Course 212: Academic Year 1991-2
Section 5: Complete and Compact Metric Spaces

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5 Complete and Compact Metric Spaces

5.1 Complete Metric Spaces

Definition Let X be a metric space with distance function d. A sequence $x_1, x_2, x_3, \ldots$ of points of X is said to be a *Cauchy sequence* in X if and only if, given any $\varepsilon > 0$, there exists some natural number N such that $d(x_j, x_k) < \varepsilon$ for all j and k satisfying $j \geq N$ and $k \geq N$.

Every convergent sequence in a metric space is a Cauchy sequence. Indeed let X be a metric space with distance function d, and let $x_1, x_2, x_3, \ldots$ be a sequence of points in X which converges to some point p of X. Given any $\varepsilon > 0$, there exists some natural number N such that $d(x_n, p) < \varepsilon/2$ whenever $n \geq N$. But then it follows from the Triangle Inequality that

$$d(x_j, x_k) \leq d(x_j, p) + d(p, x_k) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $j \geq N$ and $k \geq N$.

Definition A metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges to some point of X.

The spaces $\mathbb{R}$ and $\mathbb{C}$ are complete metric spaces with respect to the distance function given by $d(z, w) = |z - w|$. Indeed this result is *Cauchy's Criterion for Convergence.* However the space $\mathbb{Q}$ of rational numbers (with distance function $d(q, r) = |q - r|$) is not complete. Indeed one can construct an infinite sequence $q_1, q_2, q_3, \ldots$ of rational numbers which converges (in $\mathbb{R}$) to $\sqrt{2}$. Such a sequence of rational numbers is a Cauchy sequence in both $\mathbb{R}$ and $\mathbb{Q}$. However this Cauchy sequence does not converge to an point of the metric space $\mathbb{Q}$ (since $\sqrt{2}$ is an irrational number). Thus the metric space $\mathbb{Q}$ is not complete.

Lemma 5.1 *Let X be a complete metric space, and let A be a subset of X. Then A is complete if and only if A is closed in X.*

Proof Suppose that A is closed in X. Let $a_1, a_2, a_3, \ldots$ be a Cauchy sequence in A. This Cauchy sequence must converge to some point p of X, since X is complete. But the limit of every sequence of points of A must belong to A, since A is closed (see Lemma 1.10). In particular $p \in A$. We deduce that A is complete.

Conversely, suppose that A is complete. Suppose that A were not closed. Then the complement $X \setminus A$ of A would not be open, and therefore there would exist a point p of $X \setminus A$ with the property that $B_X(p, \delta) \cap A$ is non-empty for all $\delta > 0$, where $B_X(p, \delta)$ denotes the open ball in X of radius δ centred at p. We could then find a sequence $a_1, a_2, a_3, \ldots$ of points of A satisfying $d(a_j, p) < 1/j$ for all natural numbers j. This sequence would be a Cauchy sequence in A which did not converge to a point of A, contradicting the completeness of A. Thus A must be closed, as required. ∎

Theorem 5.2 *The metric space $\mathbb{R}^n$ (with the Euclidean distance function) is a complete metric space.*

Proof Let $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots$ be a Cauchy sequence in $\mathbb{R}^n$. Then for each integer m between 1 and n, the sequence $(\mathbf{p}_1)_m, (\mathbf{p}_2)_m, (\mathbf{p}_3)_m, \ldots$ is a Cauchy sequence of real numbers, where $(\mathbf{p}_j)_m$ denotes the mth component of $\mathbf{p}_j$. But every Cauchy sequence of real numbers is convergent (Cauchy's criterion for convergence). Let $q_m = \lim\limits_{j \to +\infty} (\mathbf{p}_j)_m$ for $m = 1, 2, \ldots, n$, and let $\mathbf{q} = (q_1, q_2, \ldots, q_n)$. We claim that $\mathbf{p}_j \to \mathbf{q}$ as $j \to +\infty$.

