquad \|x - u\| < \frac{\varepsilon}{2(\|\mu\| + 1)}, \qquad |\lambda| < |\mu| + 1,$$

and hence

$$|\lambda x - \mu u| \leq |\lambda| \, \|x - u\| + |\lambda - \mu| \, \|u\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that the function $(\lambda, x) \mapsto \lambda x$ is continuous at $(\mu, u) \in \mathbb{F} \times X$, as required. ∎

Corollary 6.3 *Let X be a normed vector space over the field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let (x_n) and (y_n) be convergent sequences in X, and let (λ_n) be a convergent sequence in $\mathbb{F}$. Then the sequences $(x_n + y_n)$ and $(\lambda_n x_n)$ are convergent in X, and*

$$\begin{aligned} \lim_{n \to +\infty} (x_n + y_n) &= \lim_{n \to +\infty} x_n + \lim_{n \to +\infty} y_n, \\ \lim_{n \to +\infty} (\lambda_n x_n) &= \left(\lim_{n \to +\infty} \lambda_n\right) \left(\lim_{n \to +\infty} x_n\right). \end{aligned}$$

Proof Let $x = \lim_{n \to +\infty} x_n$, $y = \lim_{n \to +\infty} y_n$ and $\lambda = \lim_{n \to +\infty} \lambda_n$. Using Lemma 6.1, together with the definition of convergence in metric spaces, it follows easily that the sequences (x_n, y_n) and (λ_n, x_n) converge to (x, y) and (λ, x) in $X \times X$ and $\mathbb{F} \times X$ respectively. The convergence of $(x_n + y_n)$ and $\lambda_n x_n$ to $x + y$ and λx respectively now follows from Proposition 6.2 (using Lemma 1.3). ∎

Let X be a normed vector space, and let $x_1, x_2, x_3, \ldots$ be elements of X. The infinite series $\lim_{n \to +\infty} x_n$ is said to *converge* to some element s of X if, given any $\varepsilon > 0$, there exists some natural number N such that

$$\|s - \sum_{n=1}^{m} x_n\| < \varepsilon$$

for all $m \geq N$ (where $\|.\|$ denotes the norm on X).

We say that a normed vector space X is *complete* if and only if every Cauchy sequence in X is convergent. (A sequence $x_1, x_2, x_3, \ldots$ is a *Cauchy sequence* if and only if, given any $\varepsilon > 0$, there exists some natural number N such that $\|x_j - x_k\| < \varepsilon$ for all j and k satisfying $j \geq N$ and $k \geq N$.) A complete normed vector space is referred to as a *Banach space*. (The basic theory of such spaces was extensively developed by the famous Polish mathematician Stefan Banach and his co-workers.)

Lemma 6.4 *Let X be a Banach space, and let $x_1, x_2, x_3, \ldots$ be elements of X. Suppose that $\lim\limits_{n\to+\infty} \|x_n\|$ is convergent. Then $\lim\limits_{n\to+\infty} x_n$ is convergent, and*

$$\left\| \lim_{n\to+\infty} x_n \right\| \leq \lim_{n\to+\infty} \|x_n\|.$$

Proof For each natural number n, let

$$s_n = x_1 + x_2 + \cdots + x_n.$$

Let $\varepsilon > 0$ be given. We can find N such that $\sum\limits_{n=N}^{+\infty} \|x_n\| < \varepsilon$, since $\lim\limits_{n\to+\infty} \|x_n\|$ is convergent. Let $s_n = x_1 + x_2 + \cdots + x_n$. If $j \geq N$, $k \geq N$ and $j < k$ then

$$\|s_k - s_j\| = \left\| \sum_{n=j+1}^{k} x_n \right\| \leq \sum_{n=j+1}^{k} \|x_n\| \leq \sum_{n=N}^{+\infty} \|x_n\| < \varepsilon.$$

Thus $s_1, s_2, s_3, \ldots$ is a Cauchy sequence in X, and therefore converges to some element s of X, since X is complete. But then $s = \lim\limits_{j=1}^{+\infty} x_j$. Moreover, on choosing m large enough to ensure that $s - s_m < \varepsilon$, we deduce that

$$\|s\| \leq \left\| \sum_{n=1}^{m} x_n \right\| + \left\| s - \sum_{n=1}^{m} x_n \right\| \leq \sum_{n=1}^{m} \|x_n\| + \left\| s - \sum_{n=1}^{m} x_n \right\| < \sum_{n=1}^{+\infty} \|x_n\| + \varepsilon.$$

Since this inequality holds for all $\varepsilon > 0$, we conclude that

$$\|s\| \leq \sum_{n=1}^{+\infty} \|x_n\|,$$

as required. ∎

6.2 Bounded Linear Transformations

Let X and Y be real or complex vector spaces. A function $T: X \to Y$ is said to be a *linear transformation* if $T(x+y) = Tx + Ty$ and $T(\lambda x) = \lambda Tx$ for all elements x and y of X and scalars λ. A linear transformation mapping X into itself is referred to as a *linear operator* on X.

