Course 212: Academic Year 1991-2
Section 6: Normed Vector Spaces

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6 Normed Vector Spaces

6.1 Norms on Real and Complex Vector Spaces

A set $X$ is a vector space over some field $\mathbb{F}$ if

- given any $x, y \in X$ and $\lambda \in \mathbb{F}$, there are well-defined elements $x + y$ and $\lambda x$ of $X$,
- $X$ is an Abelian group with respect to the operation $+$ of addition,
- the identities
  \[
  \lambda(x + y) = \lambda x + \lambda y, \quad (\lambda + \mu)x = \lambda x + \mu x, \\
  (\lambda \mu)x = \lambda(\mu x), \quad 1x = x
  \]
  are satisfied for all $x, y \in X$ and $\lambda, \mu \in \mathbb{F}$.

Elements of the field $\mathbb{F}$ are referred to as scalars. We consider here only real vector spaces and complex vector spaces: these are vector spaces over the fields of real numbers and complex numbers respectively.

**Definition** A norm $\| \cdot \|$ on a real or complex vector space $X$ is a function, associating to each element $x$ of $X$ a corresponding real number $\| x \|$, such that the following conditions are satisfied:—

(i) $\| x \| \geq 0$ for all $x \in X$,
(ii) $\| x + y \| \leq \| x \| + \| y \|$ for all $x, y \in X$,
(iii) $\| \lambda x \| = |\lambda| \| x \|$ for all $x \in X$ and for all scalars $\lambda$.
(iv) $\| x \| = 0$ if and only if $x = 0$.

A normed vector space $(X, \| \cdot \|)$ consists of a real or complex vector space $X$, together with a norm $\| \cdot \|$ on $X$.

Note that any normed complex vector space can also be regarded as a normed real vector space.

If $x_1, x_2, \ldots, x_m$ are elements of a normed vector space $X$ then

\[
\left\| \sum_{k=1}^{m} x_k \right\| \leq \sum_{k=1}^{m} \| x_k \|,
\]

where $\| \cdot \|$ denotes the norm on $X$. (This follows directly using induction on $m$.)
Example The field \( \mathbb{R} \) is a one-dimensional normed vector space over itself: the norm \(|t|\) of \( t \in \mathbb{R} \) is the absolute value of \( t \).

Example The field \( \mathbb{C} \) is a one-dimensional normed vector space over itself: the norm \(|z|\) of \( z \in \mathbb{R} \) is the modulus of \( z \). The field \( \mathbb{C} \) is also a two-dimensional normed vector space over \( \mathbb{R} \).

Example Let \( \|z\|_1, \|z\|_2 \) and \( \|z\|_\infty \) be the real-valued functions on \( \mathbb{C}^n \) defined by

\[
\|z\|_1 = \sum_{j=1}^{n} |z_j|,
\]

\[
\|z\|_2 = \left( \sum_{j=1}^{n} |z_j|^2 \right)^{\frac{1}{2}},
\]

\[
\|z\|_\infty = \max(|z_1|, |z_2|, \ldots, |z_n|),
\]

for each \( z \in \mathbb{C}^n \), where \( z = (z_1, z_2, \ldots, z_n) \). Then \( \|z\|_1, \|z\|_2 \) and \( \|z\|_\infty \) are norms on \( \mathbb{C}^n \). In particular, if we regard \( \mathbb{C}^n \) as a \( 2n \)-dimensional real vector space naturally isomorphic to \( \mathbb{R}^{2n} \) (via the isomorphism

\[
( z_1, z_2, \ldots, z_n ) \mapsto (x_1, y_1, x_2, y_2, \ldots, x_n, y_n),
\]

where \( x_j \) and \( y_j \) are the real and imaginary parts of \( z_j \) for \( j = 1, 2, \ldots, n \) then \( \|z\|_2 \) represents the Euclidean norm on this space. The inequality \( \|z+w\|_2 \leq \|z\|_2 + \|w\|_2 \) satisfied for all \( z, w \in \mathbb{C}^n \) is therefore just the standard Triangle Inequality for the Euclidean norm.

Example The space \( \mathbb{R}^n \) is also an \( n \)-dimensional real normed vector space with respect to the norms \( \|z\|_1, \|z\|_2 \) and \( \|z\|_\infty \) defined above. Note that \( \|z\|_2 \) is the standard Euclidean norm on \( \mathbb{R}^n \).