Let $\varepsilon > 0$ be given. Then there exist natural numbers $N_1, N_2, \ldots, N_n$ such that $|(\mathbf{p}_j)_m - q_m| < \varepsilon/\sqrt{n}$ whenever $j \geq N_m$ (where $m = 1, 2, \ldots, n$). Let N be the maximum of $N_1, N_2, \ldots, N_n$. If $j \geq N$ then

$$|\mathbf{p}_j - \mathbf{q}|^2 = \sum_{m=1}^{n} ((\mathbf{p}_j)_m - q_m)^2 < \varepsilon^2.$$

Thus $\mathbf{p}_j \to \mathbf{q}$ as $j \to +\infty$. Thus every Cauchy sequence in $\mathbb{R}^n$ is convergent, as required. ∎

The following result follows directly from Lemma 5.1 and Theorem 5.2.

Corollary 5.3 *A subset X of $\mathbb{R}^n$ is complete if and only if it is closed.*

Example The n-sphere S^n (with the chordal distance function given by $d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$) is a complete metric space, where

$$S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}.$$

5.2 Characterizations of Compact Metric Spaces

We recall that a metric or topological space is said to be *compact* if every open cover of the space has a finite subcover. We shall obtain some equivalent characterizations of compactness for *metric spaces* (Theorem 5.9); these characterizations do not generalize to arbitrary topological spaces.

Proposition 5.4 *Every sequence of points in a compact metric space has a convergent subsequence.*

Proof Let X be a compact metric space, and let $x_1, x_2, x_3, \ldots$ be a sequence of points of X. We must show that this sequence has a convergent subsequence. Let F_n denote the closure of $\{x_n, x_{n+1}, x_{n+2}, \ldots\}$. We claim that the intersection of the sets $F_1, F_2, F_3, \ldots$ is non-empty. For suppose that this intersection were the empty set. Then X would be the union of the sets $V_1, V_2, V_3, \ldots$, where $V_n = X \setminus F_n$ for all n. But $V_1 \subset V_2 \subset V_3 \subset \cdots$, and each set V_n is open. It would therefore follow from the compactness of X that X would be covered by finitely many of the sets $V_1, V_2, V_3, \ldots$, and therefore $X = V_n$ for some sufficiently large n. But this is impossible, since F_n is non-empty for all natural numbers n. Thus the intersection of the sets $F_1, F_2, F_3, \ldots$ is non-empty, as claimed, and therefore there exists a point p of X which belongs to F_n for all natural numbers n.

We now obtain, by induction on n, a subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ which satisfies $d(x_{n_j}, p) < 1/j$ for all natural numbers j. Now p belongs to the closure F_1 of the set $\{x_1, x_2, x_3, \ldots\}$. Therefore there exists some natural number n_1 such that $d(x_{n_1}, p) < 1$. Suppose that x_{n_j} has been chosen so that $d(x_{n_j}, p) < 1/j$. The point p belongs to the closure F_{n_j+1} of the set $\{x_n : n > n_j\}$. Therefore there exists some natural number n_{j+1} such that $n_{j+1} > n_j$ and $d(x_{n_{j+1}}, p) < 1/(j+1)$. The subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ constructed in this manner converges to the point p, as required. ∎

We shall also prove the converse of Proposition 5.4: if X is a metric space, and if every sequence of points of X has a convergent subsequence, then X is compact (see Theorem 5.9 below).

Proposition 5.5 *Let X be a metric space with the property that every sequence of points of X has a convergent subsequence. Then X is complete.*

Proof Let $x_1, x_2, x_3, \ldots$ be a Cauchy sequence in X. This sequence then has a subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ which converges to some point p of X. We claim that the given Cauchy sequence also converges to p.