Definition Let X and Y be normed vector spaces. A linear transformation $T: X \to Y$ is said to be *bounded* if there exists some non-negative real number C with the property that $\|Tx\| \leq C\|x\|$ for all $x \in X$. If T is bounded, then the smallest non-negative real number C with this property is referred to as the *operator norm* of T, and is denoted by $\|T\|$.

Lemma 6.5 *Let X and Y be normed vector spaces, and let $S: X \to Y$ and $T: X \to Y$ be bounded linear transformations. Then $S + T$ and λS are bounded linear transformations for all scalars λ, and*

$$\|S + T\| \leq \|S\| + \|T\|, \qquad \|\lambda S\| = |\lambda|\|S\|.$$

Moreover $\|S\| = 0$ if and only if $S = 0$. Thus the vector space $B(X, Y)$ of bounded linear transformations from X to Y is a normed vector space (with respect to the operator norm).

Proof $\|(S+T)x\| \leq \|Sx\|+\|Tx\| \leq (\|S\|+\|T\|)\|x\|$ for all $x \in X$. Therefore $S+T$ is bounded, and $\|S+T\| \leq \|S\| + \|T\|$. Using the fact that $\|(\lambda S)x\| = |\lambda|\,\|Sx\|$ for all $x \in X$, we see that λS is bounded, and $\|\lambda S\| = |\lambda|\,\|S\|$. If $S = 0$ then $\|S\| = 0$. Conversely if $\|S\| = 0$ then $\|Sx\| \leq \|S\|\,\|x\| = 0$ for all $x \in X$, and hence $S = 0$. The result follows. ∎

Lemma 6.6 *Let X, Y and Z be normed vector spaces, and let $S: X \to Y$ and $T: Y \to Z$ be bounded linear transformations. Then the composition TS of S and T is also bounded, and $\|TS\| \leq \|T\|\,\|S\|$.*

Proof $\|TSx\| \leq \|T\|\,\|Sx\| \leq \|T\|\,\|S\|\,\|x\|$ for all $x \in X$. The result follows. ∎

Proposition 6.7 *Let X and Y be normed vector spaces, and let $T: X \to Y$ be a linear transformation from X to Y. Then the following are equivalent:—*

(i) *$T: X \to Y$ is continuous,*

(ii) *$T: X \to Y$ is continuous at 0,*

(iii) $T: X \to Y$ *is bounded.*

Proof Obviously (i) implies (ii). We show that (ii) implies (iii) and (iii) implies (i). The equivalence of the three conditions then follows immediately.

Suppose that $T: X \to Y$ is continuous at 0. Then there exists $\delta > 0$ such that $\|Tx\| < 1$ for all $x \in X$ satisfying $\|x| < \delta$. Let C be any positive real number satisfying $C > 1/\delta$. If x is any non-zero element of X then $\|\lambda x\| < \delta$, where $\lambda = 1/(C\|x\|)$, and hence

$$\|Tx\| = C\|x\|\,\|\lambda Tx\| = C\|x\|\,\|T(\lambda x)\| < C\|x\|.$$

Thus $\|Tx\| \leq C\|x\|$ for all $x \in X$, and hence $T: X \to Y$ is bounded. Thus (ii) implies (iii).

Finally suppose that $T: X \to Y$ is bounded. Let x be a point of X, and let $\varepsilon > 0$ be given. Choose $\delta > 0$ satisfying $\|T\|\delta < \varepsilon$. If $x' \in X$ satisfies $\|x' - x\| < \delta$ then

$$\|Tx' - Tx\| = T(x' - x)\| \leq \|T\|\,\|x' - x\| < \|T\|\delta < \varepsilon.$$

Thus $T: X \to Y$ is continuous. Thus (iii) implies (i), as required. ∎

Let X be a normed vector space, and let $x_1, x_2, x_3, \ldots$ be elements of X. The infinite series $\lim_{n \to +\infty} x_n$ is said to *converge* to some element s of X if, given any $\varepsilon > 0$, there exists some natural number N such that

$$\|s - \sum_{n=1}^{m} x_n\| < \varepsilon$$

for all $m \geq N$ (where $\|.\|$ denotes the norm on X).

