Course 212: Academic Year 1991-2
Section 4: Compact Topological Spaces

D. R. Wilkins

Contents

4 Compact Topological Spaces.

4.1 Open Covers and Compactness

Let X be a topological space, and let A be a subset of X. A collection of open sets in X is said to *cover* A if and only if every point of A belongs to at least one of these open sets. In particular, an *open cover* of X is collection of open sets in X that covers X.

If $\mathcal{U}$ and $\mathcal{V}$ are open covers of some topological space X then $\mathcal{V}$ is said to be a *subcover* of $\mathcal{U}$ if and only if every open set belonging to $\mathcal{V}$ also belongs to $\mathcal{U}$.

Definition A topological space X is said to be *compact* if and only if every open cover of X possesses a finite subcover.

Lemma 4.1 *Let X be a topological space. A subset A of X is compact (with respect to the subspace topology on A) if and only if, given any collection $\mathcal{U}$ of open sets in X covering A, there exists a finite collection $V_1, V_2, \ldots, V_r$ of open sets belonging to $\mathcal{U}$ such that*

$$A \subset V_1 \cup V_2 \cup \cdots \cup V_r.$$

Proof A subset B of A is open in A (with respect to the subspace topology on A) if and only if $B = A \cap V$ for some open set V in X. The desired result therefore follows directly from the definition of compactness. ∎

4.2 The Heine-Borel Theorem

We now show that any closed bounded interval in the real line is compact. This result is known as the *Heine-Borel Theorem.* The proof of this theorem uses the *least upper bound principle* which states that, given any non-empty set S of real numbers which is bounded above, there exists a *least upper bound* (or *supremum*) $\sup S$ for the set S.

Theorem 4.2 (Heine-Borel) *Let a and b be real numbers satisfying $a < b$. Then the closed bounded interval $[a, b]$ is a compact subset of $\mathbb{R}$.*

Proof Let $\mathcal{U}$ be a collection of open sets in $\mathbb{R}$ with the property that each point of the interval $[a, b]$ belongs to at least one of these open sets. We must show that $[a, b]$ is covered by finitely many of these open sets.

Let S be the set of all $\tau \in [a, b]$ with the property that $[a, \tau]$ is covered by some finite collection of open sets belonging to $\mathcal{U}$, and let $s = \sup S$. Now

$s \in W$ for some open set W belonging to $\mathcal{U}$. Moreover W is open in $\mathbb{R}$, and therefore there exists some $\delta > 0$ such that $(s - \delta, s + \delta) \subset W$. Moreover $s - \delta$ is not an upper bound for the set S, hence there exists some $\tau \in S$ satisfying $\tau > s - \delta$. It follows from the definition of S that $[a, \tau]$ is covered by some finite collection $V_1, V_2, \ldots, V_r$ of open sets belonging to $\mathcal{U}$.

Let $t \in [a, b]$ satisfy $\tau \leq t < s + \delta$. Then

$$[a, t] \subset [a, \tau] \cup (s - \delta, s + \delta) \subset V_1 \cup V_2 \cup \cdots \cup V_r \cup W,$$

and thus $t \in S$. In particular $s \in S$, and moreover $s = b$, since otherwise s would not be an upper bound of the set S. Thus $b \in S$, and therefore $[a, b]$ is covered by a finite collection of open sets belonging to $\mathcal{U}$, as required. ∎

4.3 Basic Properties of Compact Topological Spaces

Lemma 4.3 *Let A be a closed subset of some compact topological space X. Then A is compact.*

Proof Let $\mathcal{U}$ be any collection of open sets in X covering A. On adjoining the open set $X \setminus A$ to $\mathcal{U}$, we obtain an open cover of X. This open cover of X possesses a finite subcover, since X is compact. Moreover A is covered by the open sets in the collection $\mathcal{U}$ that belong to this finite subcover. It follows from Lemma 4.1 that A is compact, as required. ∎

Lemma 4.4 *Let $f: X \to Y$ be a continuous function between topological spaces X and Y, and let A be a compact subset of X. Then $f(A)$ is a compact subset of Y.*