Example Let

\[
\ell_1 = \{(z_1, z_2, z_3, \ldots) \in \mathbb{C}^\infty : |z_1| + |z_2| + |z_3| + \cdots \text{ converges}\},
\]

\[
\ell_2 = \{(z_1, z_2, z_3, \ldots) \in \mathbb{C}^\infty : |z_1|^2 + |z_2|^2 + |z_3|^2 + \cdots \text{ converges}\},
\]

\[
\ell_\infty = \{(z_1, z_2, z_3, \ldots) \in \mathbb{C}^\infty : \text{the sequence } |z_1|, |z_2|, |z_3|, \ldots \text{ is bounded}\},
\]

where \( \mathbb{C}^\infty \) denotes the set of all sequences \((z_1, z_2, z_3, \ldots)\) of complex numbers. Then \( \ell_1, \ell_2 \) and \( \ell_\infty \) are infinite-dimensional normed vector spaces, with norms
\[ \|z\|_1 = \sum_{j=1}^{+\infty} |z_j|, \]
\[ \|z\|_2 = \left( \sum_{j=1}^{+\infty} |z_j|^2 \right)^{\frac{1}{2}}, \]
\[ \|z\|_\infty = \sup\{|z_1|, |z_2|, |z_3|, \ldots\}. \]

(For example, to show that \[ \|z + w\|_2 \leq \|z\|_2 + \|w\|_2 \]
for all \( z, w \in \ell_2 \), we note that
\[ \left( \sum_{j=1}^{n} |z_j + w_j|^2 \right)^{\frac{1}{2}} = \left( \sum_{j=1}^{n} |z_j|^2 \right)^{\frac{1}{2}} + \left( \sum_{j=1}^{n} |w_j|^2 \right)^{\frac{1}{2}} \leq \|z\|_2 + \|w\|_2 \]
for all natural numbers \( n \), by the Triangle Inequality in \( C^n \). Taking limits as \( n \to +\infty \), we deduce that \[ \|z + w\|_2 \leq \|z\|_2 + \|w\|_2 \], as required.)

If \( x_1, x_2, \ldots, x_m \) are elements of a normed vector space \( X \) then
\[ \left\| \sum_{n=1}^{m} x_n \right\| \leq \sum_{n=1}^{m} \|x_n\|, \]
where \( \|\cdot\| \) denotes the norm on \( X \). (This follows directly using induction on \( m \).)

A norm \( \|\cdot\| \) on a vector space \( X \) induces a corresponding distance function on \( X \): the distance \( d(x, y) \) between elements \( x \) and \( y \) of \( X \) is defined by \( d(x, y) = \|x - y\| \). This distance function satisfies the metric space axioms. Thus any vector space with a given norm can be regarded as a metric space. A norm on a vector space \( X \) therefore generates a topology on \( X \): a subset \( U \) of \( X \) is an open set if and only if, given any point \( u \) of \( U \), there exists some \( \delta > 0 \) such that
\[ \{ x \in X : \|x - u\| < \delta \} \subset U. \]

The function \( x \mapsto \|x\| \) is a continuous function from \( X \) to \( \mathbb{R} \), since
\[ \|x\| - \|y\| = \|(x - y) + y\| = \|x - y\| \leq (\|x - y\| + \|y\|) - \|y\| = \|x - y\|, \]
and \( \|y\| - \|x\| \leq \|x - y\|, \) and therefore \( \|\|x\| - \|y\|| \leq \|x - y\|. \)

The Cartesian product \( X_1 \times X_2 \times \cdots \times X_n \) of vector spaces \( X_1, X_2, \ldots, X_n \) can itself be regarded as a vector space: if \( (x_1, x_2, \ldots, x_n) \) and \( (y_1, y_2, \ldots, y_n) \) are points of \( X_1 \times X_2 \times \cdots \times X_n \), and if \( \lambda \) is any scalar, then
\[ \lambda(x_1, x_2, \ldots, x_n) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n). \]
Lemma 6.1 Let $X_1, X_2, \ldots, X_n$ be normed vector spaces, and let $\|\cdot\|_{\text{max}}$ be the norm on $X_1 \times X_2 \times \cdots \times X_n$ defined by

$$\|(x_1, x_2, \ldots, x_n)\|_{\text{max}} = \max(\|x_1\|_1, \|x_2\|_2, \ldots, \|x_n\|_n),$$

where $\|\cdot\|_i$ is the norm on $X_i$ for $i = 1, 2, \ldots, n$. Then the topology on $X_1 \times X_2 \times \cdots \times X_n$ generated by the norm $\|\cdot\|_{\text{max}}$ is the product topology on $X_1 \times X_2 \times \cdots \times X_n$.