Let $\varepsilon > 0$ be given. Then there exists some natural number N such that $d(x_m, x_n) < \frac{1}{2}\varepsilon$ whenever $m \geq N$ and $n \geq N$, since $x_1, x_2, x_3, \ldots$ is a Cauchy sequence. Moreover n_j can be chosen large enough to ensure that $n_j \geq N$ and $d(x_{n_j}, p) < \frac{1}{2}\varepsilon$. If $n \geq N$ then

$$d(x_n, p) \leq d(x_n, x_{n_j}) + d(x_{n_j}, p) < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

This shows that the Cauchy sequence $x_1, x_2, x_3, \ldots$ converges to the point p. Thus X is complete, as required. ∎

Definition Let X be a metric space with distance function d. A subset A of X is said to be *bounded* if there exists a non-negative real number K such that $d(x,y) \leq K$ for all $x, y \in A$. The smallest real number K with this property is referred to as the *diameter* of A, and is denoted by $\operatorname{diam} A$. (Note that $\operatorname{diam} A$ is the supremum of the values of $d(x,y)$ as x and y range over all points of A.)

Lemma 5.6 *Let X be a metric space, and let A be a subset of X. Then* $\operatorname{diam} A = \operatorname{diam} \overline{A}$*, where $\overline{A}$ is the closure of A.*

Proof Clearly $\operatorname{diam} A \leq \operatorname{diam} \overline{A}$. Let x and y be points of $\overline{A}$. Then, given any $\varepsilon > 0$, there exist points x' and y' of A satisfying $d(x,x') < \varepsilon$ and $d(y,y') < \varepsilon$ (see Lemma 1.11). It follows from the Triangle Inequality that

$$d(x,y) \leq d(x,x') + d(x',y') + d(y',y) < \operatorname{diam} A + 2\varepsilon.$$

Thus $d(x,y) < \operatorname{diam} A + 2\varepsilon$ for all $\varepsilon > 0$, and hence $d(x,y) \leq \operatorname{diam} A$. This shows that $\operatorname{diam} \overline{A} \leq \operatorname{diam} A$, as required. ∎

Definition A metric space X is said to be *totally bounded* if, given any $\varepsilon > 0$, the set X can be expressed as a finite union of subsets of X, each of which has diameter less than ε.

Any subset A of a totally bounded metric space X is itself totally bounded. For if X is the union of the subsets $B_1, B_2, \ldots, B_k$, where $\operatorname{diam} B_n < \varepsilon$ for $n = 1, 2, \ldots, k$, then A is the union of $A \cap B_n$ for $n = 1, 2, \ldots, k$, and $\operatorname{diam} A \cap B_n < \varepsilon$.

Proposition 5.7 *Let X be a metric space. Suppose that every sequence of points of X has a convergent subsequence. Then X is totally bounded.*

Proof Suppose that X were not totally bounded. Then there would exist some $\varepsilon > 0$ with the property that no finite collection of subsets of X of diameter less than 3ε covers the set X. There would then exist an infinite sequence $x_1, x_2, x_3, \ldots$ of points of X with the property that $d(x_m, x_n) \geq \varepsilon$ whenever $m \neq n$. Indeed suppose that points $x_1, x_2, \ldots, x_{k-1}$ of X have already been chosen satisfying $d(x_m, x_n) \geq \varepsilon$ whenever $m < k$, $n < k$ and $m \neq n$. The diameter of each open ball $B_X(x_m, \varepsilon)$ is less than or equal to 2ε. Therefore X could not be covered by the sets $B_X(x_m, \varepsilon)$ for $m < k$, and thus there would exist a point x_k of X which does not belong to $B(x_m, \varepsilon)$ for any $m < k$. Then $d(x_m, x_k) \geq \varepsilon$ for all $m < k$. In this way we can successively choose points $x_1, x_2, x_3, \ldots$ to form an infinite sequence with the required property. However such an infinite sequence would have no convergent subsequence, which is impossible. This shows that X must be totally bounded, as required. ∎

Proposition 5.8 *Every complete totally bounded metric space is compact.*

Proof Let X be some totally bounded metric space. Suppose that there exists an open cover $\mathcal{V}$ of X which has no finite subcover. We shall prove the existence of a Cauchy sequence $x_1, x_2, x_3, \ldots$ in X which cannot converge to any point of X. (Thus if X is not compact, then X cannot be complete.)