Proof Let $\mathcal{V}$ be a collection of open sets in Y which covers $f(A)$. Then A is covered by the collection of all open sets of the form $f^{-1}(V)$ for some $V \in \mathcal{V}$. It follows from the compactness of A that there exists a finite collection $V_1, V_2, \ldots, V_k$ of open sets belonging to $\mathcal{V}$ such that

$$A \subset f^{-1}(V_1) \cup f^{-1}(V_2) \cup \cdots \cup f^{-1}(V_k).$$

But then $f(A) \subset V_1 \cup V_2 \cup \cdots \cup V_k$. This shows that $f(A)$ is compact. ∎

Lemma 4.5 *Let $f: X \to \mathbb{R}$ be a continuous real-valued function on a compact topological space X. Then f is bounded above and below on X.*

Proof The range $f(X)$ of the function f is covered by some finite collection $I_1, I_2, \ldots, I_k$ of open intervals of the form $(-m, m)$, where $m \in \mathbb{N}$, since $f(X)$ is compact (Lemma 4.4) and $\mathbb{R}$ is covered by the collection of all intervals of this form. It follows that $f(X) \subset (-M, M)$, where $(-M, M)$ is the largest of the intervals $I_1, I_2, \ldots, I_k$. Thus the function f is bounded above and below on X, as required. ∎

Proposition 4.6 *Let $f\colon X \to \mathbb{R}$ be a continuous real-valued function on a compact topological space X. Then there exist points u and v of X such that $f(u) \leq f(x) \leq f(v)$ for all $x \in X$.*

Proof Let $m = \inf\{f(x) : x \in X\}$ and $M = \sup\{f(x) : x \in X\}$. There must exist $v \in X$ satisfying $f(v) = M$, for if $f(x) < M$ for all $x \in X$ then the function $x \mapsto 1/(M - f(x))$ would be a continuous real-valued function on X that was not bounded above, contradicting Lemma 4.5. Similarly there must exist $u \in X$ satisfying $f(u) = m$, since otherwise the function $x \mapsto 1/(f(x)-m)$ would be a continuous function on X that was not bounded above, again contradicting Lemma 4.5. But then $f(u) \leq f(x) \leq f(v)$ for all $x \in X$, as required. ∎

4.4 Compact Hausdorff Spaces

Proposition 4.7 *Let X be a Hausdorff topological space, and let K be a compact subset of X. Let x be a point of $X \setminus K$. Then there exist open sets V and W in X such that $x \in V$, $K \subset W$ and $V \cap W = \emptyset$.*

Proof For each point $y \in K$ there exist open sets $V_{x,y}$ and $W_{x,y}$ such that $x \in V_{x,y}$, $y \in W_{x,y}$ and $V_{x,y} \cap W_{x,y} = \emptyset$ (since X is a Hausdorff space). But then there exists a finite set $\{y_1, y_2, \ldots, y_r\}$ of points of K such that

$$K \subset W_{x,y_1} \cup W_{x,y_2} \cup \cdots \cup W_{x,y_r},$$

since K is compact. Define

$$V = V_{x,y_1} \cap V_{x,y_2} \cap \cdots \cap V_{x,y_r}, \qquad W = W_{x,y_1} \cup W_{x,y_2} \cup \cdots \cup W_{x,y_r}.$$

Then V and W are open sets, $x \in V$, $K \subset W$ and $V \cap W = \emptyset$, as required. ∎

Corollary 4.8 *Let X be a Hausdorff topological space, and let K be a compact subset of X. Then K is closed.*

Proof It follows immediately from Proposition 4.7 that, for each $x \in X \setminus K$, there exists an open set V_x such that $x \in V_x$ and $V_x \cap K = \emptyset$. But then $X \setminus K$ is equal to the union of the open sets V_x as x ranges over all points of $X \setminus K$, and any set that is a union of open sets is itself an open set. We conclude that $X \setminus K$ is open, and thus K is closed. ∎

Proposition 4.9 *Let X be a Hausdorff topological space, and let K_1 and K_2 be compact subsets of X, where $K_1 \cap K_2 = \emptyset$. Then there exist open sets U_1 and U_2 such that $K_1 \subset U_1$, $K_2 \subset U_2$ and $U_1 \cap U_2 = \emptyset$.*

Proof It follows from Proposition 4.7 that, for each point x of K_1, there exist open sets V_x and W_x such that $x \in V_x$, $K_2 \subset W_x$ and $V_x \cap W_x = \emptyset$. But then there exists a finite set $\{x_1, x_2, \ldots, x_r\}$ of points of K_1 such that

$$K_1 \subset V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_r},$$

since K_1 is compact. Define

$$U_1 = V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_r}, \qquad U_2 = W_{x_1} \cap W_{x_2} \cap \cdots \cap W_{x_r}.$$