Proposition 6.8 *Let X be a normed vector space and let Y be a Banach space. Then the space $B(X, Y)$ of bounded linear transformations from X to Y is also a Banach space.*

Proof We have already shown that $B(X, Y)$ is a normed vector space (see Lemma 6.5). Thus it only remains to show that $B(X, Y)$ is complete.

Let $S_1, S_2, S_3, \ldots$ be a Cauchy sequence in $B(X, Y)$. Let $x \in X$. We claim that $S_1 x, S_2 x, S_3 x, \ldots$ is a Cauchy sequence in Y. This result is trivial if $x = 0$. If $x \neq 0$, and if $\varepsilon > 0$ is given then there exists some natural number N such that $\|S_j - S_k\| < \varepsilon/\|x\|$ whenever $j \geq N$ and $k \geq N$. But then $\|S_j x - S_k x\| \leq \|S_j - S_k\|\,\|x\| < \varepsilon$ whenever $j \geq N$ and $k \geq N$. This shows that $S_1 x, S_2 x, S_3 x, \ldots$ is indeed a Cauchy sequence. It therefore converges to some element of Y, since Y is a Banach space.

Let the function $S: X \to Y$ be defined by $Sx = \lim_{n \to +\infty} S_n x$. Then

$$S(x+y) = \lim_{n\to+\infty} (S_n x + S_n y) = \lim_{n\to+\infty} S_n x + \lim_{n\to+\infty} S_n y = Sx + Sy,$$

(see Corollary 6.3), and

$$S(\lambda x) = \lim_{n\to+\infty} S_n(\lambda x) = \lambda \lim_{n\to+\infty} S_n x = \lambda S x,$$

Thus $S: X \to Y$ is a linear transformation.

Next we show that $S_n \to S$ in $B(X,Y)$ as $n \to +\infty$. Let $\varepsilon > 0$ be given. Then there exists some natural number N such that $\|S_j - S_n\| < \frac{1}{2}\varepsilon$ whenever $j \geq N$ and $n \geq N$, since the sequence $S_1, S_2, S_3, \ldots$ is a Cauchy sequence in $B(X,Y)$. But then $\|S_j x - S_n x\| < \frac{1}{2}\varepsilon\|x\|$ for all $j \geq N$ and $n \geq N$, and thus

$$\begin{aligned} \|Sx - S_n x\| &= \left\| \lim_{j\to+\infty} S_j x - S_n x \right\| = \lim_{j\to+\infty} \|S_j x - S_n x\| \\ &\leq \lim_{j\to+\infty} \|S_j - S_n\| \, \|x\| \leq \tfrac{1}{2}\varepsilon\|x\| \end{aligned}$$

for all $n \geq N$ (since the norm is a continuous function on Y). But then

$$\|Sx\| \leq \|S_n x\| + \|Sx - S_n x\| \leq \left(\|S_n\| + \tfrac{1}{2}\varepsilon\right)\|x\|$$

for any $n \geq N$, showing that $S: X \to Y$ is a bounded linear transformation, and $\|S - S_n\| \leq \frac{1}{2}\varepsilon < \varepsilon$ for all $n \geq N$, showing that $S_n \to S$ in $B(X,Y)$ as $n \to +\infty$. Thus the Cauchy sequence $S_1, S_2, S_3, \ldots$ is convergent in $B(X,Y)$, as required. ∎

Corollary 6.9 *Let X and Y be Banach spaces, and let $T_1, T_2, T_3, \ldots$ be bounded linear transformations from X to Y. Suppose that $\lim_{n\to+\infty} \|T_n\|$ is convergent. Then $\lim_{n\to+\infty} T_n$ is convergent, and*

$$\left\| \lim_{n\to+\infty} T_n \right\| \leq \lim_{n\to+\infty} \|T_n\|.$$

Proof The space $B(X,Y)$ of bounded linear maps from X to Y is a Banach space by Proposition 6.8. The result therefore follows immediately on applying Lemma 6.4. ∎

Example Let T be a bounded linear operator on a Banach space X (i.e., a bounded linear transformation from X to itself). The infinite series

$$\sum_{n=0}^{+\infty} \frac{\|T\|^n}{n!}$$

converges to $\exp(\|T\|)$. It follows immediately from Lemma 6.6 (using induction on n) that $\|T^n\| \leq \|T\|^n$ for all $n \geq 0$ (where T^0 is the identity operator on X). It therefore follows from Corollary 6.7 that there is a well-defined bounded linear operator $\exp T$ on X, defined by

$$\exp T = \sum_{n=0}^{+\infty} \frac{1}{n!} T^n$$

(where T_0 is the identity operator I on X).