Proof It is a straightforward exercise to verify that $\|\cdot\|_{\text{max}}$ is indeed a norm on $X$, where $X = X_1 \times X_2 \times \cdots \times X_n$.

Let $U$ be a subset of $X$. Suppose that $U$ is open with respect to the product topology. Let $u$ be any point of $U$, given by $u = (u_1, u_2, \ldots, u_n)$. We must show that there exists some $\delta > 0$ such that

$$\{x \in X : \|x - u\|_{\text{max}} < \delta\} \subset U.$$

Now it follows from the definition of the product topology that there exist open sets $V_1, V_2, \ldots, V_n$ in $X_1, X_2, \ldots, X_n$ such that $u_i \in V_i$ for all $i$ and $V_1 \times V_2 \times \cdots \times V_n \subset U$. We can then take $\delta$ to be the minimum of $\delta_1, \delta_2, \ldots, \delta_n$, where $\delta_1, \delta_2, \ldots, \delta_n$ are chosen such that

$$\{x_i \in X_i : \|x_i - u_i\| < \delta_i\} \subset V_i$$

for $i = 1, 2, \ldots, n$.

Conversely suppose that $U$ is open with respect to the topology generated by the norm $\|\cdot\|_{\text{max}}$. Let $u$ be any point of $U$. Then there exists $\delta > 0$ such that

$$\{x \in X : \|x - u\|_{\text{max}} < \delta\} \subset U.$$

Let $V_i = \{x_i \in X_i : \|x_i - u_i\| < \delta_i\}$ for $i = 1, 2, \ldots, n$. Then, for each $i$, $V_i$ is an open set in $X_i$, $u_i \in V_i$, and $V_1 \times V_2 \times \cdots \times V_n \subset U$. We deduce that $U$ is also open with respect to the product topology, as required.

Proposition 6.2 Let $X$ be a normed vector space over the field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then the function from $X \times X$ to $X$ sending $(x, y) \in X \times X$ to $x + y$ is continuous. Also the function from $\mathbb{F} \times X$ to $X$ sending $(\lambda, x) \in \mathbb{F} \times X$ to $\lambda x$ is continuous.

Proof Let $(u, v) \in X \times X$, and let $\varepsilon > 0$ be given. Let $\delta = \frac{1}{2} \varepsilon$. If $(x, y) \in X \times X$ satisfies $\|(x, y) - (u, v)\|_{\text{max}} < \delta$, then $\|x - u\| < \delta$ and $\|y - v\| < \delta$, and hence

$$\|(x + y) - (u + v)\| \leq \|x - u\| + \|y - v\| < \varepsilon.$$
This shows that the function \((x, y) \mapsto x + y\) is continuous at \((u, v) \in X \times X\). Next let \((\mu, u) \in F \times X\), and let \(\varepsilon > 0\) be given. Let
\[
\delta = \min \left( \frac{\varepsilon}{2(\|u\| + 1)}, \frac{\varepsilon}{2(\|\mu\| + 1)}, 1 \right).
\]
Now \(\lambda x - \mu u = \lambda(x - u) + (\lambda - \mu) u\) for all \(\lambda \in F\) and \(x \in X\). Thus if \((\lambda, x) \in F \times X\) satisfies \(\| (\lambda, x) - (\mu, u) \|_{\text{max}} < \delta\), then
\[
|\lambda - \mu| < \frac{\varepsilon}{2(\|u\| + 1)}, \quad \|x - u\| < \frac{\varepsilon}{2(\|\mu\| + 1)}, \quad |\lambda| < |\mu| + 1,
\]
and hence
\[
|\lambda x - \mu u| \leq |\lambda| \|x - u\| + |\lambda - \mu| \|u\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
This shows that the function \((\lambda, x) \mapsto \lambda x\) is continuous at \((\mu, u) \in F \times X\), as required.

Corollary 6.3

Let \(X\) be a normed vector space over the field \(F\), where \(F = \mathbb{R}\) or \(\mathbb{C}\). Let \((x_n)\) and \((y_n)\) be convergent sequences in \(X\), and let \((\lambda_n)\) be a convergent sequence in \(F\). Then the sequences \((x_n + y_n)\) and \((\lambda_n x_n)\) are convergent in \(X\), and
\[
\lim_{n \to +\infty} (x_n + y_n) = \lim_{n \to +\infty} x_n + \lim_{n \to +\infty} y_n, \\
\lim_{n \to +\infty} (\lambda_n x_n) = \left( \lim_{n \to +\infty} \lambda_n \right) \left( \lim_{n \to +\infty} x_n \right).
\]

Proof

Let \(x = \lim_{n \to +\infty} x_n\), \(y = \lim_{n \to +\infty} y_n\) and \(\lambda = \lim_{n \to +\infty} \lambda_n\). Using Lemma 6.1, together with the definition of convergence in metric spaces, it follows easily that the sequences \((x_n, y_n)\) and \((\lambda_n, x_n)\) converge to \((x, y)\) and \((\lambda, x)\) in \(X \times X\) and \(F \times X\) respectively. The convergence of \((x_n + y_n)\) and \(\lambda_n x_n\) to \(x + y\) and \(\lambda x\) respectively now follows from Proposition 6.2 (using Lemma 1.3).