Then U_1 and U_2 are open sets, $K_1 \subset U_1$, $K_2 \subset U_2$ and $U_1 \cap U_2 = \emptyset$, as required. ∎

Lemma 4.10 *Let $f: X \to Y$ be a continuous function from a compact topological space X to a Hausdorff space Y. Then $f(K)$ is closed in Y for every closed set K in X.*

Proof If K is a closed set in X, then K is compact (Lemma 4.3), and therefore $f(K)$ is compact (Lemma 4.4). But any compact subset of a Hausdorff space is closed (Corollary 4.8). Thus $f(K)$ is closed in Y, as required. ∎

Theorem 4.11 *Let $f: X \to Y$ be a continuous bijection from a compact topological space X to a Hausdorff space Y. Then $f: X \to Y$ is a homeomorphism.*

Proof The function f is invertible, since it is a bijection. Let $g: Y \to X$ be the inverse of $f: X \to Y$. If U is open in X then $X \setminus U$ is closed in X, and hence $f(X \setminus U)$ is closed in Y, by Lemma 4.10. But

$$f(X \setminus U) = g^{-1}(X \setminus U) = Y \setminus g^{-1}(U).$$

It follows that $g^{-1}(U)$ is open in Y for every open set U in X. Therefore $g: Y \to X$ is continuous, and thus $f: X \to Y$ is a homeomorphism. ∎

We recall that a function $f: X \to Y$ from a topological space X to a topological space Y is said to be an *identification map* if it is surjective and satisfies the following condition: a subset U of Y is open in Y if and only if $f^{-1}(U)$ is open in X.

Proposition 4.12 *Let $f: X \to Y$ be a continuous surjection from a compact topological space X to a Hausdorff space Y. Then $f: X \to Y$ is an identification map.*

Proof Let U be a subset of Y. We claim that $Y \setminus U = f(K)$, where $K = X \setminus f^{-1}(U)$. Clearly $f(K) \subset Y \setminus U$. Also, given any $y \in Y \setminus U$, there exists $x \in X$ satisfying $y = f(x)$, since $f\colon X \to Y$ is surjective. Moreover $x \in K$, since $f(x) \notin U$. Thus $Y \setminus U \subset f(K)$, and hence $Y \setminus U = f(K)$, as claimed.

We must show that the set U is open in Y if and only if $f^{-1}(U)$ is open in X. First suppose that $f^{-1}(U)$ is open in X. Then K is closed in X, and hence $f(K)$ is closed in Y, by Lemma 4.10. It follows that U is open in Y. Conversely if U is open in Y then $f^{-1}(Y)$ is open in X, since $f\colon X \to Y$ is continuous. Thus the surjection $f\colon X \to Y$ is an identification map. ∎

Example Let S^1 be the unit circle in $\mathbb{R}^2$, defined by

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\},$$

and let $q\colon [0, 1] \to S^1$ be the function defined by

$$q(t) = (\cos 2\pi t, \sin 2\pi t) \qquad (t \in [0, 1]).$$

Now the closed interval $[0, 1]$ is compact, by the Heine-Borel Theorem (Theorem 4.2), the circle S^1 is Hausdorff, and the function $q\colon [0, 1] \to S^1$ is a continuous surjection. It follows from Proposition 4.12 that the function $q\colon [0, 1] \to S^1$ is an identification map. Thus any function $f\colon S^1 \to Z$ from the circle S^1 to some topological space Z is continuous if and only if the composition function $f \circ q\colon [0, 1] \to Z$ is continuous (see Lemma 2.16).

4.5 Finite Products of Compact Spaces

We shall show that a finite Cartesian product of compact spaces is compact. To prove this, we apply the following result, known as the *Tube Lemma*.

Lemma 4.13 *Let X and Y be topological spaces, let K be a compact subset of Y, and U be an open set in $X \times Y$. Let V be the subset of X defined by*

$$V = \{x \in X : \{x\} \times K \subset U\}.$$

Then V is an open set in X.

