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Section 4: Compact Topological Spaces

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4 Compact Topological Spaces.

4.1 Open Covers and Compactness

Let $X$ be a topological space, and let $A$ be a subset of $X$. A collection of open sets in $X$ is said to cover $A$ if and only if every point of $A$ belongs to at least one of these open sets. In particular, an open cover of $X$ is collection of open sets in $X$ that covers $X$.

If $\mathcal{U}$ and $\mathcal{V}$ are open covers of some topological space $X$ then $\mathcal{V}$ is said to be a subcover of $\mathcal{U}$ if and only if every open set belonging to $\mathcal{V}$ also belongs to $\mathcal{U}$.

Definition A topological space $X$ is said to be compact if and only if every open cover of $X$ possesses a finite subcover.

Lemma 4.1 Let $X$ be a topological space. A subset $A$ of $X$ is compact (with respect to the subspace topology on $A$) if and only if, given any collection $\mathcal{U}$ of open sets in $X$ covering $A$, there exists a finite collection $V_1, V_2, \ldots, V_r$ of open sets belonging to $\mathcal{U}$ such that

$$A \subset V_1 \cup V_2 \cup \cdots \cup V_r.$$ 

Proof A subset $B$ of $A$ is open in $A$ (with respect to the subspace topology on $A$) if and only if $B = A \cap V$ for some open set $V$ in $X$. The desired result therefore follows directly from the definition of compactness.

4.2 The Heine-Borel Theorem

We now show that any closed bounded interval in the real line is compact. This result is known as the Heine-Borel Theorem. The proof of this theorem uses the least upper bound principle which states that, given any non-empty set $S$ of real numbers which is bounded above, there exists a least upper bound (or supremum) $\sup S$ for the set $S$.

Theorem 4.2 (Heine-Borel) Let $a$ and $b$ be real numbers satisfying $a < b$. Then the closed bounded interval $[a, b]$ is a compact subset of $\mathbb{R}$.

Proof Let $\mathcal{U}$ be a collection of open sets in $\mathbb{R}$ with the property that each point of the interval $[a, b]$ belongs to at least one of these open sets. We must show that $[a, b]$ is covered by finitely many of these open sets.

Let $S$ be the set of all $\tau \in [a, b]$ with the property that $[a, \tau]$ is covered by some finite collection of open sets belonging to $\mathcal{U}$, and let $s = \sup S$. Now
Let $s \in W$ for some open set $W$ belonging to $\mathcal{U}$. Moreover $W$ is open in $\mathbb{R}$, and therefore there exists some $\delta > 0$ such that $(s - \delta, s + \delta) \subset W$. Moreover $s - \delta$ is not an upper bound for the set $S$, hence there exists some $\tau \in S$ satisfying $\tau > s - \delta$. It follows from the definition of $S$ that $[a, \tau]$ is covered by some finite collection $V_1, V_2, \ldots, V_r$ of open sets belonging to $\mathcal{U}$.

Let $t \in [a, b]$ satisfy $\tau \leq t < s + \delta$. Then 

$$[a, t] \subset [a, \tau] \cup (s - \delta, s + \delta) \subset V_1 \cup V_2 \cup \cdots \cup V_r \cup W,$$

and thus $t \in S$. In particular $s \in S$, and moreover $s = b$, since otherwise $s$ would not be an upper bound of the set $S$. Thus $b \in S$, and therefore $[a, b]$ is covered by a finite collection of open sets belonging to $\mathcal{U}$, as required. 

### 4.3 Basic Properties of Compact Topological Spaces

**Lemma 4.3** Let $A$ be a closed subset of some compact topological space $X$. Then $A$ is compact.

**Proof** Let $\mathcal{U}$ be any collection of open sets in $X$ covering $A$. On adjoining the open set $X \setminus A$ to $\mathcal{U}$, we obtain an open cover of $X$. This open cover of $X$ possesses a finite subcover, since $X$ is compact. Moreover $A$ is covered by the open sets in the collection $\mathcal{U}$ that belong to this finite subcover. It follows from Lemma 4.1 that $A$ is compact, as required.

**Lemma 4.4** Let $f: X \to Y$ be a continuous function between topological spaces $X$ and $Y$, and let $A$ be a compact subset of $X$. Then $f(A)$ is a compact subset of $Y$.

**Proof** Let $\mathcal{V}$ be a collection of open sets in $Y$ which covers $f(A)$. Then $A$ is covered by the collection of all open sets of the form $f^{-1}(V)$ for some $V \in \mathcal{V}$. It follows from the compactness of $A$ that there exists a finite collection $V_1, V_2, \ldots, V_k$ of open sets belonging to $\mathcal{V}$ such that

$$A \subset f^{-1}(V_1) \cup f^{-1}(V_2) \cup \cdots \cup f^{-1}(V_k).$$

But then $f(A) \subset V_1 \cup V_2 \cup \cdots \cup V_k$. This shows that $f(A)$ is compact.