Let \(X\) be a normed vector space, and let \(x_1, x_2, x_3, \ldots\) be elements of \(X\). The infinite series \(\lim_{n \to +\infty} x_n\) is said to converge to some element \(s\) of \(X\) if, given any \(\varepsilon > 0\), there exists some natural number \(N\) such that
\[
\|s - \sum_{n=1}^{m} x_n\| < \varepsilon
\]
for all \(m \geq N\) (where \(\|\cdot\|\) denotes the norm on \(X\)).
We say that a normed vector space $X$ is complete if and only if every Cauchy sequence in $X$ is convergent. (A sequence $x_1, x_2, x_3, \ldots$ is a Cauchy sequence if and only if, given any $\varepsilon > 0$, there exists some natural number $N$ such that $\|x_j - x_k\| < \varepsilon$ for all $j$ and $k$ satisfying $j \geq N$ and $k \geq N$.)

A complete normed vector space is referred to as a Banach space. (The basic theory of such spaces was extensively developed by the famous Polish mathematician Stefan Banach and his co-workers.)

**Lemma 6.4** Let $X$ be a Banach space, and let $x_1, x_2, x_3, \ldots$ be elements of $X$. Suppose that $\lim_{n \to +\infty} \|x_n\|$ is convergent. Then $\lim_{n \to +\infty} x_n$ is convergent, and

$$\left\| \lim_{n \to +\infty} x_n \right\| \leq \lim_{n \to +\infty} \|x_n\|.$$ 

**Proof** For each natural number $n$, let

$$s_n = x_1 + x_2 + \cdots + x_n.$$ 

Let $\varepsilon > 0$ be given. We can find $N$ such that $\sum_{n=N}^{+\infty} \|x_n\| < \varepsilon$, since $\lim_{n \to +\infty} \|x_n\|$ is convergent. Let $s_n = x_1 + x_2 + \cdots + x_n$. If $j \geq N$, $k \geq N$ and $j < k$ then

$$\|s_k - s_j\| = \left\| \sum_{n=j+1}^{k} x_n \right\| \leq \sum_{n=j+1}^{k} \|x_n\| \leq \sum_{n=N}^{+\infty} \|x_n\| < \varepsilon.$$ 

Thus $s_1, s_2, s_3, \ldots$ is a Cauchy sequence in $X$, and therefore converges to some element $s$ of $X$, since $X$ is complete. But then $s = \lim_{j \to +\infty} x_j$. Moreover, on choosing $m$ large enough to ensure that $s - s_m < \varepsilon$, we deduce that

$$\|s\| \leq \left\| \sum_{n=1}^{m} x_n \right\| + \left\| s - \sum_{n=1}^{m} x_n \right\| \leq \sum_{n=1}^{m} \|x_n\| + \left\| s - \sum_{n=1}^{m} x_n \right\| < \sum_{n=1}^{+\infty} \|x_n\| + \varepsilon.$$ 

Since this inequality holds for all $\varepsilon > 0$, we conclude that

$$\|s\| \leq \sum_{n=1}^{+\infty} \|x_n\|,$$

as required. 

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6.2 Bounded Linear Transformations

Let $X$ and $Y$ be real or complex vector spaces. A function $T: X \to Y$ is said to be a linear transformation if $T(x + y) = Tx + Ty$ and $T(\lambda x) = \lambda Tx$ for all elements $x$ and $y$ of $X$ and scalars $\lambda$. A linear transformation mapping $X$ into itself is referred to as a linear operator on $X$.

**Definition** Let $X$ and $Y$ be normed vector spaces. A linear transformation $T: X \to Y$ is said to be bounded if there exists some non-negative real number $C$ with the property that $\|Tx\| \leq C\|x\|$ for all $x \in X$. If $T$ is bounded, then the smallest non-negative real number $C$ with this property is referred to as the operator norm of $T$, and is denoted by $\|T\|$.