Proposition 6.10 *Let T be a bounded linear operator on a Banach space X. Suppose that $\|T\| < 1$. Then the operator $I - T$ has a bounded inverse $(I - T)^{-1}$ (where I denotes the identity operator on X). Moreover*

$$(I - T)^{-1} = I + T + T^2 + T^3 + \cdots.$$

Proof $\|T^n\| \leq \|T\|^n$ for all n, and the geometric series

$$1 + \|T\| + \|T\|^2 + \|T\|^3 + \cdots$$

is convergent (since $\|T\| < 1$). It follows from Corollary 6.9 that the infinite series

$$(I - T)^{-1} = I + T + T^2 + T^3 + \cdots$$

converges to some bounded linear operator S on X. Now

$$\begin{aligned}(I - T)S &= \lim_{n\to+\infty} (I - T)(I + T + T^2 + \cdots + T^n) = \lim_{n\to+\infty} (I - T^{n+1}) \\ &= I - \lim_{n\to+\infty} T^{n+1} = I,\end{aligned}$$

since $\|T\|^{n+1} \to 0$ and therefore $T^{n+1} \to 0$ as $n \to +\infty$. Similarly $S(I-T) = I$. This shows that $I - T$ is invertible, with inverse S, as required. ∎

6.3 Equivalence of Norms on a Finite-Dimensional Vector Space

Let $\|.\|$ and $\|.\|_*$ be norms on a real or complex vector space X. The norms $\|.\|$ and $\|.\|_*$ are said to be *equivalent* if and only if there exist constants c and C, where $0 < c \leq C$, such that

$$c\|x\| \leq \|x\|_* \leq C\|x\|$$

for all $x \in X$.

Lemma 6.11 *Two norms $\|.\|$ and $\|.\|_*$ on a real or complex vector space X are equivalent if and only if they induce the same topology on X.*

Proof Suppose that the norms $\|.\|$ and $\|.\|_*$ induce the same topology on X. Then there exists some $\delta > 0$ such that

$$\{x \in X : \|x\| < \delta\} \subset \{x \in X : \|x\|_* < 1\},$$

since the set $\{x \in X : \|x\|_* < 1\}$ is open with respect to the topology on X induced by both $\|.\|_*$ and $\|.\|$. Let C be any positive real number satisfying $C\delta > 1$. Then

$$\left\| \frac{1}{C\|x\|} x \right\| = \frac{1}{C} < \delta,$$

and hence

$$\|x\|_* = C\|x\| \left\| \frac{1}{C\|x\|} x \right\|_* < C\|x\|.$$

for all non-zero elements x of X, and thus $\|x\|_* \leq C\|x\|$ for all $x \in X$. On interchanging the roles of the two norms, we deduce also that there exists a positive real number c such that $\|x\| \leq (1/c)\|x\|_*$ for all $x \in X$. But then $c\|x\| \leq \|x\|_* \leq C\|x\|$ for all $x \in X$. We conclude that the norms $\|.\|$ and $\|.\|_*$ are equivalent.

Conversely suppose that the norms $\|.\|$ and $\|.\|_*$ are equivalent. Then there exist constants c and C, where $0 < c \leq C$, such that $c\|x\| \leq \|x\|_* \leq C\|x\|$ for all $x \in X$. Let U be a subset of X that is open with respect to the topology on X induced by the norm $\|.\|_*$, and let $u \in U$. Then there exists some $\delta > 0$ such that

$$\{x \in X : \|x - u\|_* < C\delta\} \subset U.$$

But then

$$\{x \in X : \|x - u\| < \delta\} \subset \{x \in X : \|x - u\|_* < C\delta\} \subset U,$$

showing that U is open with respect to the topology induced by the norm $\|.\|$. Similarly any subset of X that is open with respect to the topology induced by the norm $\|.\|$ must also be open with respect to the topology induced by $\|.\|_*$. Thus equivalent norms induce the same topology on X. ∎

It follows immediately from Lemma 6.11 that if $\|.\|$, $\|.\|_*$ and $\|.\|_\sharp$ are norms on a real (or complex) vector space X, if the norms $\|.\|$ and $\|.\|_*$ are equivalent, and if the norms $\|.\|_*$ and $\|.\|_\sharp$ are equivalent, then the norms $\|.\|$ and $\|.\|_\sharp$ are also equivalent. This fact can easily be verified directly from the definition of equivalence of norms.

We recall that the usual topology on $\mathbb{R}^n$ is that generated by the Euclidean norm on $\mathbb{R}^n$.