Let $\varepsilon > 0$ be given. Then X can be covered by finitely many closed sets whose diameter is less than ε, since X is totally bounded and every subset of X has the same diameter as its closure (Lemma 5.6). At least one of these closed sets cannot be covered by a finite collection of open sets belonging to $\mathcal{V}$ (since if every one of these closed sets could be covered by a such a finite collection of open sets, then we could combine these collections to obtain a finite subcover of $\mathcal{V}$). We conclude that, given any $\varepsilon > 0$, there exists a closed subset of X of diameter less than ε which cannot be covered by any finite collection of open sets belonging to $\mathcal{V}$.

We claim that there exists a sequence $F_1, F_2, F_3, \ldots$ of closed sets in X satisfying $F_1 \supset F_2 \supset F_3 \supset \cdots$ such that each closed set F_n has the following properties: $\operatorname{diam} F_n < 1/2^n$, and no finite collection of open sets belonging to $\mathcal{V}$ covers F_n. For if F_n is a closed set with these properties then F_n is itself both complete (Lemma 5.1) and totally bounded, and thus the above remarks (applied with F_n in place of X) guarantee the existence of a closed subset F_{n+1} of F_n with the required properties. Thus the existence of the required sequence of closed sets follows by induction on n.

Choose $x_n \in F_n$ for each natural number n. Then $d(x_m, x_n) < 1/2^n$ for any $m > n$, since x_m and x_n belong to F_n and $\operatorname{diam} F_n < 1/2^n$. Therefore the sequence $x_1, x_2, x_3, \ldots$ is a Cauchy sequence. Suppose that this Cauchy sequence were to converge to some point p of X. Then $p \in F_n$ for each natural number n, since F_n is closed and $x_m \in F_n$ for all $m \geq n$ (see Lemma 1.10). Moreover $p \in V$ for some open set V belonging to $\mathcal{V}$, since $\mathcal{V}$ is an open cover of X. But then there would exist $\delta > 0$ such that $B_X(p, \delta) \subset V$, where $B_X(p, \delta)$ denotes the open ball of radius δ in X centred on p. Thus if n were large enough to ensure that $1/2^n < \delta$, then $p \in F_n$ and $\operatorname{diam} F_n < \delta$, and hence $F_n \subset B_X(p, \delta) \subset V$, contradicting the fact that no finite collection of open sets belonging to $\mathcal{V}$ covers the set F_n. This contradiction shows that the Cauchy sequence $x_1, x_2, x_3, \ldots$ is not convergent.

We have thus shown that if X is a totally bounded metric space which is not compact then X is not complete. Thus every complete totally bounded metric space must be compact, as required. ∎

Theorem 5.9 *Let X be a metric space with distance function d. The following are equivalent:—*

(i) *X is compact,*

(ii) *every sequence of points of X has a convergent subsequence,*

(iii) *X is complete and totally bounded,*

Proof Propositions 5.4, 5.5 5.7 and 5.8 show that (i) implies (ii), (ii) implies (iii), and (iii) implies (i). It follows that (i), (ii) and (iii) are all equivalent to one another. ∎

Remark A subset K of $\mathbb{R}^n$ is complete if and only if it is closed in $\mathbb{R}^n$ (see Corollary 5.3). Also it is easy to see that K is totally bounded if and only if K is a bounded subset of $\mathbb{R}^n$. Thus Theorem 5.9 generalizes to arbitrary metric spaces the theorem which states that a subset K of $\mathbb{R}^n$ is compact if and only if it is both closed and bounded (Theorem 4.16).