**Lemma 6.5** Let $X$ and $Y$ be normed vector spaces, and let $S: X \to Y$ and $T: X \to Y$ be bounded linear transformations. Then $S + T$ and $\lambda S$ are bounded linear transformations for all scalars $\lambda$, and

$$\|S + T\| \leq \|S\| + \|T\|, \quad \|\lambda S\| = |\lambda|\|S\|.$$  

Moreover $\|S\| = 0$ if and only if $S = 0$. Thus the vector space $B(X,Y)$ of bounded linear transformations from $X$ to $Y$ is a normed vector space (with respect to the operator norm).

**Proof** $\|(S+T)x\| \leq \|Sx\| + \|Tx\| \leq (\|S\| + \|T\|)\|x\|$ for all $x \in X$. Therefore $S + T$ is bounded, and $\|S + T\| \leq \|S\| + \|T\|$. Using the fact that $\|\lambda S\|x\| = |\lambda|\|Sx\|$ for all $x \in X$, we see that $\lambda S$ is bounded, and $\|\lambda S\| = |\lambda|\|S\|$. If $S = 0$ then $\|S\| = 0$. Conversely if $\|S\| = 0$ then $\|Sx\| \leq \|S\|\|x\| = 0$ for all $x \in X$, and hence $S = 0$. The result follows.

**Lemma 6.6** Let $X$, $Y$ and $Z$ be normed vector spaces, and let $S: X \to Y$ and $T: Y \to Z$ be bounded linear transformations. Then the composition $TS$ of $S$ and $T$ is also bounded, and $\|TS\| \leq \|T\|\|S\|$.

**Proof** $\|TSx\| \leq \|T\|\|Sx\| \leq \|T\|\|S\|\|x\|$ for all $x \in X$. The result follows.

**Proposition 6.7** Let $X$ and $Y$ be normed vector spaces, and let $T: X \to Y$ be a linear transformation from $X$ to $Y$. Then the following are equivalent:

(i) $T: X \to Y$ is continuous,

(ii) $T: X \to Y$ is continuous at 0,
(iii) \( T: X \to Y \) is bounded.

**Proof** Obviously (i) implies (ii). We show that (ii) implies (iii) and (iii) implies (i). The equivalence of the three conditions then follows immediately.

Suppose that \( T: X \to Y \) is continuous at 0. Then there exists \( \delta > 0 \) such that \( \|Tx\| < 1 \) for all \( x \in X \) satisfying \( \|x\| < \delta \). Let \( C \) be any positive real number satisfying \( C > 1/\delta \). If \( x \) is any non-zero element of \( X \) then \( \|\lambda x\| < \delta \), where \( \lambda = 1/(C\|x\|) \), and hence

\[
\|Tx\| = C\|x\| \|\lambda Tx\| = C\|x\| \|T(\lambda x)\| < C\|x\|.
\]

Thus \( \|Tx\| \leq C\|x\| \) for all \( x \in X \), and hence \( T: X \to Y \) is bounded. Thus (ii) implies (iii).

Finally suppose that \( T: X \to Y \) is bounded. Let \( x \) be a point of \( X \), and let \( \varepsilon > 0 \) be given. Choose \( \delta > 0 \) satisfying \( \|T\|\delta < \varepsilon \). If \( x' \in X \) satisfies \( \|x' - x\| < \delta \) then

\[
\|T(x' - x)\| = \|T(\lambda x' - \lambda x)\| \leq \|T\| \|\lambda x' - \lambda x\| < \|T\|\delta < \varepsilon.
\]

Thus \( T: X \to Y \) is continuous. Thus (iii) implies (i), as required. \( \blacksquare \)

Let \( X \) be a normed vector space, and let \( x_1, x_2, x_3, \ldots \) be elements of \( X \). The infinite series \( \lim_{n \to +\infty} x_n \) is said to **converge** to some element \( s \) of \( X \) if, given any \( \varepsilon > 0 \), there exists some natural number \( N \) such that

\[
\|s - \sum_{n=1}^{m} x_n\| < \varepsilon
\]

for all \( m \geq N \) (where \( \|\cdot\| \) denotes the norm on \( X \)).

**Proposition 6.8** Let \( X \) be a normed vector space and let \( Y \) be a Banach space. Then the space \( B(X,Y) \) of bounded linear transformations from \( X \) to \( Y \) is also a Banach space.

**Proof** We have already shown that \( B(X,Y) \) is a normed vector space (see Lemma 6.5). Thus it only remains to show that \( B(X,Y) \) is complete.