**Lemma 4.5** Let $f: X \to \mathbb{R}$ be a continuous real-valued function on a compact topological space $X$. Then $f$ is bounded above and below on $X$.

**Proof** The range $f(X)$ of the function $f$ is covered by some finite collection $I_1, I_2, \ldots, I_k$ of open intervals of the form $(-m, m)$, where $m \in \mathbb{N}$, since $f(X)$ is compact (Lemma 4.4) and $\mathbb{R}$ is covered by the collection of all intervals of this form. It follows that $f(X) \subset (-M, M)$, where $(-M, M)$ is the largest of the intervals $I_1, I_2, \ldots, I_k$. Thus the function $f$ is bounded above and below on $X$, as required.
Proposition 4.6 Let $f : X \to \mathbb{R}$ be a continuous real-valued function on a compact topological space $X$. Then there exist points $u$ and $v$ of $X$ such that $f(u) \leq f(x) \leq f(v)$ for all $x \in X$.

Proof Let $m = \inf \{f(x) : x \in X\}$ and $M = \sup \{f(x) : x \in X\}$. There must exist $v \in X$ satisfying $f(v) = M$, for if $f(x) < M$ for all $x \in X$ then the function $x \mapsto 1/(M - f(x))$ would be a continuous real-valued function on $X$ that was not bounded above, contradicting Lemma 4.5. Similarly there must exist $u \in X$ satisfying $f(u) = m$, since otherwise the function $x \mapsto 1/(f(x) - m)$ would be a continuous function on $X$ that was not bounded above, again contradicting Lemma 4.5. But then $f(u) \leq f(x) \leq f(v)$ for all $x \in X$, as required.

4.4 Compact Hausdorff Spaces

Proposition 4.7 Let $X$ be a Hausdorff topological space, and let $K$ be a compact subset of $X$. Let $x$ be a point of $X \setminus K$. Then there exist open sets $V$ and $W$ in $X$ such that $x \in V$, $K \subset W$ and $V \cap W = \emptyset$.

Proof For each point $y \in K$ there exist open sets $V_{x,y}$ and $W_{x,y}$ such that $x \in V_{x,y}$, $y \in W_{x,y}$ and $V_{x,y} \cap W_{x,y} = \emptyset$ (since $X$ is a Hausdorff space). But then there exists a finite set $\{y_1, y_2, \ldots, y_r\}$ of points of $K$ such that

$$K \subset \bigcup_{i=1}^{r} V_{x,y_i} \cup \bigcup_{i=1}^{r} W_{x,y_i}.$$

since $K$ is compact. Define

$$V = V_{x,y_1} \cap V_{x,y_2} \cap \cdots \cap V_{x,y_r}, \quad W = W_{x,y_1} \cup W_{x,y_2} \cup \cdots \cup W_{x,y_r}.$$

Then $V$ and $W$ are open sets, $x \in V$, $K \subset W$ and $V \cap W = \emptyset$, as required.

Corollary 4.8 Let $X$ be a Hausdorff topological space, and let $K$ be a compact subset of $X$. Then $K$ is closed.

Proof It follows immediately from Proposition 4.7 that, for each $x \in X \setminus K$, there exists an open set $V_x$ such that $x \in V_x$ and $V_x \cap K = \emptyset$. But then $X \setminus K$ is equal to the union of the open sets $V_x$ as $x$ ranges over all points of $X \setminus K$, and any set that is a union of open sets is itself an open set. We conclude that $X \setminus K$ is open, and thus $K$ is closed.

Proposition 4.9 Let $X$ be a Hausdorff topological space, and let $K_1$ and $K_2$ be compact subsets of $X$, where $K_1 \cap K_2 = \emptyset$. Then there exist open sets $U_1$ and $U_2$ such that $K_1 \subset U_1$, $K_2 \subset U_2$ and $U_1 \cap U_2 = \emptyset$. 

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Proof It follows from Proposition 4.7 that, for each point \( x \) of \( K_1 \), there exist open sets \( V_x \) and \( W_x \) such that \( x \in V_x \), \( K_2 \subset W_x \) and \( V_x \cap W_x = \emptyset \). But then there exists a finite set \( \{x_1, x_2, \ldots, x_r\} \) of points of \( K_1 \) such that

\[
K_1 \subset V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_r},
\]
since \( K_1 \) is compact. Define

\[
U_1 = V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_r}, \quad U_2 = W_{x_1} \cap W_{x_2} \cap \cdots \cap W_{x_r}.
\]

Then \( U_1 \) and \( U_2 \) are open sets, \( K_1 \subset U_1 \), \( K_2 \subset U_2 \) and \( U_1 \cap U_2 = \emptyset \), as required.