Proof Let $x \in V$. For each $y \in K$ there exist open subsets D_y and E_y of X and Y respectively such that $(x, y) \in D_y \times E_y$ and $D_y \times E_y \subset U$. Now there exists a finite set $\{y_1, y_2, \ldots, y_k\}$ of points of K such that

$$K \subset E_{y_1} \cup E_{y_2} \cup \cdots \cup E_{y_k},$$

since K is compact. Set

$$N_x = D_{y_1} \cap D_{y_2} \cap \cdots \cap D_{y_k}.$$

Then N_x is an open set in X. Moreover

$$N_x \times K \subset \bigcup_{i=1}^{k} (N_x \times E_{y_i}) \subset \bigcup_{i=1}^{k} (D_{y_i} \times E_{y_i}) \subset U,$$

so that $N_x \subset V$. It follows that V is the union of the open sets N_x for all $x \in V$. Thus V is itself an open set in X, as required. ∎

Theorem 4.14 *The Cartesian product $X \times Y$ of compact topological spaces X and Y is itself compact.*

Proof Let $\mathcal{U}$ be an open cover of $X \times Y$. We must show that this open cover possesses a finite subcover.

Let x be a point of X. The set $\{x\} \times Y$ is a compact subset of $X \times Y$, since it is the image of the compact space Y under the continuous map from Y to $X \times Y$ which sends $y \in Y$ to (x, y), and the image of any compact set under a continuous map is itself compact (Lemma 4.4). Therefore there exists a finite collection $U_1, U_2, \ldots, U_r$ of open sets belonging to the open cover $\mathcal{U}$ such that

$$\{x\} \times Y \subset U_1 \cup U_2 \cup \cdots \cup U_r.$$

Let

$$V_x = \{x' \in X : \{x'\} \times Y \subset U_1 \cup U_2 \cup \cdots \cup U_r\}.$$

It follows from Lemma 4.13 that V_x is an open set in X. We have therefore shown that, for each point x in X, there exists an open set V_x in X containing the point x such that $V_x \times Y$ is covered by finitely many of the open sets belonging to the open cover $\mathcal{U}$.

Now $\{V_x : x \in X\}$ is an open cover of the space X. It follows from the compactness of X that there exists a finite set $\{x_1, x_2, \ldots, x_r\}$ of points of X such that

$$X = V_{x_1} \cup V_{x_2} \cup \cdots \cup V_{x_r}.$$

Now $X \times Y$ is the union of the sets $V_{x_j} \times Y$ for $j = 1, 2, \ldots, r$, and each of these sets can be covered by a finite collection of open sets belonging to the open cover $\mathcal{U}$. On combining these finite collections, we obtain a finite collection of open sets belonging to $\mathcal{U}$ which covers $X \times Y$. This shows that $X \times Y$ is compact. ∎

Corollary 4.15 *The Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ of a finite number of compact topological spaces $X_1, X_2, \ldots, X_n$ is itself compact.*

Proof It follows easily from the definition of the product topology that the product topologies on $X_1 \times X_2 \times \cdots \times X_n$ and $(X_1 \times X_2 \times \cdots \times X_{n-1}) \times X_n$ coincide. The desired result therefore follows from Theorem 4.14 by induction on n. ∎

Theorem 4.16 *Let K be a subset of $\mathbb{R}^n$. Then K is compact if and only if K is both closed and bounded.*

Proof Suppose that K is compact. Then K is closed, since $\mathbb{R}^n$ is Hausdorff, and a compact subset of a Hausdorff space is closed (by Corollary 4.8). For each natural number m, let B_m be the open ball of radius m about the origin, given by

$$B_m = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < m\}.$$

Then $\{B_m : m \in \mathbb{N}\}$ is an open cover of $\mathbb{R}$. It follows from the compactness of K that there exist natural numbers $m_1, m_2, \ldots, m_k$ such that

$$K \subset B_{m_1} \cup B_{m_2} \cup \cdots \cup B_{m_k}.$$

But then $K \subset B_M$, where M is the maximum of $m_1, m_2, \ldots, m_k$, and thus K is bounded.

Conversely suppose that K is both closed and bounded. Then there exists some real number L such that K is contained within the closed cube C given by

$$C = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : -L \leq x_j \leq L \text{ for } j = 1, 2, \ldots, n\}.$$

Now the closed interval $[-L, L]$ is compact by the Heine-Borel Theorem (Theorem 4.2), and C is the Cartesian product of n copies of the compact set $[-L, L]$. It follows from Corollary 4.15 that C is compact. But K is a closed subset of C, and a closed subset of a compact topological space is itself compact, by Lemma 4.3. Thus K is compact, as required. ∎