Let \( S_1, S_2, S_3, \ldots \) be a Cauchy sequence in \( B(X,Y) \). Let \( x \in X \). We claim that \( S_1x, S_2x, S_3x, \ldots \) is a Cauchy sequence in \( Y \). This result is trivial if \( x = 0 \). If \( x \neq 0 \), and if \( \varepsilon > 0 \) is given then there exists some natural number \( N \) such that \( \|S_j - S_k\| < \varepsilon/\|x\| \) whenever \( j \geq N \) and \( k \geq N \). But then \( \|S_jx - S_kx\| \leq \|S_j - S_k\| \|x\| < \varepsilon \) whenever \( j \geq N \) and \( k \geq N \). This shows that \( S_1x, S_2x, S_3x, \ldots \) is indeed a Cauchy sequence. It therefore converges to some element of \( Y \), since \( Y \) is a Banach space.
Let the function $S: X \to Y$ be defined by $Sx = \lim_{n \to +\infty} S_n x$. Then

$$S(x + y) = \lim_{n \to +\infty} (S_n x + S_n y) = \lim_{n \to +\infty} S_n x + \lim_{n \to +\infty} S_n y = Sx + Sy,$$

(see Corollary 6.3), and

$$S(\lambda x) = \lim_{n \to +\infty} S_n (\lambda x) = \lambda \lim_{n \to +\infty} S_n x = \lambda Sx,$$

Thus $S: X \to Y$ is a linear transformation.

Next we show that $S_n \to S$ in $B(X, Y)$ as $n \to +\infty$. Let $\varepsilon > 0$ be given. Then there exists some natural number $N$ such that $\|S_j - S_n\| < \frac{1}{2} \varepsilon$ whenever $j \geq N$ and $n \geq N$, since the sequence $S_1, S_2, S_3, \ldots$ is a Cauchy sequence in $B(X, Y)$. But then $\|S_j x - S_n x\| < \frac{1}{2} \varepsilon \|x\|$ for all $j \geq N$ and $n \geq N$, and thus

$$\|Sx - S_n x\| = \left\| \lim_{j \to +\infty} S_j x - S_n x \right\| = \lim_{j \to +\infty} \|S_j x - S_n x\| \leq \lim_{j \to +\infty} \|S_j - S_n\| \|x\| \leq \frac{1}{2} \varepsilon \|x\|$$

for all $n \geq N$ (since the norm is a continuous function on $Y$). But then

$$\|Sx\| \leq \|S_n x\| + \|Sx - S_n x\| \leq (\|S_n\| + \frac{1}{2} \varepsilon) \|x\|$$

for any $n \geq N$, showing that $S: X \to Y$ is a bounded linear transformation, and $\|S - S_n\| \leq \frac{1}{2} \varepsilon < \varepsilon$ for all $n \geq N$, showing that $S_n \to S$ in $B(X, Y)$ as $n \to +\infty$. Thus the Cauchy sequence $S_1, S_2, S_3, \ldots$ is convergent in $B(X, Y)$, as required.

**Corollary 6.9** Let $X$ and $Y$ be Banach spaces, and let $T_1, T_2, T_3, \ldots$ be bounded linear transformations from $X$ to $Y$. Suppose that $\lim_{n \to +\infty} \|T_n\|$ is convergent. Then $\lim_{n \to +\infty} T_n$ is convergent, and

$$\left\| \lim_{n \to +\infty} T_n \right\| \leq \lim_{n \to +\infty} \|T_n\|.$$

**Proof** The space $B(X, Y)$ of bounded linear maps from $X$ to $Y$ is a Banach space by Proposition 6.8. The result therefore follows immediately on applying Lemma 6.4.

**Example** Let $T$ be a bounded linear operator on a Banach space $X$ (i.e., a bounded linear transformation from $X$ to itself). The infinite series

$$\sum_{n=0}^{+\infty} \frac{\|T\|^n}{n!}$$
converges to \( \exp(\|T\|) \). It follows immediately from Lemma 6.6 (using induction on \( n \)) that \( \|T^n\| \leq \|T\|^n \) for all \( n \geq 0 \) (where \( T^0 \) is the identity operator on \( X \)). It therefore follows from Corollary 6.7 that there is a well-defined bounded linear operator \( \exp T \) on \( X \), defined by

\[
\exp T = \sum_{n=0}^{+\infty} \frac{1}{n!} T^n
\]

(where \( T_0 \) is the identity operator \( I \) on \( X \)).