5.3 The Lebesgue Lemma and Uniform Continuity

Lemma 5.10 (Lebesgue Lemma) *Let (X, d) be a compact metric space. Let $\mathcal{U}$ be an open cover of X. Then there exists a positive real number δ such that every subset of X whose diameter is less than δ is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$.*

Proof Every point of X is contained in at least one of the open sets belonging to the open cover $\mathcal{U}$. It follows from this that, for each point x of X, there exists some $\delta_x > 0$ such that the open ball $B(x, 2\delta_x)$ of radius $2\delta_x$ about the point x is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$. But then the collection consisting of the open balls $B(x, \delta_x)$ of radius δ_x about the points x of X forms an open cover of the compact space X. Therefore there exists a finite set $x_1, x_2, \ldots, x_r$ of points of X such that

$$B(x_1, \delta_1) \cup B(x_2, \delta_2) \cup \cdots \cup B(x_r, \delta_r) = X,$$

where $\delta_i = \delta_{x_i}$ for $i = 1, 2, \ldots, r$. Let $\delta > 0$ be given by

$$\delta = \text{minimum}(\delta_1, \delta_2, \ldots, \delta_r).$$

Suppose that A is a subset of X whose diameter is less than δ. Let u be a point of A. Then u belongs to $B(x_i, \delta_i)$ for some integer i between 1 and r. But then it follows that $A \subset B(x_i, 2\delta_i)$, since, for each point v of A,

$$d(v, x_i) \leq d(v, u) + d(u, x_i) < \delta + \delta_i \leq 2\delta_i.$$

But $B(x_i, 2\delta_i)$ is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$. Thus A is contained wholly within one of the open sets belonging to $\mathcal{U}$, as required. ∎

Let $\mathcal{U}$ be an open cover of a compact metric space X. A *Lebesgue number* for the open cover $\mathcal{U}$ is a positive real number δ such that every subset of X whose diameter is less than δ is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$. The Lebesgue Lemma thus states that there exists a Lebesgue number for every open cover of a compact metric space.

Let X and Y be metric spaces with distance functions d_X and d_Y respectively, and let $f\colon X \to Y$ be a function from X to Y. The function f is said to be *uniformly continuous* on X if and only if, given $\varepsilon > 0$, there exists some $\delta > 0$ such that $d_Y(f(x), f(x')) < \varepsilon$ for all points x and x' of X satisfying $d_X(x, x') < \delta$. (The value of δ should be independent of both x and x'.)

Theorem 5.11 *Let X and Y be metric spaces. Suppose that X is compact. Then every continuous function from X to Y is uniformly continuous.*

Proof Let d_X and d_Y denote the distance functions for the metric spaces X and Y respectively. Let $f\colon X \to Y$ be a continuous function from X to Y. We must show that f is uniformly continuous.

Let $\varepsilon > 0$ be given. For each $y \in Y$, define

$$V_y = \{x \in X : d_Y(f(x), y) < \tfrac{1}{2}\varepsilon\}.$$

Note that $V_y = f^{-1}\left(B_Y(y, \frac{1}{2}\varepsilon)\right)$, where $B_Y(y, \frac{1}{2}\varepsilon)$ denotes the open ball of radius $\frac{1}{2}\varepsilon$ about y in Y. Now $B_Y(y, \frac{1}{2}\varepsilon)$ is open in Y (see Lemma 1.4), and f is continuous. Therefore V_y is open in X for all $y \in Y$. Note that $x \in V_{f(x)}$ for all $x \in X$.

Now $\{V_y : y \in Y\}$ is an open cover of the compact metric space X. It follows from the Lebesgue Lemma (Lemma 5.10) that there exists some $\delta > 0$ such that every subset of X whose diameter is less than δ is a subset of some set V_y. Let x and x' be points of X satisfying $d_X(x, x') < \delta$. The diameter of the set $\{x, x'\}$ is $d_X(x, x')$, which is less than δ. Therefore there exists some $y \in Y$ such that $x \in V_y$ and $x' \in V_y$. But then $d_Y(f(x), y) < \frac{1}{2}\varepsilon$ and $d_Y(f(x'), y) < \frac{1}{2}\varepsilon$, and hence

$$d_Y(f(x), f(x')) \leq d_Y(f(x), y) + d_Y(y, f(x')) < \varepsilon.$$

This shows that $f\colon X \to Y$ is uniformly continuous, as required. ∎

Let K be a closed bounded subset of $\mathbb{R}^n$. It follows from Theorem 4.16 and Theorem 5.11 that any continuous function $f\colon K \to \mathbb{R}^k$ is uniformly continuous.