**Lemma 4.10** Let \( f: X \to Y \) be a continuous function from a compact topological space \( X \) to a Hausdorff space \( Y \). Then \( f(K) \) is closed in \( Y \) for every closed set \( K \) in \( X \).

Proof If \( K \) is a closed set in \( X \), then \( K \) is compact (Lemma 4.3), and therefore \( f(K) \) is compact (Lemma 4.4). But any compact subset of a Hausdorff space is closed (Corollary 4.8). Thus \( f(K) \) is closed in \( Y \), as required.

**Theorem 4.11** Let \( f: X \to Y \) be a continuous bijection from a compact topological space \( X \) to a Hausdorff space \( Y \). Then \( f: X \to Y \) is a homeomorphism.

Proof The function \( f \) is invertible, since it is a bijection. Let \( g: Y \to X \) be the inverse of \( f: X \to Y \). If \( U \) is open in \( X \) then \( X \setminus U \) is closed in \( X \), and hence \( f(X \setminus U) \) is closed in \( Y \), by Lemma 4.10. But

\[
f(X \setminus U) = g^{-1}(X \setminus U) = Y \setminus g^{-1}(U).
\]

It follows that \( g^{-1}(U) \) is open in \( Y \) for every open set \( U \) in \( X \). Therefore \( g: Y \to X \) is continuous, and thus \( f: X \to Y \) is a homeomorphism.

We recall that a function \( f: X \to Y \) from a topological space \( X \) to a topological space \( Y \) is said to be an **identification map** if it is surjective and satisfies the following condition: a subset \( U \) of \( Y \) is open in \( Y \) if and only if \( f^{-1}(U) \) is open in \( X \).

**Proposition 4.12** Let \( f: X \to Y \) be a continuous surjection from a compact topological space \( X \) to a Hausdorff space \( Y \). Then \( f: X \to Y \) is an identification map.
**Proof** Let $U$ be a subset of $Y$. We claim that $Y \setminus U = f(K)$, where $K = X \setminus f^{-1}(U)$. Clearly $f(K) \subset Y \setminus U$. Also, given any $y \in Y \setminus U$, there exists $x \in X$ satisfying $y = f(x)$, since $f: X \to Y$ is surjective. Moreover $x \in K$, since $f(x) \not\in U$. Thus $Y \setminus U \subset f(K)$, and hence $Y \setminus U = f(K)$, as claimed.

We must show that the set $U$ is open in $Y$ if and only if $f^{-1}(U)$ is open in $X$. First suppose that $f^{-1}(U)$ is open in $X$. Then $K$ is closed in $X$, and hence $f(K)$ is closed in $Y$, by Lemma 4.10. It follows that $U$ is open in $Y$.

Conversely if $U$ is open in $Y$ then $f^{-1}(U)$ is open in $X$, since $f: X \to Y$ is continuous. Thus the surjection $f: X \to Y$ is an identification map.

**Example** Let $S^1$ be the unit circle in $\mathbb{R}^2$, defined by

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$

and let $q: [0, 1] \to S^1$ be the function defined by

$$q(t) = (\cos 2\pi t, \sin 2\pi t) \quad (t \in [0, 1]).$$

Now the closed interval $[0, 1]$ is compact, by the Heine-Borel Theorem (Theorem 4.2), the circle $S^1$ is Hausdorff, and the function $q: [0, 1] \to S^1$ is a continuous surjection. It follows from Proposition 4.12 that the function $q: [0, 1] \to S^1$ is an identification map. Thus any function $f: S^1 \to Z$ from the circle $S^1$ to some topological space $Z$ is continuous if and only if the composition function $f \circ q: [0, 1] \to Z$ is continuous (see Lemma 2.16).

### 4.5 Finite Products of Compact Spaces

We shall show that a finite Cartesian product of compact spaces is compact. To prove this, we apply the following result, known as the **Tube Lemma**.

**Lemma 4.13** Let $X$ and $Y$ be topological spaces, let $K$ be a compact subset of $Y$, and $U$ be an open set in $X \times Y$. Let $V$ be the subset of $X$ defined by

$$V = \{x \in X : \{x\} \times K \subset U\}.$$ 

Then $V$ is an open set in $X$.