**Proposition 6.10** Let \( T \) be a bounded linear operator on a Banach space \( X \). Suppose that \( \|T\| < 1 \). Then the operator \( I - T \) has a bounded inverse \((I - T)^{-1}\) (where \( I \) denotes the identity operator on \( X \)). Moreover

\[
(I - T)^{-1} = I + T + T^2 + T^3 + \cdots.
\]

**Proof** \( \|T^n\| \leq \|T\|^n \) for all \( n \), and the geometric series

\[
1 + \|T\| + \|T\|^2 + \|T\|^3 + \cdots
\]

is convergent (since \( \|T\| < 1 \)). It follows from Corollary 6.9 that the infinite series

\[
(I - T)^{-1} = I + T + T^2 + T^3 + \cdots
\]

converges to some bounded linear operator \( S \) on \( X \). Now

\[
(I - T)S = \lim_{n \to +\infty} (I - T)(I + T + T^2 + \cdots + T^n) = \lim_{n \to +\infty} (I - T^{n+1}) = I - \lim_{n \to +\infty} T^{n+1} = I,
\]

since \( \|T\|^{n+1} \to 0 \) and therefore \( T^{n+1} \to 0 \) as \( n \to +\infty \). Similarly \( S(I - T) = I \). This shows that \( I - T \) is invertible, with inverse \( S \), as required. □

### 6.3 Equivalence of Norms on a Finite-Dimensional Vector Space

Let \( \| \cdot \| \) and \( \| \cdot \|_* \) be norms on a real or complex vector space \( X \). The norms \( \| \cdot \| \) and \( \| \cdot \|_* \) are said to be *equivalent* if and only if there exist constants \( c \) and \( C \), where \( 0 < c \leq C \), such that

\[
c\|x\| \leq \|x\|_* \leq C\|x\|
\]

for all \( x \in X \).
Lemma 6.11 Two norms \(|\cdot|\) and \(|\cdot|_*\) on a real or complex vector space \(X\) are equivalent if and only if they induce the same topology on \(X\).

Proof Suppose that the norms \(|\cdot|\) and \(|\cdot|_*\) induce the same topology on \(X\). Then there exists some \(\delta > 0\) such that

\[
\{ x \in X : |x| < \delta \} \subset \{ x \in X : |x|_* < 1 \},
\]

since the set \(\{ x \in X : |x|_* < 1 \}\) is open with respect to the topology on \(X\) induced by both \(|\cdot|_*\) and \(|\cdot|\). Let \(C\) be any positive real number satisfying \(C\delta > 1\). Then

\[
\frac{1}{C|x|} x = \frac{1}{C} < \delta,
\]

and hence

\[
|x|_* = C|x| \left( \frac{1}{C|x|} x \right) < C|x|.
\]

for all non-zero elements \(x\) of \(X\), and thus \(|x|_* \leq C|x|\) for all \(x \in X\). On interchanging the roles of the two norms, we deduce also that there exists a positive real number \(c\) such that \(c|x| \leq (1/c)|x|_*\) for all \(x \in X\). But then \(c|x| \leq |x|_* \leq C|x|\) for all \(x \in X\). We conclude that the norms \(|\cdot|\) and \(|\cdot|_*\) are equivalent.

Conversely suppose that the norms \(|\cdot|\) and \(|\cdot|_*\) are equivalent. Then there exist constants \(c\) and \(C\), where \(0 < c \leq C\), such that \(c|x| \leq |x|_* \leq C|x|\) for all \(x \in X\). Let \(U\) be a subset of \(X\) that is open with respect to the topology on \(X\) induced by the norm \(|\cdot|_*\), and let \(u \in U\). Then there exists some \(\delta > 0\) such that

\[
\{ x \in X : |x - u|_* < C\delta \} \subset U.
\]

But then

\[
\{ x \in X : |x - u| < \delta \} \subset \{ x \in X : |x - u|_* < C\delta \} \subset U,
\]

showing that \(U\) is open with respect to the topology induced by the norm \(|\cdot|\). Similarly any subset of \(X\) that is open with respect to the topology induced by the norm \(|\cdot|_*\) must also be open with respect to the topology induced by \(|\cdot|_*\). Thus equivalent norms induce the same topology on \(X\). 

It follows immediately from Lemma 6.11 that if \(|\cdot|, |\cdot|_*\) and \(|\cdot|_2\) are norms on a real (or complex) vector space \(X\), if the norms \(|\cdot|\) and \(|\cdot|_*\) are equivalent, and if the norms \(|\cdot|_*\) and \(|\cdot|_2\) are equivalent, then the norms \(|\cdot|\) and \(|\cdot|_2\) are also equivalent. This fact can easily be verified directly from the definition of equivalence of norms.