Lemma 6.12 *Let $\|.\|$ be a norm on $\mathbb{R}^n$. Then the function $\mathbf{x} \mapsto \|x\|$ is continuous with respect to the usual topology on on $\mathbb{R}^n$.*

Proof Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ denote the basis of $\mathbb{R}^n$ given by

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \cdots, \quad \mathbf{e}_n = (0, 0, 0, \ldots, 1).$$

Let $\mathbf{x}$ and $\mathbf{y}$ be points of $\mathbb{R}^n$, given by

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \qquad \mathbf{y} = (y_1, y_2, \ldots, y_n).$$

Using Schwarz' Inequality, we see that

$$\begin{aligned} \|x - y\| &= \left\| \sum_{j=1}^{n} (x_j - y_j)\mathbf{e}_j \right\| \leq \sum_{j=1}^{n} |x_j - y_j| \, \|\mathbf{e}_j\| \\ &\leq \left(\sum_{j=1}^{n} (x_j - y_j)^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} \|\mathbf{e}_j\|^2 \right)^{\frac{1}{2}} = C\|\mathbf{x} - \mathbf{y}\|_2, \end{aligned}$$

where

$$C^2 = \|\mathbf{e}_1\|^2 + \|\mathbf{e}_2\|^2 + \cdots + \|\mathbf{e}_n\|^2$$

and $\|\mathbf{x} - \mathbf{y}\|_2$ denotes the Euclidean norm of $\mathbf{x} - \mathbf{y}$, defined by

$$\|\mathbf{x} - \mathbf{y}\|_2 = \left(\sum_{j=1}^{n} (x_j - y_j)^2 \right)^{\frac{1}{2}}.$$

Also $|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq \|\mathbf{x} - \mathbf{y}\|$, since

$$\|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\|, \qquad \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{x}\|.$$

We conclude therefore that

$$|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq C\|\mathbf{x} - \mathbf{y}\|_2,$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and thus the function $\mathbf{x} \mapsto \|\mathbf{x}\|$ is continuous on $\mathbb{R}^n$ (with respect to the usual topology on $\mathbb{R}^n$). ∎

Theorem 6.13 *Any two norms on $\mathbb{R}^n$ are equivalent, and induce the usual topology on $\mathbb{R}^n$.*

Proof Let $\|.\|$ be any norm on $\mathbb{R}^n$. We show that $\|.\|$ is equivalent to the Euclidean norm $\|.\|_2$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, defined by

$$S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 = 1\}.$$

Now S^{n-1} is a compact subset of $\mathbb{R}^n$, since it is both closed and bounded (see Theorem 4.16). Also the function $\mathbf{x} \mapsto \|\mathbf{x}\|$ is continuous (Lemma 6.12). But any continuous real-valued function on a compact topological space attains both its maximum and minimum values on that space (Proposition 4.6). Therefore there exist points $\mathbf{u}$ and $\mathbf{v}$ of S^{n-1} such that $\|\mathbf{u}\| \leq \|\mathbf{x}\| \leq \|\mathbf{v}\|$ for all $\mathbf{x} \in S^{n-1}$. Set $c = \|\mathbf{u}\|$ and $C = \|\mathbf{v}\|$. Then $0 < c \leq C$ (since it follows from the definition of norms that the norm of any non-zero element of $\mathbb{R}^n$ is necessarily non-zero).

If $\mathbf{x}$ is any non-zero element of $\mathbb{R}^n$ then $\lambda\mathbf{x} \in S^{n-1}$, where $\lambda = 1/\|\mathbf{x}\|_2$. But $\|\lambda\mathbf{x}\| = |\lambda|\,\|\mathbf{x}\|$ (see the the definition of norms). Therefore $c \leq |\lambda|\,\|\mathbf{x}\| \leq C$, and hence $c\|\mathbf{x}\|_2 \leq \|\mathbf{x}\| \leq C\|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{R}^n$, showing that the norm $\|.\|$ is equivalent to the Euclidean norm $\|.\|_2$ on $\mathbb{R}^n$. Therefore any two norms on $\mathbb{R}^n$ are equivalent, and thus generate the same topology on $\mathbb{R}^n$ (Lemma 6.11). This topology must then be the usual topology on $\mathbb{R}^n$. ∎

Let X be a finite-dimensional real vector space. Then X is isomorphic to $\mathbb{R}^n$, where n is the dimension of X. It follows immediately from Theorem 6.13 and Lemma 6.11 that all norms on X are equivalent and therefore generate the same topology on X. This result does not generalize to infinite-dimensional vector spaces.