**Proof** Let $x \in V$. For each $y \in K$ there exist open subsets $D_y$ and $E_y$ of $X$ and $Y$ respectively such that $(x, y) \in D_y \times E_y$ and $D_y \times E_y \subset U$. Now there exists a finite set $\{y_1, y_2, \ldots, y_k\}$ of points of $K$ such that

$$K \subset E_{y_1} \cup E_{y_2} \cup \cdots \cup E_{y_k}.$$
since $K$ is compact. Set

$$N_x = D_{y_1} \cap D_{y_2} \cap \cdots \cap D_{y_k}.$$  

Then $N_x$ is an open set in $X$. Moreover

$$N_x \times K \subset \bigcup_{i=1}^k (N_x \times E_{y_i}) \subset \bigcup_{i=1}^k (D_{y_i} \times E_{y_i}) \subset U,$$

so that $N_x \subset V$. It follows that $V$ is the union of the open sets $N_x$ for all $x \in V$. Thus $V$ is itself an open set in $X$, as required. 

**Theorem 4.14** The Cartesian product $X \times Y$ of compact topological spaces $X$ and $Y$ is itself compact.

**Proof** Let $U$ be an open cover of $X \times Y$. We must show that this open cover possesses a finite subcover.

Let $x$ be a point of $X$. The set $\{x\} \times Y$ is a compact subset of $X \times Y$, since it is the image of the compact space $Y$ under the continuous map from $Y$ to $X \times Y$ which sends $y \in Y$ to $(x,y)$, and the image of any compact set under a continuous map is itself compact (Lemma 4.4). Therefore there exists a finite collection $U_1, U_2, \ldots, U_r$ of open sets belonging to the open cover $U$ such that

$$\{x\} \times Y \subset U_1 \cup U_2 \cup \cdots \cup U_r.$$  

Let

$$V_x = \{x' \in X : \{x'\} \times Y \subset U_1 \cup U_2 \cup \cdots \cup U_r\}.$$  

It follows from Lemma 4.13 that $V_x$ is an open set in $X$. We have therefore shown that, for each point $x$ in $X$, there exists an open set $V_x$ in $X$ containing the point $x$ such that $V_x \times Y$ is covered by finitely many of the open sets belonging to the open cover $U$.

Now $\{V_x : x \in X\}$ is an open cover of the space $X$. It follows from the compactness of $X$ that there exists a finite set $\{x_1, x_2, \ldots, x_r\}$ of points of $X$ such that

$$X = V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_r}.$$  

Now $X \times Y$ is the union of the sets $V_{x_j} \times Y$ for $j = 1, 2, \ldots, r$, and each of these sets can be covered by a finite collection of open sets belonging to the open cover $U$. On combining these finite collections, we obtain a finite collection of open sets belonging to $U$ which covers $X \times Y$. This shows that $X \times Y$ is compact. 

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**Corollary 4.15** The Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ of a finite number of compact topological spaces $X_1, X_2, \ldots, X_n$ is itself compact.

**Proof** It follows easily from the definition of the product topology that the product topologies on $X_1 \times X_2 \times \cdots \times X_n$ and $(X_1 \times X_2 \times \cdots \times X_{n-1}) \times X_n$ coincide. The desired result therefore follows from Theorem 4.14 by induction on $n$.

**Theorem 4.16** Let $K$ be a subset of $\mathbb{R}^n$. Then $K$ is compact if and only if $K$ is both closed and bounded.

**Proof** Suppose that $K$ is compact. Then $K$ is closed, since $\mathbb{R}^n$ is Hausdorff, and a compact subset of a Hausdorff space is closed (by Corollary 4.8). For each natural number $m$, let $B_m$ be the open ball of radius $m$ about the origin, given by

$$B_m = \{x \in \mathbb{R}^n : |x| < m\}.$$  

Then $\{B_m : m \in \mathbb{N}\}$ is an open cover of $\mathbb{R}$. It follows from the compactness of $K$ that there exist natural numbers $m_1, m_2, \ldots, m_k$ such that

$$K \subset B_{m_1} \cup B_{m_2} \cup \cdots \cup B_{m_k}.$$  

But then $K \subset B_M$, where $M$ is the maximum of $m_1, m_2, \ldots, m_k$, and thus $K$ is bounded.

Conversely suppose that $K$ is both closed and bounded. Then there exists some real number $L$ such that $K$ is contained within the closed cube $C$ given by

$$C = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : -L \leq x_j \leq L \text{ for } j = 1, 2, \ldots, n\}.$$  

Now the closed interval $[-L, L]$ is compact by the Heine-Borel Theorem (Theorem 4.2), and $C$ is the Cartesian product of $n$ copies of the compact set $[-L, L]$. It follows from Corollary 4.15 that $C$ is compact. But $K$ is a closed subset of $C$, and a closed subset of a compact topological space is itself compact, by Lemma 4.3. Thus $K$ is compact, as required.