We recall that the usual topology on \(\mathbb{R}^n\) is that generated by the Euclidean norm on \(\mathbb{R}^n\).
Lemma 6.12. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. Then the function $x \mapsto \|x\|$ is continuous with respect to the usual topology on $\mathbb{R}^n$.

Proof. Let $e_1, e_2, \ldots, e_n$ denote the basis of $\mathbb{R}^n$ given by
\[ e_1 = (1, 0, 0, \ldots, 0), \quad e_2 = (0, 1, 0, \ldots, 0), \ldots, \quad e_n = (0, 0, 0, \ldots, 1). \]

Let $x$ and $y$ be points of $\mathbb{R}^n$, given by
\[ x = (x_1, x_2, \ldots, x_n), \quad y = (y_1, y_2, \ldots, y_n). \]

Using Schwarz' Inequality, we see that
\[
\|x - y\| = \left\| \sum_{j=1}^{n} (x_j - y_j) e_j \right\| \leq \sum_{j=1}^{n} |x_j - y_j| \|e_j\| \\
\leq \left( \sum_{j=1}^{n} (x_j - y_j)^2 \right)^{\frac{1}{2}} \left( \sum_{j=1}^{n} \|e_j\|^2 \right)^{\frac{1}{2}} = C \|x - y\|_2,
\]
where
\[ C^2 = \|e_1\|^2 + \|e_2\|^2 + \cdots + \|e_n\|^2 \]
and $\|x - y\|_2$ denotes the Euclidean norm of $x - y$, defined by
\[ \|x - y\|_2 = \left( \sum_{j=1}^{n} (x_j - y_j)^2 \right)^{\frac{1}{2}}. \]

Also $\|\|x\| - \|y\|| \leq \|x - y\|$, since
\[ \|x\| \leq \|x - y\| + \|y\|, \quad \|y\| \leq \|x - y\| + \|x\|. \]

We conclude therefore that
\[ \|\|x\| - \|y\|| \leq C \|x - y\|_2, \]
for all $x, y \in \mathbb{R}^n$, and thus the function $x \mapsto \|x\|$ is continuous on $\mathbb{R}^n$ (with respect to the usual topology on $\mathbb{R}^n$).

Theorem 6.13. Any two norms on $\mathbb{R}^n$ are equivalent, and induce the usual topology on $\mathbb{R}^n$. 

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Proof Let \( \| \cdot \| \) be any norm on \( \mathbb{R}^n \). We show that \( \| \cdot \| \) is equivalent to the Euclidean norm \( \| \cdot \|_2 \). Let \( S^{n-1} \) denote the unit sphere in \( \mathbb{R}^n \), defined by

\[
S^{n-1} = \{ x \in \mathbb{R}^n : \| x \|_2 = 1 \}.
\]

Now \( S^{n-1} \) is a compact subset of \( \mathbb{R}^n \), since it is both closed and bounded (see Theorem 4.16). Also the function \( x \mapsto \| x \| \) is continuous (Lemma 6.12). But any continuous real-valued function on a compact topological space attains both its maximum and minimum values on that space (Proposition 4.6). Therefore there exist points \( u \) and \( v \) of \( S^{n-1} \) such that \( \| u \| \leq \| x \| \leq \| v \| \) for all \( x \in S^{n-1} \). Set \( c = \| u \| \) and \( C = \| v \| \). Then \( 0 < c \leq C \) (since it follows from the definition of norms that the norm of any non-zero element of \( \mathbb{R}^n \) is necessarily non-zero).

If \( x \) is any non-zero element of \( \mathbb{R}^n \) then \( \lambda x \in S^{n-1} \), where \( \lambda = 1/\| x \|_2 \). But \( \| \lambda x \| = |\lambda| \| x \| \) (see the the definition of norms). Therefore \( c \leq |\lambda| \| x \| \leq C \), and hence \( c \| x \|_2 \leq \| x \| \leq C \| x \|_2 \) for all \( x \in \mathbb{R}^n \), showing that the norm \( \| \cdot \| \) is equivalent to the Euclidean norm \( \| \cdot \|_2 \) on \( \mathbb{R}^n \). Therefore any two norms on \( \mathbb{R}^n \) are equivalent, and thus generate the same topology on \( \mathbb{R}^n \) (Lemma 6.11). This topology must then be the usual topology on \( \mathbb{R}^n \). 

Let \( X \) be a finite-dimensional real vector space. Then \( X \) is isomorphic to \( \mathbb{R}^n \), where \( n \) is the dimension of \( X \). It follows immediately from Theorem 6.13 and Lemma 6.11 that all norms on \( X \) are equivalent and therefore generate the same topology on \( X \). This result does not generalize to infinite-dimensional vector spaces.